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On the pseudoconvexity and pseudolinearity of some classes of fractional functions

Laura Carosi *- Laura Martein†

Abstract

The aim of the paper is to characterize the pseudoconvexity (pseudoconcavity) of the ratio between a quadratic function and the square of an affine function. First of all, we study the pseudoconvexity of the quadratic function, defined on a suitable halfspace, obtained applying the Charnes-Cooper transformation of variables. The obtained results allow to give necessary and sufficient conditions for the pseudoconvexity and pseudolinearity of the ratio in terms of the initial data.

KeyWords Pseudoconvexity, pseudolinearity, fractional programming.
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1 Introduction

Pseudoconvexity and pseudolinearity of functions are widely studied in the literature for their nice properties and for their applications in Economics [1]. In particular, these classes of functions play an important role in Optimization because of the fundamental property that a local minimum is also global and it is reached at an extremum point in case of pseudolinearity. On the other hand, it is a difficult tool to test if a given function is pseudoconvex or pseudolinear. For such a reason, and taking into account that many applications give rise to multi-ratio fractional programs [15], some approaches to study pseudoconvexity and pseudolinearity of some particular classes of fractional functions have been recently suggested ([2, 4, 5, 8]). In this framework,

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the Charnes-Cooper transformation has shown to be an useful tool because of its property to preserve pseudoconvexity and pseudolinearity ([4, 5]).
In this paper we consider the ratio between a quadratic function and the square of an affine function. For such class of functions a complete characterization of pseudoconvexity and pseudolinearity is given. More precisely, in section 2, by means of the Charnes-Cooper transformation, the ratio is transformed in a quadratic function, so that the study of pseudoconvexity (pseudolinearity) is reduced to the study of pseudoconvexity (pseudolinearity) of a quadratic function on a suitable halfspace which is performed in section 3. The obtained results allow to give, in section 4, a characterization of the pseudoconvexity of the ratio in terms of the initial data. Based on this charcterization, a procedure for testing pseudoconvexity is given in section 5 and it is illustrated by several numerical examples. Special cases and pseudolinearity of the ratio are studied in sections 6, 7.

2 Statement of the problem

The aim of this paper is to study the pseudoconvexity of the function

$$f(x) = \frac{\frac{1}{2}x^T Ax + a^T x + a_0}{(b^T x + b_0)^2} \tag{1}$$

on the halfspace $S = \{x \in \Re^n : b^T x + b_0 > 0\}$, $b_0 \neq 0$.
We recall that a differentiable function h defined on an open convex set X is pseudoconvex if for $x^1,\ x^2 \in X$

$$h(x^1) > h(x^2) \Rightarrow \nabla h(x^1)^T (x^2 - x^1) < 0$$

It is known [1] that a pseudoconvex function verifies the property given in the following theorem.

Theorem 2.1 *Let h be a continuously differentiable function defined on the open convex set $X \subseteq \Re^n$. Then h is pseudoconvex if and only if for every $x^0 \in X$ and $v \in \Re^n$ such that $\nabla h(x^0)^T v = 0$, the function $\varphi(t) = h(x^0 + tv)$ attains a local minimum at $t = 0$.*

Remark 2.1 *From Theorem 2.1 it follows that the pseudoconvexity of h implies, with respect to the restriction $\varphi(t) = h(kz + tv)$, that the two conditions $\varphi'(\bar{t}) = v^T \nabla h(kz + \bar{t}v) = 0,\ \varphi''(\bar{t}) < 0$ cannot occur simultaneously.*

Performing the Charnes and Cooper transformation $y = \frac{x}{b^T x + b_0}$, whose inverse is $x = \frac{b_0 y}{1 - b^T y}$ (see [9]), we obtain the following quadratic function

$$f(x(y)) = Q(y) = y^T Q y + q^T y + q_0$$

where:

$$Q = \frac{1}{2}A - \frac{ab^T + ba^T}{2b_0} + \frac{a_0}{b_0^2}bb^T \tag{2}$$

$$q = \frac{1}{b_0}\left(a - 2\frac{a_0}{b_0}b\right), \quad q_0 = \frac{a_0}{b_0^2} \tag{3}$$

Taking into account that the previous Charnes-Cooper transformation preserves pseudoconvexity and pseudoconcavity [4, 5], we have the following result:

Theorem 2.2 *The function $f(x)$ is pseudoconvex (pseudoconcave) on the halfspace S if and only if the quadratic function $Q(y)$ is pseudoconvex (pseudoconcave) on the halfspace $S^* = \left\{y \in \Re^n : \frac{1-b^Ty}{b_0} > 0\right\}$.*

In order to find conditions which ensure the pseudoconvexity of f, in the next section we will study the pseudoconvexity of a quadratic function defined on an halfspace.
Trough the paper we will use the following notations:
$\nu_-(C)$ ($\nu_+(C)$) denotes the number of negative (positive) eigenvalues of a matrix C of order s;
$\ker C$ denotes the kernel of C that is $\ker C = \{v : Cv = 0\}$;
$\operatorname{Im} C$ denotes the set $\operatorname{Im} C = \{z = Cv,\ v \in \Re^s\}$;
$v^\perp$ denotes the orthogonal space to the vector v, that is $v^\perp = \{w : v^Tw = 0\}$.

