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A note on the connectedness of the efficiente frontier

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A note on the connectedness of the efficient frontier

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Abstract

The aim of this paper is to study sufficient conditions for the connectedness of the efficient frontier. We propose an unifying approach, based on a new regularity concept, which allows to generalize the results in [1, 6]. Connectedness results are studied for both V-compact and not V-compact sets. The obtained results are finally used to study the connectedness of the efficient frontier of continuous multiobjective functions.

1 Introduction

One of the main topics in multiobjective optimization is the study of the properties of the efficient frontier. Among all the topological properties of these sets, it is worth studying their connectedness, which allows to continuously move from one efficient solution to any other without leaving the efficient frontier. This problem has been approached in the literature in various way (see for example [1, 4, 5, 6, 7, 9, 10, 11, 12, 13]), that is assuming convexity hypothesis, strict quasiconcavity ones, studying both efficient and weak efficient points.

In this paper we aim to study some sufficient conditions for the connectedness of the efficient frontier which generalize the ones by [1, 6]. With this purpose, see Section 2, we first present some general preliminary results related to the efficient frontier; note that the efficient points are obtained with a partial order given by any closed, convex, pointed cone with nonempty interior.

Then, see Section 3, in order to study the connectedness of the efficient frontier we suggest an unifying approach based on the use of "compactness-connectedness regularity conditions", that is to say conditions guaranteeing the connectedness of the efficient frontier of compact sets. This general approach can be used when the partial order is given by any closed, convex, pointed cone with nonempty interior.

The introduced general approach, see Section 4, is then applied in the particular case of a partial ordering given by polyhedral/paretian cones. This allow us to generalize both the result by Benoist [1] and the one by Hu and Sun [6] under the assumption of the V-compactedness of the set. The generalization of the result by Benoist is given using also a generalization of sequentially strictly quasiconcave sets. It is worth noticing that the two obtained generalizations cannot be applied to the same sets, that is to say that none of them generalizes the other.

The results obtained in Section 4, allow us in Section 5 to state connectedness sufficient conditions also for non V-compacted sets, thus extending furthermore the class of connected efficient frontiers which can be recognized. Finally, in Section 6 we show how the obtained results may be used in order to study the connectedness of the efficient frontier in the image of a continuous vector valued function.

2 Definitions and preliminary results

In this preliminary section we aim to provide some general definitions and results which will be used in the rest of the paper. Let us first recall the definitions of efficient frontier and V-compact sets.

Definition 2.1 Let $X \subset \mathbb{R}^n$, let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior, let $a \in \mathbb{R}^n$. By introducing the following notation:

$$V(a) = \{a\} + V$$

we define the set of the **efficient points** of X with respect to the cone V, also called the **efficient frontier** of X with respect to V, as:

$$Max(X,V) = \{a \in X \mid V(a) \cap X = \{a\}\}$$

Definition 2.2 Let $X \subset \mathbb{R}^n$ and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. The set X is said to be **V-compact** if for all $a \in \mathbb{R}^n$ the sets $V(a) \cap X$ are compact, while it is said to be **weakly V-compact** if for all $a \in X$ the sets $V(a) \cap X$ are compact.

Note that the weak V-compactedness has been studied in [3]. Note also that a V-compact set is also weak V-compact while the converse is not true, as it is pointed out in the next example.

Example 2.1 Let $X = \{(x,y) \in \mathbb{R}^2 \mid x+y=1,\ 0<x<1\}$ and $V = \mathbb{R}^2_+$. Clearly, the set X is weakly V-compact since $V(a) \cap X$ is compact $\forall a \in X$, while it is not V-compact since $V(a) \cap X$ with $a = (0,0)$ is not compact.

It is worth noticing also the next known result which guarantees a nonempty efficient frontier.

Property 2.1 *Let $X \subset \mathbb{R}^n$ be nonempty and closed and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. If there exists $a \in X$ such that $V(a) \cap X$ is compact then $Max(X,V) \neq \emptyset$.*

In order to provide the following further useful results related to V-compact sets, from now on we will denote with V^+ the positive polar cone of V, that is to say that

$$V^+ = \{u \in \mathbb{R}^n \mid v^T u \geq 0\ \forall v \in V\}$$

Property 2.2 *Let $X \subset \mathbb{R}^n$ be nonempty and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. Consider also the following conditions:*

(C1) there exists a compact set C in $\mathbb{R}^n$ such that $X = C - V$;

(C2) X is closed and there exists $v \in \mathbb{R}^n$ such that $X \subset \{v\} - V$;

(C3) X is V-compact.