3 Pseudoconvexity of a quadratic function on an halfspace.

Let $Q(y) = y^TQy + q^Ty + q_0$, $y \in \Re^n$ be a quadratic function. It is well known that $Q(y)$ is pseudoconvex if and only if it is convex, so that pseudoconvexity of a quadratic function can differ from convexity only if it is restricted on a proper subset of $\Re^n$ (see for instance [1]).
Our first aim is to characterize the pseudoconvexity of $Q(y)$ on the halfspace $H = \{y \in \Re^n : c^Ty + c_0 > 0\}$.

Lemma 3.1 *If the function $Q(y)$ is pseudoconvex on the halfspace H, then $\nu_-(Q) \leq 1$.*

Proof. Assume that $\nu_-(Q) > 1$. Then there exist two orthogonal eigenvectors v_1, v_2 such that $v_1^TQv_1 < 0$, $v_2^TQv_2 < 0$. Let W be the linear subspace generated by v_1, v_2. We have $\dim(W \cap c^\perp) \geq 1$; infact if $W \subset c^\perp$ then $\dim(W \cap c^\perp) = \dim W = 2$, otherwise $\dim(W \cap c^\perp) = \dim W + \dim c^\perp -$

$\dim(W + c^\perp) = 2 + n - 1 - n = 1$.
Let $v \in c^\perp \cap W$ and consider $y_0 \in H$. The restriction $Q(y_0 + tv) = t^2 v^T Q v + (2v^T Q y_0 + q^T v)t + Q(y_0)$ is a concave function since $v^T Q v < 0,\ \forall v \in W$; on the other hand, the line $y = y_0 + tv,\ t \in \Re$ is contained in H, so that the maximum point of $Q(y_0+tv)$ is feasible and this contradicts the pseudoconvexity of $Q(y)$. ■

Theorem 3.1 *The function $Q(y)$ is pseudoconvex on the halfspace H if and only if one of the following conditions holds:*
i) $\nu_-(Q) = 0$;
ii) $\nu_-(Q) = 1,\ \ker Q = c^\perp,\ q = \beta c,\ c_0 \le \frac{\|c\|^4 \beta}{2c^T Q c}$.

Proof. $\Rightarrow$ From Lemma 3.1, we have two possible cases : $\nu_-(Q) = 0$, so that $i)$ holds or $\nu_-(Q) = 1$. In this last case, let α be the negative eigenvalue of Q and let v such that $Qv = \alpha v,\ \| v \| = 1,\ c^T v > 0$.
In order to prove that $\ker Q = c^\perp$, first of all we will show that:
a) $v \notin c^\perp$;
b) $v = \lambda c,\ \lambda \in \Re$.
a) Assume that $v \in c^\perp$ and consider the restriction $\varphi(t) = Q(y_0 + tv)$ with $y_0 \in H$. It results $c^T(y_0+tv) = c^T y_0 + c_0,\ \forall t \in \Re$, so that the line $y = y_0 + tv$ is contained in H.
We have $\varphi'(t) = v^T \nabla Q(y_0 + tv) = v^T[2Q(y_0 + tv) + q]$ so that $\varphi'(\bar{t}) = 0$ for $\bar{t} = \frac{-q^T v - 2\alpha v^T y_0}{2\alpha \|v\|^2}$; on the other hand, $\varphi''(\bar{t}) = v^T Q v = \alpha < 0,\ \bar{y} = y_0 + \bar{t}v \in H$ and this contradicts the pseudoconvexity of $Q(y)$ on H (see Remark 2.1).
b) If $v \neq \lambda c,\ \forall \lambda \in \Re$, then $v^\perp \neq c^\perp$, so that there exists $z \in v^\perp,\ z \notin c^\perp$, with $c^T z > 0$.
Consider the restriction $\varphi(t) = Q(kz + tv)$; we have $\varphi'(t) = v^T[2Q(kz + tv) + q] = 2\alpha t + v^T q$, so that $\varphi'(\bar{t}) = 0$ for $\bar{t} = -\frac{q^T v}{2\alpha}$ and $\varphi''(\bar{t}) = v^T Q v = \alpha < 0$.
It is easy to verify that the point $y = kz + \bar{t}v$ is feasible for k large enough (it is sufficient to choose $k > \left(\frac{v^T q c^T v}{2\alpha} - c_0\right)\frac{1}{c^T z}$). Taking into account Remark 2.1, the function $Q(y)$ is not pseudoconvex on H and this is absurd.
Now we prove that $\ker Q = c^\perp$.
Since $v = \lambda c$, c is an eigenvector of Q, so that any other eigenvector w of Q with $\| w \| = 1$ belongs to $c^\perp$. We will prove that $Qw = 0$, so that it results $Qz = 0,\ \forall z \in c^\perp$, that is $\ker Q = c^\perp$.
Set $Qw = \delta w,\ \delta > 0$ and consider the restriction $\varphi(t) = Q(kw + td)$, with $d = c + \frac{1}{2} \| c \| \sqrt{\frac{|\alpha|}{\delta}} w$. It results $\varphi'(t) = d^T[2Q(kw+td)+q] = k\delta \| c \| \sqrt{\frac{|\alpha|}{\delta}} + \frac{3}{2}\alpha \| c \|^2 t + q^T d$, so that $\varphi'(\bar{t}) = 0$ for $\bar{t} = -\frac{2}{3\alpha\|c\|^2}\left(q^T d + k\delta \| c \| \sqrt{\frac{|\alpha|}{\delta}}\right)$ and $\varphi''(\bar{t}) = d^T Q d = \frac{3}{4}\alpha \| c \|^2 < 0$.

The point $y = kw + \bar{t}d$ is feasible for k large enough (it is sufficient to choose $k > \frac{3\alpha}{\delta\|c\|}\sqrt{\frac{\delta}{|\alpha|}}\left(c_0 - \frac{2d^Tq}{3\alpha}\right)$). Taking into account Remark 2.1, we get a contradiction.
It remains to prove that $q = \beta c$, $c_0 \leq \frac{\|c\|^4\beta}{2c^TQc}$.
Consider the restriction $\varphi(t) = Q(y_0 + t(w + \epsilon c))$, $y_0 \in H$, $w \in c^\perp$, $\epsilon \neq 0$. It results $\varphi'(t) = 2\epsilon c^TQy_0 + 2t\epsilon^2c^TQc + w^Tq + \epsilon c^Tq$, so that $\varphi'(\bar{t}) = 0$ for $\bar{t} = \frac{-q^Tw - \epsilon c^Tq - 2\epsilon c^TQy_0}{2\epsilon^2c^TQc}$ and $\varphi''(\bar{t}) = \epsilon^2c^TQc$.
Taking into account that $Qc = \alpha c$, $\varphi''(\bar{t}) = \epsilon^2c^TQc = \alpha\epsilon^2 \| c \|^2 < 0$, the pseudoconvexity of $Q(y)$ on H implies that the point $y_0 + \bar{t}(w + \epsilon c)$ is not feasible that is

$$c_0 \leq \frac{q^Tw + \epsilon c^Tq}{2\epsilon c^TQc} \| c \|^2 = \psi(\epsilon) \tag{4}$$

If $q^Tw > 0$ ($q^Tw < 0$), it results $\psi(\epsilon) \to -\infty$ when $\epsilon \to 0^+$ ($\epsilon \to 0^-$) and this is absurd. Consequently $q^Tw = 0$, $\forall w \in c^\perp$, so that $q = \beta c$ and (4) reduces to $c_0 \leq \frac{\beta}{2c^TQc} \| c \|^4$.
$\Leftarrow$ If $i)$ holds, $Q(y)$ is a convex function and, in particular, pseudoconvex on $\Re^n$. It remains to prove that $ii)$ implies the pseudoconvexity of $Q(y)$ on H. The assumptions imply $Qc = \alpha c$, $\alpha < 0$, $Qw = 0$, $\forall w \in c^\perp$.
Since $\Re^n = c^\perp \oplus [c]$, any element $y \in \Re^n$ can be expressed in the form $y = w + kc$, $w \in c^\perp$, $k \in \Re$.
We have $Q(y) = k^2c^TQc + kq^Tc + q^Tw + q_0$ and $c^Ty + c_0 = k\|c\|^2 + c_0$. Obviously, $Q(y)$ is pseudoconvex on the halfspace H if and only if the function $\varphi(k) = k^2c^TQc + kq^Tc + q^Tw + q_0$ is pseudoconvex on the halfline of equation $k\|c\|^2 + c_0 > 0$ and this occurs if and only if the maximum point $\bar{k} = -\frac{q^Tc}{2c^TQc}$ is not feasible that is $c_0 \leq \frac{\|c\|^2c^Tq}{2c^TQc} = \frac{\|c\|^4\beta}{2c^TQc}$.
This completes the proof. ■

Remark 3.1 *Let us note that* $\ker Q = c^\perp$ *implies the existence of* $\mu \in \Re$ *such that* $y^TQy = \mu(c^Ty)^2$ *or, equivalently,* $Q = \mu cc^T$. *In fact,* $\ker Q = c^\perp$ *implies that* c *is an eigenvector of* Q *so that there exists* $\mu^* \in \Re$ *with* $Qc = \mu^*c$. *Any element* $y \in \Re^n$ *can be expressed in the form* $y = kc + w$, $w \in c^\perp$, $k \in \Re$ *and, consequently, taking into account that* $Qw = 0$, *we have* $y^TQy = k^2c^TQc$. *On the other hand,* $c^TQc = \mu^* \| c \|^2$, $c^Ty = k \| c \|^2$ *so that* $y^TQy = \mu(c^Ty)^2$ *with* $\mu = \frac{\mu^*}{\|c\|^2}$.

In the special case $\ker Q = c^\perp$, $q = \beta c$, Theorem 3.1 can be expressed equivalently in the following way:

Theorem 3.2 *Consider the function* $Q(y)$ *with* $Q = \mu cc^T$, $q = \beta c$, $\mu, \beta \in \Re$. *Then* $Q(y)$ *is pseudoconvex on* H *if and only if* $\mu \geq 0$ *or* $\mu < 0$ *and* $c_0 \leq \frac{\beta}{2\mu}$.

Corollary 3.1 *Consider the function $h(y) = y^TQy$. Then $h(y)$ is pseudoconvex on H if and only if Q is positive semidefinite or $Q = \mu cc^T$ with $\mu < 0$ and $c_0 < 0$.*

4 Pseudoconvexity of the function $f(x)$

The results given in the previous sections allow to find a characterization of the pseudoconvexity of the function $f(x)$ in terms of the data A, a, a_0, b, b_0. The following theorem points out that f is not pseudoconvex if A has at least two negative eigenvalues.