Then:

$$(C1) \implies (C2) \quad \textit{and} \quad (C2) \implies (C3)$$

Proof $(C1) \implies (C2)$. Let us first note that X is a closed set since it is the sum of a compact set and a closed cone. Given a vector $u \in int(V)$ and a point $a \in C$, we just have to prove that there exists $k \in \mathbb{R}$, $k \geq 0$, such that $C \subset (a+ku)-V$. Suppose by contradiction that for all $k \geq 0$ there exists $b_k \in C$ such that $b_k \notin (a+ku)-V$. Since $(a+ku)-V$ is a closed convex set for a known separation theorem there exists $\alpha \in V^+$, $\|\alpha\|=1$, such that $\alpha^T(b_k-a-ku)>0$. Hence:

$$\alpha^T a + k\alpha^T u < \alpha^T b_k < \|\alpha\| \, \|b_k\| = \|b_k\|$$

note also that since $u \in int(V)$ and $\alpha \in V^+$ then $\alpha^T u > 0$. Since the previous inequalities hold for all $k \geq 0$ then:

$$+\infty = \lim_{k\to+\infty} \alpha^T a + k\alpha^T u < \lim_{k\to+\infty} \|b_k\|$$

and this contradicts the compactness of C.

$(C2) \implies (C3)$. Let $v \in \mathbb{R}^n$ be a point such that $X \subset \{v\}-V$ and let us verify that for all $a \in \mathbb{R}^n$ the sets $V(a)\cap X$ are compact. Let $a \in \mathbb{R}^n$ such that $V(a)\cap X = (\{a\}+V)\cap X$ is nonempty. Since X and $V(a)$ are closed then $V(a)\cap X$ is closed. Suppose now, by contradiction, that $V(a)\cap X$ is not compact, then there exists a sequence $\{x_k\} \subset \{a\}+V$ and $\{x_k\} \subset X \subset \{v\}-V$ such that $\|x_k\| \to +\infty$. As a consequence, there exist two sequences $\{u_k\} \subset V$ and $\{w_k\} \subset V$ such that for all $k \geq 0$ it is $x_k = a+u_k = v-w_k$. Since $\|x_k\| \to +\infty$ then $\|u_k\| \to +\infty$ and $\|w_k\| \to +\infty$ too. We can then extract a subsequence of $\{u_k\}$ such that $\lim_{k\to+\infty} u_k/\|u_k\| = \bar{u} \in V$ with $\|\bar{u}\|=1$ and this yields, for the closure of V, that:

$$\bar{u} = \lim_{k\to+\infty} \frac{u_k}{\|u_k\|} = \lim_{k\to+\infty} \frac{a-v}{\|u_k\|} + \frac{u_k}{\|u_k\|} = \lim_{k\to+\infty} \frac{-w_k}{\|u_k\|} \in -V$$

and since $\bar{u} \in V$ with $\|\bar{u}\|=1$ this contradicts the pointedness of V. □

Property 2.3 *Let $X \subset \mathbb{R}^n$ be nonempty and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. If X is V-compact then X is closed.*

Proof If the set X is finite the result is obvious. Let now X be not finite and assume by contradiction that there exists a sequence $\{b_k\} \subset X$ such that $\{b_k\} \to \bar{b} \notin X$. Given any vector v in the interior of V let us define the point $a = \bar{b} - v$; as a consequence we have that $b_k \in V(a) \cap X$ for k great enough and this implies that $V(a) \cap X$ is not compact since $\{b_k\} \to \bar{b} \notin X$. □

For the sake of completeness, let us provide also the following useful known results regarding to the efficient frontier.

Property 2.4 *Let $X \subset \mathbb{R}^n$ be nonempty and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. Then,*

$$Max(X, V) = Max(X - V, V)$$

Proof ($\subseteq$) Let $a \in Max(X, V) \subseteq X \subseteq X - V$ and suppose by contradiction that $a \notin Max(X - V, V)$, that is to say that $\exists w \in V$, $w \neq 0$, such that $a + w \in X - V$. Then, $\exists b \in X$ and $\exists v \in V$ such that $a + w = b - v$. As a consequence, $a + w + v = b \in X$ and since $w + v \neq 0$ this implies that $a \notin Max(X, V)$, which is a contradiction.

($\supseteq$) Let $a \in Max(X - V, V) \subseteq X - V$, then $\exists b \in X$ and $\exists v \in V$ such that $a = b - v$. As a consequence, $a + v = b \in X \subseteq X - V$ and since $a \in Max(X - V, V)$ this implies $v = 0$ and $a = b \in X$. Suppose now by contradiction that $a \notin Max(X, V)$, that is to say that $\exists w \in V$, $w \neq 0$, such that $a + w \in X \subseteq X - V$. Then, $a \notin Max(X - V, V)$ which is a contradiction. □

Property 2.5 *Let $X \subset \mathbb{R}^n$ be nonempty and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. Defining the sets $Y_X = X - V$ and $Y_M = Max(X, V) - V$ it results:*

$$Max(X, V) = Max(Y_X, V) = Max(Y_M, V)$$

Proof By means of Property 2.4 we have:

$$\begin{aligned} Max(Y_X, V) &= Max(X - V, V) = Max(X, V) \\ Max(Y_M, V) &= Max(Max(X, V) - V, V) \\ &= Max(Max(X, V), V) = Max(X, V) \end{aligned}$$

and the result is then proved. □

Finally, let us state the next property which provides a condition more general than the closedness of the efficient frontier of a V-compact set.