Theorem 4.1 *If f is pseudoconvex on S then A has at most one negative eigenvalue.*

Proof. If $\ker A = b^\perp$ then $A = \delta bb^T$, so that A has only one non null eigenvalue given by $\delta \parallel b \parallel^2$.
Consider now the case $\ker A \neq b^\perp$. Suppose by contradiction $v_-(A) > 1$ and let v_1 and v_2 be two linearly independent eigenvectors associated with two negative eigenvalues of A, such that $v_1^T v_2 = 0$. Let W be the linear subspace generated by v_1 and v_2. Let us note that $W \cap b^\perp \neq \emptyset$ since or $W \subset b^\perp$ or $\dim (W + b^\perp) = n$, so that $\dim (W \cap b^\perp) = \dim W + \dim b^\perp - \dim (W + b^\perp) = 1$. Let $v \in W \cap b^\perp$, $v \neq 0$. Since v is a linear combination of v_1 and v_2, we have $v^TAv < 0$. Consider the line $x = x_0 + tv$, $x_0 \in S$, $t \in \Re$ which is contained in S since $b^Tx + b_0 = b^Tx_0 + b_0 > 0$. It is easy to verify that the restriction $\varphi(t) = f(x_0 + tv)$ is of the kind $\varphi(t) = \alpha t^2 + \beta t + \gamma$ with $\alpha < 0$ and this contradicts the pseudoconvexity of f. ∎

Consider now the special case corresponding to Theorem 3.2.

Theorem 4.2 *Consider the function $f(x)$ with $A = \delta bb^T$, $a = \gamma b$, $\delta, \gamma \in \Re$. Then $f(x)$ is pseudoconvex on $S = \{x \in \Re^n : b^Tx + b_0 > 0\}$ if and only if $\delta b_0^2 - 2\gamma b_0 + 2a_0 \geq 0$ or $\delta b_0^2 - 2\gamma b_0 + 2a_0 < 0$ and $\gamma \leq \delta b_0$.*

Proof. Taking into account (2) and (3), we have $Q = (\frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0)\frac{bb^T}{b_0^2}$, $q = (\frac{2a_0}{b_0} - \gamma)(-\frac{b}{b_0})$.
Setting $c = -\frac{b}{b_0}$, $c_0 = \frac{1}{b_0}$, from Theorem 3.2, $f(x)$ is pseudoconvex on S if and only if $\mu = \frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0$ is non negative or $\mu < 0$ and $c_0 \leq \frac{\beta}{2\mu}$ with $\beta = \frac{2a_0}{b_0} - \gamma$. This last inequality is equivalent to $\frac{1}{b_0} \leq \frac{\frac{2a_0}{b_0} - \gamma}{2(\frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0)}$, that is $\frac{\gamma - \delta b_0}{\delta b_0^2 - 2\gamma b_0 + 2a_0} \geq 0$.
As a consequence $\delta b_0^2 - 2\gamma b_0 + 2a_0 < 0$ implies $\gamma - \delta b_0 \leq 0$ and the thesis is achieved. ∎

Corollary 4.1 *The function $f(x)$ with $A = \delta bb^T$, $a = \gamma b$, $\delta, \gamma \in \Re$, is pseudoconvex on the halfspace S if and only if it can be reduced in the following canonical form*

$$f(x) = \frac{B}{b^T x + b_0} + \frac{C}{(b^T x + b_0)^2} + D$$

where $C \geq 0$ or $C < 0$ and $B \leq 0$.

Proof. The thesis follows by simple calculations. ∎

The following theorem gives a complete characterization of the pseudoconvexity of f in the general case $\ker A \neq b^\perp$.

Theorem 4.3 *When $\ker A \neq b^\perp$, the function f is pseudoconvex on the halfspace S if and only if A is positive semidefinite on $b^\perp$ and one of the following conditions holds:*
i) there exists $\alpha \in \Re$ such that $Ab - \frac{\|b\|^2}{b_0} a = \alpha b$ with

$$\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2} \tag{5}$$

ii) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ for every $\alpha \in \Re$, there exist $a^*, b^* \in \Re^n$ such that $Ab^* = b$, $Aa^* = a$, $b^* \in b^\perp$, $b^T a^* = b_0$ and

$$a^{*T} a \leq 2a_0 \tag{6}$$

iii) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ for every $\alpha \in \Re$, there exist $a^*, b^* \in \Re^n$ such that $Ab^* = b$, $Aa^* = a$, $b^{*T} b \neq 0$ and

$$a_0 - \frac{a^{*T} a}{2} + \frac{1}{2b^T b^*} \left(b_0 - b^T a^*\right)^2 \geq 0 \tag{7}$$

iv) $Ab - \frac{\|b\|^2}{b_0} a \neq \alpha b$ for every $\alpha \in \Re$ and there exist $\mu^* \in \Re$, $a^* \in \Re^n$ such that $a = Aa^* + \mu^* b$, $b \notin \operatorname{Im} A$ and

$$a_0 - \mu^* b_0 - \frac{1}{2} a^{*T} A a^* \geq 0 \tag{8}$$

Proof. From Theorem 2.2, $f(x)$ is pseudoconvex on S if and only if the function $Q(y)$ is pseudoconvex on $S^* = \{y \in \Re^n : c^T y + c_0 > 0\}$, with $c = -\frac{1}{b_0} b$, $c_0 = \frac{1}{b_0}$.
The case *ii)* of Theorem 3.1 corresponds to the case $\ker A = b^\perp$, $a = \gamma b$ and the characterization of the pseudoconvexity of f is given in Theorem 4.2.