Property 2.6 *Let $X \subset \mathbb{R}^n$ be nonempty and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. If X is V-compact and $Max(X, V)$ is closed then $Max(X, V)$ is nonempty and V-compact.*

Proof Since X is nonempty and V-compact then $Max(X, V)$ is nonempty. Notice also that for all $a \in \mathbb{R}^n$ the sets $V(a) \cap Max(X, V)$ are closed for the closedness of V and $Max(X, V)$. Suppose now by contradiction that $Max(X, V)$ is not V-compact, that is to say that there exists $a \in \mathbb{R}^n$ such that $V(a) \cap Max(X, V)$ is closed but not compact. As a consequence, the set $V(a) \cap X$, which contains $V(a) \cap Max(X, V)$, is not compact too and this contradicts the V-compactness of X. □

Note that the forthcoming Example 4.2 points out the importance in Property 2.6 of the closedness of the efficient frontier. Note also that the converse of the implication stated in Property 2.6 does not hold, as it is pointed out in the forthcoming Example 5.1.

3 A unifying approach

In this section we are going to describe the unifying approach which will be used in the rest of the paper in order to study the connectedness of the efficient frontier.

The concept of connectedness which will be used through this paper is the usual one (see for example [13]).

Definition 3.1 A set $X \subset \mathbb{R}^n$ is said to be **disconnected** if there exists two nonempty sets $X_1, X_2 \subset \mathbb{R}^n$ such that $X = X_1 \cup X_2$ and $cl(X_1) \cap X_2 = X_1 \cap cl(X_2) = \emptyset$. A set $X \subset \mathbb{R}^n$ is said to be **connected** if it is not disconnected.

In order to study the connectedness of the efficient frontier let us now introduce the following new concept of regularity.

Definition 3.2 Let X be a nonempty and closed subset of $\mathbb{R}^n$ and let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior. From now on any condition which guarantees that the following implication holds:

$$X \text{ is compact } \Longrightarrow Max(X, V) \text{ is connected}$$

will be referred to as a **Compactness-Connectedness regularity condition**, that is a **CC-regularity condition** in short.

The use of the previously introduced regularity concept is pointed out in the following theorem, which suggests how to obtain new sufficient conditions for the connectedness of the efficient frontier. With this aim, a technical lemma is needed.

Lemma 3.1 *Let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior and let X be a nonempty subset of $\mathbb{R}^n$. Let also $\bar{a} \in X$, $u \in int(V)$ and $a_k = \bar{a} - ku$ with $k \in \mathbb{N}$. Then, $\{Max(V(a_k) \cap X, V)\}$ is an increasing sequence of sets such that*

$$\cup_{k \in \mathbb{N}} Max(V(a_k) \cap X, V) = Max(X, V).$$

Proof It is easy to check that for all $k \in \mathbb{N}$ it is:

$$\begin{aligned} V(a_k) &\subseteq V(a_{k+1}) \ , \\ V(a_k) \cap X &\subseteq V(a_{k+1}) \cap X \ , \\ Max(V(a_k) \cap X, V) &\subseteq Max(V(a_{k+1}) \cap X, V) \ , \end{aligned}$$

so that $\{Max(V(a_k) \cap X, V)\}$ is an increasing sequence of sets. For all $k \in \mathbb{N}$ it is $V(a_k) \cap X \subseteq X$ so that $Max(V(a_k) \cap X, V) \subseteq Max(X, V)$ and hence

$$\cup_{k \in \mathbb{N}} Max(V(a_k) \cap X, V) \subseteq Max(X, V)$$

We are now left to prove that $Max(X,V) \subseteq \cup_{k\in I\!N} Max(V(a_k)\cap X, V)$, that is to say that for all $b \in Max(X,V)$ there exists $\bar{k} \in I\!N$ such that $b \in Max(V(a_{\bar{k}}) \cap X, V)$.

With this aim, given $b \in Max(X,V) \subseteq X$ let us first prove that there exists $\bar{k} \in I\!N$ such that $b \in V(a_{\bar{k}}) = \{a_{\bar{k}}\} + V$, that is to say that $a_{\bar{k}} \in \{b\} - V$. By contradiction suppose that:

$$\{a_k \in \mathbb{R}^n \mid a_k = \bar{a} - ku,\ k \in \mathbb{R}_+\} \cap (\{b\} - V) = \emptyset$$

By means of a known separation theorem between convex sets, there exists a separating hyperplane $\alpha^T z = \beta$ such that:

(i) $\alpha^T(b - v) \leq \beta$ for all $v \in V$;

(ii) $\alpha^T(\bar{a} - ku) \geq \beta$ for all $k \geq 0$;

Let now $\bar{v}$ be any nonzero vector of V so that for all $t > 0$ it is $t\bar{v} \in V$; from (i) we have:

$$\alpha^T b - \beta = \lim_{t\to+\infty} \alpha^T b - \beta \leq \lim_{t\to+\infty} \alpha^T(t\bar{v}) = \alpha^T\bar{v} \lim_{t\to+\infty} t$$

which is possible only if $\alpha^T\bar{v} \geq 0$. In other words, we have proved that $\alpha^T\bar{v} \geq 0$ for all $\bar{v} \in V$ which means that $\alpha \in V^+$. Since $u \in int(V)$ it then results $\alpha^T u > 0$ and hence:

$$\alpha^T\bar{a} - \beta = \lim_{k\to+\infty} \alpha^T\bar{a} - \beta \geq \lim_{k\to+\infty} \alpha^T(ku) = \alpha^T u \lim_{k\to+\infty} k = +\infty$$

which is a contradiction. We have then proved that there exists $\bar{k} \in I\!N$ such that $b \in V(a_{\bar{k}})$; since $b \in Max(X,V) \subseteq X$, that is $V(b) \cap X = \{b\}$, this yields:

$$V(b) \cap (V(a_{\bar{k}}) \cap X) = V(a_{\bar{k}}) \cap (V(b) \cap X) = V(a_{\bar{k}}) \cap \{b\} = \{b\}$$

so that $b \in Max(V(a_{\bar{k}}) \cap X, V)$. The whole result is then proved. □

Theorem 3.1 *Let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior and let X be a nonempty and V-compact subset of $\mathbb{R}^n$. If a CC-regularity condition holds for all the sets $V(a) \cap X$, with $a \in \mathbb{R}^n$, then $Max(X,V)$ is a nonempty and connected set.*

Proof Given $\bar{a} \in X \neq \emptyset$ and $u \in int(V)$ let us define the points $a_k = \bar{a} - ku$ with $k \in I\!N$. For the hypothesis the sets $V(a_k) \cap X$ are compact and they are also nonempty since $\bar{a} \in$

$V(a_k) \cap X$. Hence, for the CC-regularity condition, the sets $Max(V(a_k) \cap X, V)$ are connected and nonempty. By means of Lemma 3.1 we then have that $\{Max(V(a_k) \cap X, V)\}$ is an increasing sequence of nonempty connected sets such that

$$\cup_{k \in \mathbb{N}} Max(V(a_k) \cap X, V) = Max(X, V).$$

so that, for a classical result of topology (see for instance Lemma 6 in [7]), we can conclude that $Max(X, V)$ is nonempty and connected. □

Corollary 3.1 *Let $V \subset \mathbb{R}^n$ be a closed, convex, pointed cone with nonempty interior and let X be a nonempty subset of $\mathbb{R}^n$ such that $Y_X = X - V$ is V-compact. If a CC-regularity condition holds for all the sets $V(a) \cap Y_X$, with $a \in \mathbb{R}^n$, then $Max(X, V)$ is a nonempty and connected set.*

Proof We just have to apply Theorem 3.1 to the set Y_X noticing that $Max(X, V) = Max(Y_X, V)$ for Property 2.4 and that since X is nonempty then Y_X is nonempty too. □

From now on, by means of the proposed unifying approach, we can reduce our analysis to the study of CC-regularity conditions.

4 Connectedness for V_p-compact sets

The aim of this section is to generalize some results appeared in the literature concerning the connectedness of the efficient frontier when the ordering cone in the image space is the paretian one.

4.1 Basic definitions

From now on, we assume that $\{e^1, e^2, \ldots, e^n\}$ is a basis of $\mathbb{R}^n$ and we define the following polyhedral cone:

$$V_p = \sum_{i=1}^{n} \mathbb{R}_+ e^i = \{x \in \mathbb{R}^n : \ x = \sum_{i=1}^{n} \lambda_i e^i, \ \lambda_i \geq 0 \ \forall i = 1, \ldots, n\} \tag{4.1}$$

Obviously, V_p is closed, convex, pointed with nonempty interior.

Since $\{e^1, e^2, \ldots, e^n\}$ is a basis of $\mathbb{R}^n$ from now on all the elements of $\mathbb{R}^n$ will be expressed by means of components given with respect to this basis. In other words, we will use the following notation:

$$x = (x_1, \ldots, x_n) = x_1 e^1 + \ldots + x_n e^n$$

We are now able to recall the following concepts of quasiconcave and strictly quasiconcave sets (see [6] by Hu and Sun) and the stronger concept of sequentially strictly quasiconcave sets (see [1, 2] by Benoist).

Definition 4.1 A subset X of $\mathbb{R}^n$ is said to be:

(QC) **quasiconcave** if for all pairs (a, b) in X^2, $a \neq b$, there exists some $c \in X$, $c \neq a, b$, such that for all $k \in \{1, \ldots, n\}$ it results $c_k \geq \min\{a_k, b_k\}$.