When $\ker A \neq b^{\perp}$, f is pseudoconvex if and only if the matrix Q is positive semidefinite, with $Q = \frac{1}{2}A - \frac{ab^T + ba^T}{2b_0} + \frac{a_0}{b_0^2}bb^T$.
Let us note that for every $u \in b^{\perp}$ we have $u^TQu = \frac{1}{2}u^TAu$, so that Q is positive semidefinite on $b^{\perp}$ if and only if A is positive semidefinite on $b^{\perp}$.
Let $\Re^n$ be decomposed as the direct sum between the space generated by vector b and its orthogonal space, so that every $x \in \Re^n$ can be written as $x = kb + w$ where $k \in \Re$ and $w \in b^{\perp}$. We have

$$x^TQx = k^2b^TQb + k\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w + \frac{1}{2}w^TAw \tag{9}$$

where

$$b^TQb = \frac{1}{2}b^TAb - \frac{\|b\|^2}{b_0}a^Tb + \frac{a_0}{b_0^2}\|b\|^4 \tag{10}$$

Consequently, the matrix Q is positive semidefinite if and only if

$$\varphi(k,w) = k^2b^TQb + k\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w + \frac{1}{2}w^TAw \geq 0,\ \forall w \in b^{\perp}, \forall k \in \Re. \tag{11}$$

We are going to distinguish two exhaustive cases:
Case 1. $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w = 0$ for every $w \in b^{\perp}$.
Case 2. There exists $w \in b^{\perp}$ such that $\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w \neq 0$.
Case 1. It is equivalent to say that there exists $\alpha \in \Re$, such that

$$\left(Ab - \frac{\|b\|^2}{b_0}a\right) = \alpha b \tag{12}$$

and condition (11) becomes

$$k^2b^TQb + \frac{1}{2}w^TAw \geq 0,\ \forall w \in b^{\perp},\ \forall k \in \Re. \tag{13}$$

Since $w^TAw \geq 0$ for every $w \in b^{\perp}$, (13) is verified $\forall k \in \Re$ if and only if

$$b^TQb = \frac{1}{2}b^TAb - \frac{\|b\|^2}{b_0}a^Tb + \frac{a_0}{b_0^2}\|b\|^4 \geq 0. \tag{14}$$

From (12) we obtain $b^TAb - \frac{\|b\|^2}{b_0}b^Ta = \alpha\|b\|^2$, so that $b^TAb = \frac{\|b\|^2}{b_0}b^Ta + \alpha\|b\|^2$ and consequently $b^TQb = \frac{1}{2}\frac{\|b\|^2}{b_0}b^Ta + \frac{1}{2}\alpha\|b\|^2 - \frac{\|b\|^2}{b_0}a^Tb + \frac{a_0}{b_0^2}\|b\|^4 =$

$\frac{1}{2}\|b\|^2\left(\alpha-\frac{1}{b_0}b^Ta+\frac{2a_0}{b_0^2}\|b\|^2\right)$. So condition (14) is satisfied if and only if

$$\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2}$$

Consequently, if A is positive semidefinite on $b^\perp$ and $\left(Ab-\frac{\|b\|^2}{b_0}a\right)^T w=0$ for every $w \in b^\perp$, Q is positive semidefinite if and only if (5) is verified.
Case 2. Let us note that, corresponding to an element $w \in b^\perp$ such that $\left(Ab-\frac{\|b\|^2}{b_0}a\right)^T w \neq 0$, necessarily we have $w^TAw>0$, otherwise (11) is not verified $\forall k \in \Re$. Furthermore, (11) is equivalent to

$$\inf_{(k,w)\in\Re\times b^\perp}\varphi(k,w)=\inf_{k\in\Re}\inf_{w\in b^\perp}\varphi(k,w)\geq 0.$$

It is well known that a quadratic convex function either has minimum value or its infimum is equal to $-\infty$ and consequently Q is positive semidefinite if and only if $\inf_{w\in b^\perp}\varphi(k,w)=\min_{w\in b^\perp}\varphi(k,w)$ and $\inf_{k\in\Re}\min_{w\in b^\perp}\varphi(k,w)\geq 0$.
Now, for any given $k \in \Re$, consider the following minimization problem

$$\begin{cases}\min\limits_{w}\ [\varphi(k,w)=k^2b^TQb+k\left(Ab-\frac{\|b\|^2}{b_0}a\right)^T w+\frac{1}{2}w^TAw] \\ b^Tw=0\end{cases} \tag{15}$$

Since A is positive semidefinite on the orthogonal space $b^\perp$, w^* is the solution of Problem (15) if and only if there exists (w^*,λ^*) which satisfies the following necessary and sufficient optimality conditions

$$\begin{cases}Aw^*+kAb-k\frac{\|b\|^2}{b_0}a=\lambda^*b & (1)\\ b^Tw^*=0 & (2)\end{cases} \tag{16}$$

Let us note that (16) implies $w^{*T}Aw^*+k(Ab-\frac{\|b\|^2}{b_0}a)^Tw^*=0$, so that

$$\varphi(k,w^*)=k^2b^TQb-\frac{1}{2}w^{*T}Aw^* \tag{17}$$

Furthermore, from (16.1), we have

$$k\frac{\|b\|^2}{b_0}a=A(w^*+kb)-\lambda^*b. \tag{18}$$

We are going to distinguish the two cases: $b\in \operatorname{Im}A$, $b\notin \operatorname{Im}A$.
If $b\in\operatorname{Im}A$, there exists b^* such that $Ab^*=b$, so that condition (18) implies