(SQ) **strictly quasiconcave** if for all pairs (a, b) in X^2, $a \neq b$, there exists some $c \in X$, $c \neq a, b$, such that for all $k \in \{1, \ldots, n\}$ it results:

$$c_k > \min\{a_k, b_k\} \text{ if } a_k \neq b_k,$$
$$c_k \geq a_k = b_k \text{ if } a_k = b_k.$$

(SSQ) **sequentially strictly quasiconcave** if for all pairs (a, b) in X^2 there exists a sequence $\{c^i\} \subset X$, $\{c^i\} \to a$, such that $\forall k \in \{1, \ldots, n\}$ and $\forall i \in \mathbb{N}$ it results:

$$c^i_k > \min\{a_k, b_k\} \text{ if } a_k \neq b_k,$$
$$c^i_k \geq a_k = b_k \text{ if } a_k = b_k.$$

It is worth noticing that Benoist in [1] uses an orthogonal basis $\{e^1, e^2, \ldots, e^n\}$ of $\mathbb{R}^n$ while Hu and Sun in [6] use the canonical one. In this paper we prefer to use a general basis of $\mathbb{R}^n$ (that is a not necessarily orthogonal one) since by means of just an homeomorphism it is equivalent to the canonical one.

4.2 An extension of sequential strict quasiconcavity

In order to generalize a result given by Benoist in [1] ([1]) let us now introduce the following extension of the sequential strictly quasiconcavity.

Definition 4.2 Let V_p as defined in (4.1). A subset X of $\mathbb{R}^n$ is said to be **extended sequentially strictly quasiconcave** with respect to V_p if it verifies the next property:

(ESSQ) for all pairs (a, b) in $Max(X, V_p)^2$ there exists a sequence $\{c^i\} \subset X$, $\{c^i\} \to a$, such that for all $k \in \{1, \dots, n\}$ and for all $i \in \mathbb{N}$ it results:

$$c_k^i > \min\{a_k, b_k\} \quad \text{if} \quad a_k \neq b_k,$$
$$c_k^i \geq a_k = b_k \quad \text{if} \quad a_k = b_k.$$

Clearly, the previous definition generalizes the class of sequentially strictly quasiconcave sets defined by Benoist. The following examples points out that these classes differ.

Example 4.1 Let us consider the set $X = X_1 \cup X_2$ where:

$$X_1 = \{(x, y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq 0,\ x + y \leq 4\}$$
$$X_2 = \{(x, y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq -1,\ x + y \leq 0\}.$$

This set is extended sequentially strictly quasiconcave but not sequentially strictly quasiconcave (just consider $a = (1, -1)$ and $b = (4, 0)$). □

In order to state a new CC-regularity condition, the following lemma is needed.

Lemma 4.1 *Let X be a nonempty and compact subset of $\mathbb{R}^n$. If X is extended sequentially strictly quasiconcave then the set $Y_X = X - V_p$ is sequentially strictly quasiconcave.*

Proof We have to prove that for all pairs (a, b) in Y_X^2 there exists a sequence $\{c^i\} \subset Y_X$, $\{c^i\} \to a$, such that for all $k \in \{1, \dots, n\}$ and for all $i \in \mathbb{N}$ it results

$$c_k^i > \min\{a_k, b_k\} \text{ if } a_k \neq b_k \text{ while } c_k^i \geq a_k = b_k \text{ if } a_k = b_k. \tag{4.2}$$

[1]**Theorem** *[Benoist [1], Theorem 4.1, page 633] Let X be a nonempty and closed subset of $\mathbb{R}^n$ such that for all $a \in \mathbb{R}^n$ the sets $V_p(a) \cap X$ are compact. If X is sequentially strictly quasiconcave then $Max(X, V_p)$ is a connected set.*

With this aim, let us first recall that since $Y_X = X - V_p$ it is $Max(Y_X, V_p) = Max(X, V_p)$.

First consider the case $a \in Y_X$ but $a \notin Max(Y_X, V_p)$. The set $Y_X \cap V_p(a)$ is a compact set for Property 2.2 and hence there exists

$$\bar{a} \in Max(Y_X \cap V_p(a), V_p) \subseteq Max(Y_X, V_p) = Max(X, V_p).$$

Since $Y_X = X - V_p$ and V_p is convex, then $a \in \bar{a} - V_p$ and the segment $[a, \bar{a}] \subset Y_X \cap V_p(a)$. Hence a sequence $\{c^i\} \to a$, $\{c^i\} \subset Y_X$, verifying (4.2) obviously exists for all $b \in Y_X$.

Consider now the case $a \in Max(Y_X, V_p) = Max(X, V_p)$ and let $b \in Y_X$. Since $Y_X \cap V_p(b)$ is a compact set, $Y_X = X - V_p$ and V_p is convex, then there exists

$$\bar{b} \in Max(Y_X \cap V_p(b), V_p) \subseteq Max(Y_X, V_p) = Max(X, V_p)$$

such that $b \in \bar{b} - V_p$. By means of the extended strictly quasiconcavity of X applied to the pair $(a, \bar{b})$ there there exists a sequence $\{c^i\} \to a$, $\{c^i\} \subset X \subseteq Y_X$, verifying (4.2) and the result is proved. □

Theorem 4.1 *Let X be a nonempty and closed subset of $\mathbb{R}^n$. The following condition is a CC-regularity one:*

X is extended sequentially strictly quasiconcave.