$a \in \operatorname{Im} A$, i.e. there exists a^* such that $Aa^* = a$. Therefore equation (16.1) can be written as follows

$$A\left(w^* + kb - \lambda^* b^* - k\frac{\|b\|^2}{b_0}a^*\right) = 0.$$

As a consequence $w^* + kb - \lambda^* b^* - k\frac{\|b\|^2}{b_0}a^* \in \ker A$, so that

$$w^* = \lambda^* b^* + k\frac{\|b\|^2}{b_0}a^* - kb + e, \tag{19}$$

with $e \in \ker A$. Taking into account that $b^T e = b^{*T} A e = 0$, substituting (19) and (10) in (17) we get

$$\varphi(k, w^*) = k^2\frac{\|b\|^4}{b_0^2}(a_0 - \frac{1}{2}a^{*T}a) - \frac{1}{2}\lambda^{*2} b^T b^* + \lambda^* k\frac{\|b\|^2}{b_0}(b_0 - b^T a^*) \tag{20}$$

From (16.2) we have

$$\lambda^* b^T b^* + k\frac{\|b\|^2}{b_0}b^T a^* - kb^T b + b^T e = \lambda^* b^T b^* + k\frac{\|b\|^2}{b_0}b^T a^* - k\|b\|^2 = 0 \tag{21}$$

If $b^T b^* = 0$, (21) is equivalent to $k(b^T a^* - b_0) = 0$, so that from (20) we get

$$\varphi(k, w^*) = k^2\frac{1}{2b_0^2}\|b\|^4\left(2a_0 - a^{*T}a\right)$$

Therefore

$$\inf_{k\in\Re}\min_{w\in b^\perp}\varphi(k, w) = \inf_{k\in\Re}\varphi(k, w^*) = \inf_{k\in\Re}\left[k^2\frac{1}{2b_0^2}\|b\|^4\left(2a_0 - a^{*T}a\right)\right] \geq 0$$

if and only if $\frac{1}{2b_0^2}\|b\|^4\left(2a_0 - a^{*T}a\right) \geq 0$. Thus, Q is positive semidefinite if and only if *ii)* holds.
Consider now the case $b^T b^* \neq 0$; from equation (21) we obtain

$$\lambda^* = \frac{k}{b_0}\frac{\|b\|^2}{b^T b^*}\left(b_0 - b^T a^*\right)$$

so that (20) becomes

$$\varphi(k, w^*) = k^2\frac{\|b\|^4}{b_0^2}(a_0 - \frac{1}{2}a^{*T}a) + \frac{1}{2}\lambda^{*2} b^T b^*$$

that is

$$\varphi(k,w^*) = k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2\right)$$

Therefore

$$\inf_{k\in\Re}\min_{w\in b^\perp}\varphi(k,w) = \inf_{k\in\Re}\varphi(k,w^*) =$$
$$= \inf_{k\in\Re}\left[k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2\right)\right] \geq 0$$

if and only if $a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2 \geq 0$. Consequently, Q is positive semidefinite if and only if *iii*) holds.
Finally we deal with the case $b \notin \operatorname{Im} A$. From (18), system (16) has a solution if and only if there exist $a^* \in \Re^n$ and μ^* such that $a = Aa^* + \mu^* b$ and hence equation (16.1) can be written as follows

$$k\frac{\|b\|^2}{b_0}\left(Aa^* + \mu^* b\right) = A\left(w^* + kb\right) - \lambda^* b$$

or equivalently

$$A\left(w^* + kb - k\frac{\|b\|^2}{b_0}a^*\right) = \left(k\frac{\|b\|^2}{b_0}\mu^* + \lambda^*\right)b$$

Since $b \notin \operatorname{Im} A$, the above equation holds if and only if $k\frac{\|b\|^2}{b_0}\mu^* + \lambda^* = 0$ and hence (λ^*, w^*) is the solution of system (16) if and only if

$$\lambda^* = -k\frac{\|b\|^2}{b_0}\mu^*$$
$$w^* = k\frac{\|b\|^2}{b_0}a^* - kb + e,\ e \in \ker A$$

Again, from $k\left(Ab - \frac{\|b\|^2}{b_0}a\right)^T w^* = -w^{*T}Aw^*$, we have

$$\varphi(k,w^*) =$$
$$= k^2\left(-\left(\frac{\|b\|^2}{b_0}Aa^* + \frac{\|b\|^2}{b_0}\mu^* b\right)^T b + \frac{a_0}{b_0^2}\|b\|^4 - \frac{1}{2}\frac{\|b\|^4}{b_0^2}a^{*T}Aa^* + \frac{\|b\|^2}{b_0}a^{*T}Ab\right)$$
$$= k^2\left(-\mu^*\frac{\|b\|^4}{b_0} + \frac{a_0}{b_0^2}\|b\|^4 - \frac{1}{2}\frac{\|b\|^4}{b_0^2}a^{*T}Aa^*\right) = k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - b_0\mu^* - \tfrac{1}{2}a^{*T}Aa^*\right)$$

Therefore

$$\inf_{k\in\Re}\min_{w\in b^\perp}\varphi(k,w) = \inf_{k\in\Re}\varphi(k,w^*) = \inf_{k\in\Re}\left[k^2\frac{\|b\|^4}{b_0^2}\left(a_0 - b_0\mu^* - \frac{1}{2}a^{*T}Aa^*\right)\right] \geq 0$$

if and only if $a_0 - b_0\mu^* - \frac{1}{2}a^{*T}Aa^* \geq 0$. Consequently, Q is positive semidefinite if and only if *iv*) holds.
The proof is complete. ■

Remark 4.1 *Let us note that in ii) and iii) of Theorem 4.3, necessarily we have* $\ker A \subset a^\perp \cap b^\perp$. *In fact,* $Aa^* = a$, $Ab^* = b$, *imply* $z^T Aa^* = z^T a = 0$, $z^T Ab^* = z^T b = 0 \quad \forall z \in \ker A$. *Consequently, relations (6) and (7) are independent from the particular choice of* a^*, b^*.
With respect to iv) of Theorem 4.3, let $\mu^*, \mu_1^* \in \Re$ *and* $a^*, a_1^* \in \Re^n$ *such that* $a = Aa^* + \mu^* b = Aa_1^* + \mu_1^* b$; *then* $A(a^* - a_1^*) = (\mu_1^* - \mu^*)b$. *Since* $b \notin \operatorname{Im} A$, *necessarily we have* $\mu_1^* = \mu^*$ *and* $a_1^* \in a^* + \ker A$. *As a consequence, in (8)* μ^* *is unique and* $(a^*)^T Aa^*$ *is independent from the particular choice of* a^*.

When the matrix A is not singular (in particular when A is positive definite) the characterization of the pseudoconvexity of the function f assumes a very simple form as is stated in the following corollaries.

Corollary 4.2 *Assume that* A *is not singular. The function* f *is pseudoconvex on the halfspace* S *if and only if* A *is positive semidefinite on* $b^\perp$ *and one of the following conditions holds:*
i) $b^T A^{-1} b = 0$ *and* $2a_0 \geq a^T A^{-1} a$;
ii) $b^T A^{-1} b \neq 0$ *and* $2a_0 - a^T A^{-1} a + \frac{\left(b_0 - b^T A^{-1} a\right)^2}{b^T A^{-1} b} \geq 0$.

Proof. Let us note that case *iv*) of Theorem 4.3 does not occur since the non singularity of A implies $b \in \operatorname{Im} A$.
Consider case *i*) of Theorem 4.3. We have

$$b = \frac{\|b\|^2}{b_0} A^{-1} a + \alpha A^{-1} b \tag{22}$$

so that

$$a^T b = \frac{\|b\|^2}{b_0} a^T A^{-1} a + \alpha a^T A^{-1} b \tag{23}$$

Substituting (23) in (5), we obtain

$$2a_0 - a^T A^{-1} a + \frac{\alpha}{\|b\|^2} \left(b_0^2 - b_0 b^T A^{-1} a\right) \geq 0 \tag{24}$$

If $b^T A^{-1} b = 0$, from (22), we have $b^T A^{-1} a = b_0$, so that (24) becomes $2a_0 - a^T A^{-1} a \geq 0$ and thus *i*) is verified.
If $b^T A^{-1} b \neq 0$, from (22), we have

$$\frac{\alpha}{\|b\|^2} = \frac{b_0 - b^T A^{-1} a}{b_0 b^T A^{-1} b} \tag{25}$$

Substituting (25) in (24), we obtain condition *ii*).
Consider now condition *ii*) of Theorem 4.3.
We have $b^* = A^{-1}b$, $a^* = A^{-1}a$, $b^T A^{-1} b = 0$, $b^T A^{-1} a = b_0$, so that (6) reduces to condition *i*).
At last consider condition *iii*) of Theorem 4.3.
We have $b^* = A^{-1}b$, $a^* = A^{-1}a$, $b^T A^{-1} b \neq 0$, so that (7) reduces to condition *ii*). ■

Corollary 4.3 *Assume that A is positive definite on $\Re^n$. Then the function f is pseudoconvex on the halfspace S if and only if*

$$2a_0 - a^T A^{-1} a + \frac{\left(b_0 - b^T A^{-1} a\right)^2}{b^T A^{-1} b} \geq 0 \tag{26}$$

Proof. It follows from Corollary 4.2, taking into account that $b^T A^{-1} b > 0$. ■

Remark 4.2 *Let us note that the function f may be not pseudoconvex even if A is positive definite (see Example 5.6).*

5 An algorithm to test for pseudoconvexity

The results obtained in the previous sections allow to state a simple algorithm for testing the pseudoconvexity of the function

$$f(x) = \frac{\frac{1}{2}x^T A x + a^T x + a_0}{\left(b^T x + b_0\right)^2}, \quad x \in S = \{x \in \Re^n : b^T x + b_0 > 0\}.$$

Step 0. If $A = \delta b b^T$ go to step 8, otherwise go to step 1.
Step 1. If A is not positive semidefinite on $b^\perp$, Stop: f is not pseudoconvex; otherwise calculate $Ab - \frac{\|b\|^2}{b_0} a$. If $Ab - \frac{\|b\|^2}{b_0} a = \alpha b$ go to step 2, otherwise go to step 3.
Step 2. If $\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2}$, Stop: f is pseudoconvex otherwise Stop: f is not pseudoconvex.
Step 3. If the system $Ax = b$ has not solutions, go to step 7, otherwise go to step 4.
Step 4. If the system $Ax = a$ has not solutions Stop: f is not pseudoconvex, otherwise let a^* such that $Aa^* = a$ and let b^* such that $Ab^* = b$. If $b^T b^* = 0$ go to step 5, otherwise go to step 6.
Step 5. If $b^T a^* = b_0$ and $a^T a^* \leq 2a_0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.

Step 6. If $a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^Tb^*}\left(b_0 - b^Ta^*\right)^2 \geq 0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.
Step 7. If there exist μ^*, a^* such that $a = Aa^* + \mu^* b$, $a_0 - \mu^* b_0 - \frac{1}{2}a^{*T}Aa^* \geq 0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.
Step 8. If $a \neq \gamma b$, Stop: f is not pseudoconvex, otherwise go to step 9.
Step 9. If $\delta b_0^2 - 2\gamma b_0 + 2a_0 \geq 0$, Stop: f is pseudoconvex, otherwise go to step 10.
Step 10. If $\gamma \leq \delta b_0$, Stop: f is pseudoconvex, otherwise Stop: f is not pseudoconvex.