Proof We have to prove that if X is compact then $Max(X, V_p)$ is connected. Let $Y_X = X - V_p$ and assume X compact; notice also that Y_X results to be nonempty and closed. By means of the previous Lemma 4.1 the set Y_X is sequentially strictly quasiconcave and hence, by means of a proposition by Benoist ([2]), $Max(Y_X, V_p)$ is connected. The result then follows since $Max(Y_X, V_p) = Max(X, V_p)$. □

The following connectedness results then follow.

[2]**Theorem** ***[Benoist [1], Proposition 4.1, page 633]*** *Let Y be a nonempty, closed and sequentially strictly quasiconcave subset of $\mathbb{R}^n$. Assume also that there exists a compact set C in $\mathbb{R}$ such that $Y = C - V_p$. Then, $Max(Y, V_p)$ is connected.*

Let us notice that this result has been rewritten in the light of Proposition 3.3 again in [1], page 633.

Theorem 4.2 *Let X be a nonempty and V_p-compact subset of $\mathbb{R}^n$. If X is extended sequentially strictly quasiconcave then $Max(X, V_p)$ is a nonempty connected set.*

Proof First notice that:

$$Max(V_p(a) \cap X, V_p) = V_p(a) \cap Max(X, V_p) \subseteq Max(X, V_p).$$

The result then follows from Theorems 3.1 and 4.1 since for all $a \in \mathbb{R}^n$ the sets $V_p(a) \cap X$ results to be extended sequentially strictly quasiconcave. □

Corollary 4.1 *Let X be a nonempty subset of $\mathbb{R}^n$ such that $Y_X = X - V_p$ is V_p-compact. If Y_X is extended sequentially strictly quasiconcave then $Max(X, V_p)$ is nonempty and connected.*

Proof We just have to apply Theorem 4.2 to the set Y_X noticing that $Max(X, V_p) = Max(Y_X, V_p)$ for Property 2.4 and that since X is nonempty then Y_X is nonempty too. □

Theorem 4.2 above generalizes the result provided by Benoist in [1], as it is pointed out in Example 4.1. In facts, the hypothesis of Theorem 4.2 above are satisfied, while the result by Benoist in [1] (Theorem 4.1, page 633) cannot be applied since the set X is not sequentially strictly quasiconcave and the efficient frontier

$$Max(X, V_p) = \{(x, y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq 0,\ x + y = 4\}$$

is connected, compact and convex.

4.3 Connectedness of closed efficient frontiers

Another CC-regularity condition can be obtained, by means of the definitions given in Section 2, just by rewriting a result by Hu and Sun (that is Theorem 3.1, p.617, in [6]).

Theorem 4.3 *Let X be a nonempty and closed subset of $\mathbb{R}^n$. The following condition is a CC-regularity one:*

$$X \text{ is strictly quasiconcave and } Max(X, V_p) \text{ is closed.}$$

The chosen approach allows to state the following result.

Theorem 4.4 *Let X be a nonempty and V_p-compact subset of $\mathbb{R}^n$. If X is strictly quasiconcave and $Max(X, V_p)$ is closed then $Max(X, V_p)$ is a nonempty connected set.*

Proof By means of Theorems 3.1 and 4.3 we just have to prove that for all $a \in \mathbb{R}^n$ the sets $V_p(a) \cap X$ are strictly quasiconcave and the sets $Max(V_p(a) \cap X, V_p)$ are closed. Let a be any element of $\mathbb{R}^n$; since X is strictly quasiconcave then, by means of the definition itself, $V_p(a) \cap X$ is strictly quasiconcave too; note also that

$$Max(V_p(a) \cap X, V_p) = V_p(a) \cap Max(X, V_p)$$

so that for the closure of $Max(X, V_p)$ and V_p we have that $Max(V_p(a) \cap X, V_p)$ is closed too. The proof is then complete. □

Corollary 4.2 *Let X be a nonempty subset of $\mathbb{R}^n$ such that $Y_X = X - V_p$ is V_p-compact. If Y_X is strictly quasiconcave and $Max(X, V_p)$ is closed then $Max(X, V_p)$ is nonempty and connected.*

Proof We just have to apply Theorem 4.4 to the set Y_X, noticing that $Max(X, V_p) = Max(Y_X, V_p)$ for Property 2.4 and that since X is nonempty then Y_X is nonempty too. □

The following example (taken from [6], pp.618) points out the importance in Theorem 4.4 of the closedness of the efficient frontier.

Example 4.2 Let us consider the set $X = X_1 \cup X_2$ where:

$$\begin{aligned} X_1 &= \left\{(x, y) \in \mathbb{R}^2 \mid x = 0,\ 0 \le y \le 1\right\} \\ X_2 &= \left\{(x, y) \in \mathbb{R}^2 \mid\ y = -x,\ 0 \le x \le 1\right\}. \end{aligned}$$

The set $Max(X, V_p) = \{(x,y) \in \mathbb{R}^2 |\ y = -x,\ 0 < x \leq 1\} \cup \{(0,1)\}$ is neither closed nor connected even if the set X is strictly quasiconcave, compact and hence V_p-compact. □

Theorem 4.4 above generalizes (see again Property 2.2) the main result provided by Hu and Sun in [6], as it is pointed out by the following example ([3]).

Example 4.3 Let us consider the set $X = \{(x,y) \in \mathbb{R}^2 |\ x + y \leq 0\}$. The efficient frontier $Max(X, V_p) = \{(x,y) \in \mathbb{R}^2 |\ x + y = 0\}$ is connected, closed and convex, and the hypothesis of Theorem 4.4 above are satisfied. On the other hand, the result by Hu and Sun in [6] (Theorem 3.2, page 618) cannot be applied since there is no $v \in \mathbb{R}^n$ such that $X \subset \{v\} - V_p$. □

The following Examples 4.4 and 4.5 point out that Theorems 4.2 and 4.4 cannot be applied to the same classes of sets, that is to say that none of them is a generalization of the other one.

Example 4.4 Let us consider the set $X = X_1 \cup X_2$ where:

$$\begin{aligned} X_1 &= \{(x,y,z) \in \mathbb{R}^3 | (x,y,z) = (2,1,-1)t + (0,1,1), 0 \leq t \leq 1\} \\ X_2 &= \{(x,y,z) \in \mathbb{R}^3 | (x,y,z) = (0,-1,1)k + (0,1,1), 0 \leq k \leq 1\}. \end{aligned}$$

This set is compact, connected and $X = Max(X, V_p)$. It is also strictly quasiconcave but not sequentially strictly quasiconcave (just consider $a = (0,0,2)$ and $b = (2,2,0)$), so that Theorem 4.4 can be used in order to recognize the connectedness of $Max(X, V_p)$ while the same cannot be done neither with Theorem 4.2 nor with the result by Benoist in [1] (Theorem 4.1, page 633). □

[3]**Theorem** *[Hu and Sun [6], Theorem 3.2, page 618] Let X be a nonempty and closed subset of $\mathbb{R}^n$ and assume that there exists $v \in \mathbb{R}^n$ such that $X \subset \{v\} - V_p$. If X is strictly quasiconcave and $Max(X, V_p)$ is closed then $Max(X, V_p)$ is a connected set.*

Example 4.5 Let us consider the set $X = X_1 \cup X_2$ where:

$$X_1 = \{(x,y) \in \mathbb{R}^2 | \ x \geq 0, \ y \geq 0, \ x+y \leq 4\}$$
$$X_2 = \{(x,y) \in \mathbb{R}^2 | \ x \geq 0, \ y \geq -4, \ x+y \leq 0\}.$$

The efficient frontier $Max(X, V_p) = \{(x,y) \in \mathbb{R}^2 | \ x \geq 0, \ y \geq 0, \ x+y = 4\}$ is connected, compact and convex. The set X is extended sequentially strictly quasiconcave but it is neither strictly quasiconcave nor sequentially strictly quasiconcave (just consider $a = (4, -4)$ and $b = (4, 0)$), hence Theorem 4.2 can be applied to X while the same does not happen neither for Theorem 4.4 nor for the result by Hu and Sun in [6] (Theorem 3.2, page 618).□

5 Connectedness for not V_p-compact sets

Theorems 4.2 and 4.4 can be used to determine the connectedness of the efficient frontier for sets X which are V_p-compact. The previous Property 2.6 and the following Property 5.1 suggest how to further generalize Theorem 4.4 in order to manage sets which are not V_p-compact.

Property 5.1 *Let X be a nonempty and closed subset of $\mathbb{R}^n$. If X is strictly quasiconcave then the set $Y_M = Max(X, V_p) - V_p$ is strictly quasiconcave too.*

Proof Let $a, b \in Y_M = Max(X, V_p) - V_p$, $a \neq b$. Then, there exist $v, w \in V_p$ such that $(a+v) \in Max(X, V_p) \subseteq X$ and $(b+w) \in Max(X, V_p) \subseteq X$. If $a+v \neq b+w$ the result follows from the strict quasiconcavity of X applied to $a+v$ and $b+w$. Otherwise, that is if $a+v = b+w$, then $a, b \in (a+v) - V_p \subseteq Y_M$ hence we just have to choose $c = (a+b)/2$ thanks to the convexity of V_p. □

Let us notice that the converse of the implication stated in Property 5.1 does not hold, as it is pointed out in the forthcoming Example 5.1.

The following generalization of Theorem 4.4 can now be stated.

Theorem 5.1 *Let X be a nonempty and closed subset of $\mathbb{R}^n$ and let also $Max(X, V_p)$ be nonempty and V_p-compact. If the set $Y_M = Max(X, V_p) - V_p$ is strictly quasiconcave then $Max(X, V_p)$ is a nonempty connected set.*

Proof For Property 2.2 since $Max(X, V_p)$ is V_p-compact then it is also closed. As a consequence, for the closure and convexity of V_p, the set Y_M is nonempty, closed and strictly quasiconcave. Note also that since $Max(X, V_p)$ is V_p-compact and V_p is closed and convex then Y_M is V_p-compact too. For Property 2.5 it is finally $Max(X, V_p) = Max(Y_M, V_p)$ so that $Max(Y_M, V_p)$ is closed. By means of Theorem 4.4 applied to the set Y_M, we then have that $Max(Y_M, V_p)$ is a nonempty connected set and the result is proved since $Max(X, V_p) = Max(Y_M, V_p)$ for Property 2.5. □

The following Example 5.1 shows that Theorem 5.1 can be applied to sets X which are not V_p-compact (just the V_p-compactedness of the efficient frontier is required).

Example 5.1 Let us consider the set $X = X_1 \cup X_2 \cup X_3$ where:

$$\begin{aligned} X_1 &= \{(x, y) \in \mathbb{R}^2 |\ x < 0,\ x + y = 1\} \\ X_2 &= \{(x, y) \in \mathbb{R}^2 |\ x = 0,\ 0 \leq y \leq 1\} \\ X_3 &= \{(x, y) \in \mathbb{R}^2 |\ x > 0,\ y = 0\}. \end{aligned}$$

The efficient frontier $Max(X, V_p) = \{(x, y) \in \mathbb{R}^2 |\ x \leq 0,\ x + y = 1\}$ is connected, V_p-compact and convex. The set X is neither strictly quasiconcave nor sequentially strictly quasiconcave (just consider $a = (0, 1)$ and $b = (1, 0)$), while $Y_M = Max(X, V_p) - V_p$ is strictly quasiconcave. Hence Theorem 5.1 can be applied to X while the same does not happen for Theorems 4.2 and 4.4. □

6 Concluding remarks

It is worth noticing that the results stated in the previous sections can be applied to study the connectedness of the efficient frontier of a vector valued function in the image space.

With this aim, let us consider the following multiobjective problem:

$$\begin{cases} max\ f(x) \\ x \in D \end{cases}$$

where D is a nonempty and closed subset of $\mathbb{R}^m$ and $f : D \to \mathbb{R}^n$ is a continuous vector valued function, that is $f(x) = (f_1(x), \ldots, f_n(x))$ with $x \in D$.

The following class of generalized quasiconcave functions can now be introduced.

Definition 6.1 A vector valued function $f : D \to \mathbb{R}^n$ is said to be **extended strictly quasiconcave** if for all pairs (a, b) in D^2 with $f(a) \neq f(b)$ and $f(a), f(b) \in Max(f(D), V_p)$, there exists some $c \in D$ with $f(c) \neq f(a), f(b)$, such that for all $k \in \{1, \ldots, n\}$ it results:

$$f_k(c) > \min\{f_k(a), f_k(b)\} \quad \text{if} \quad f_k(a) \neq f_k(b),$$
$$f_k(c) \geq f_k(a) = f_k(b) \quad \text{if} \quad f_k(a) = f_k(b).$$

The following corollary follows from Theorem 5.1.

Corollary 6.1 *Let D be a nonempty and closed set and $f : D \to E$ be a continuous vector valued function. If f is extended strictly quasiconcave and $Max(f(D), V_p)$ is nonempty and V_p-compact then $Max(f(D), V_p)$ is a nonempty connected set.*

Proof We just have to apply Theorem 5.1 to the set $f(D)$. Since f is continuous and D is nonempty and closed then $f(D)$ is nonempty and closed. For the hypothesis we now just have to verify that the set $Y_M = Max(f(D), V_p) - V_p$ is strictly quasiconcave.

Let $y, z \in Y_M = Max(f(D), V_p) - V_p$, $y \neq z$. Then, there exist $v, w \in V_p$ and $a, b \in D$ such that $y + v = f(a) \in Max(f(D), V_p)$ and $z + w = f(b) \in Max(f(D), V_p)$. If $f(a) \neq f(b)$ the result follows from the extended strict quasiconcavity of f. Otherwise, that is if $f(a) = f(b)$, then $y, z \in f(a) - V_p \subseteq Y_M$ hence we just have to choose $c = (y + z)/2$ thanks to the convexity of V_p. □

Let us finally provide the example of an extended strictly quasiconcave function having a connected efficient frontier in the image space (note, for the sake of completeness, that the provided function is not componentwise strictly quasiconcave).

Example 6.1 Let us consider the bicriteria function $f(x) = (f_1(x), f_2(x))$ with $x \in D = \mathbb{R}$, where:

$$f_1(x) = \begin{cases} x+1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 \le x \le 1 \\ x & \text{for} \quad x > 1 \end{cases}$$

$$f_2(x) = \begin{cases} -x+1 & \text{for} \quad x < 1 \\ 0 & \text{for} \quad x \ge 1 \end{cases}$$

Since f is an extended strictly quasiconcave function and its image $f(D)$ is the one described in Example 6.1, the assumptions of Corollary 6.1 hold. Hence, its efficient frontier results to be connected. □

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