The following examples point out different cases that can occur applying the previous algorithm.

Example 5.1 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_3^2 + x_1 + a_2 x_2 + x_3 + 1}{(x_2+1)^2}$$

We have

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, a = \begin{pmatrix} 1 \\ a_2 \\ 1 \end{pmatrix}, a_2 \in \Re,\ a_0 = 1, b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, b_0 = 1$$

Since $A \neq \delta bb^T$, we go to step 1. It is easy to verify that A is positive semidefinite on $\Re^3$. It results $Ab - \frac{\|b\|^2}{b_0}a = -a \neq \alpha b$, $\forall \alpha \in \Re$. We go to step 3. Since the system $Ax = b$ has not solutions, we go to step 7. A solution of the system $a = Aa^* + \mu^* b$ (see remark 4.1) is $a^* = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$, $\mu^* = a_2$.
The condition $a_0 - \mu^* b_0 - \frac{1}{2}a^{*T}Aa^* \geq 0$ becomes $\frac{1}{2} - a_2 \geq 0$, so that the function $f(x)$ is pseudoconvex for every $a_2 \leq \frac{1}{2}$ while is not pseudoconvex for every $a_2 > \frac{1}{2}$.

Example 5.2 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{\frac{1}{2}x_1^2 + x_2^2 + \frac{3}{2}x_3^2 + 2x_1x_2 + x_1 + 2x_2 + a_0}{(x_1+1)^2}$$

We have

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ a = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ a_0 \in \Re,\ b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ b_0 = 1.$$

Since $A \neq \delta bb^T$, we go to step 1. Let us note that A is not singular with a negative eigenvalue, nevertheless it is easy to verify that A is positive semi-definite on $b^\perp = \{(x_1, x_2, x_3) : x_1 = 0\}$. We have $Ab - \frac{\|b\|^2}{b_0}a = (0, 0, 0)^T = \alpha b$ with $\alpha = 0$. We go to step 2. The condition $\alpha \geq \frac{b_0 b^T a - 2\|b\|^2 a_0}{b_0^2}$ becomes $1 - 2a_0 \leq 0$, so that f is pseudoconvex for every $a_0 \geq \frac{1}{2}$ and it is not pseudoconvex for every $a_0 < \frac{1}{2}$.

Example 5.3 *Consider the function*

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 - x_3^2 + 2x_1x_2 + x_1 + x_2 - x_3 + 1}{(x_3 + b_0)^2}$$

We have

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad a = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad a_0 = 1, \ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b_0 \neq 0.$$

Since $A \neq \delta bb^T$, we go to step 1; A is singular with a negative eigenvalue, nevertheless A is positive semidefinite on $b^\perp = \{(x_1, x_2, x_3) : x_3 = 0\}$. We have $Ab - \frac{\|b\|^2}{b_0}a = (-\frac{1}{b_0}, -\frac{1}{b_0}, -2 + \frac{1}{b_0})^T \neq \alpha b$, $\forall \alpha \in \Re$; we go to step 3. A solution of the system $Ax = b$ is $b^* = (1, -1, -\frac{1}{2})^T$ and we go to step 4. A solution of $Ax = a$ is $a^* = (\frac{1}{2}, 0, \frac{1}{2})^T$. Since $b^T b^* \neq 0$, we go to step 6. The condition $a_0 - \frac{a^{*T}a}{2} + \frac{1}{2b^T b^*}\left(b_0 - b^T a^*\right)^2 \geq 0$ becomes $(b_0 - \frac{1}{2})^2 \leq 1$, so that for every $b_0 \in [-\frac{1}{2}, \frac{3}{2}]$, $b_0 \neq 0$ f is pseudoconvex, while f is not pseudoconvex for every $b_0 \in (-\infty, -\frac{1}{2}) \cup (\frac{3}{2}, +\infty)$.

Example 5.4 *Consider the function*

$$f(x_1, x_2, x_3, x_4) = \frac{\frac{1}{2}x_1^2 + 2x_2^2 - x_3^2 + \frac{1}{2}x_4^2 + 2x_1x_2 + x_1 + 2x_2 + x_4 + a_0}{(2x_1 + 4x_2 - 2\sqrt{2}x_3 + 2)^2}$$

We have

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad a_0 \in \Re, \ b = \begin{bmatrix} 2 \\ 4 \\ -2\sqrt{2} \\ 0 \end{bmatrix}, \quad b_0 = 2.$$

Since $A \neq \delta bb^T$, we go to step 1. Let us note that A is singular with a negative eigenvalue, nevertheless A is positive semidefinite on $b^\perp = \{(x_1, x_2, x_3, x_4) : 2x_1 + 4x_2 - 2\sqrt{2}x_3 = 0\}$.

We have $Ab - \frac{\|b\|^2}{b_0}a = (-4, -8, 4\sqrt{2}, -14)^T \neq \alpha b$, $\forall \alpha \in \Re$ and we go to step 3. A solution of the system $Ax = b$ is $b^* = (2, 0, \sqrt{2}, 0)^T$ and we go to step 4. A solution of the system $Ax = a$ is $a^* = (1, 0, 0, 1)^T$. Since $b^T b^* = 0$, we go to step 5. We have $b^T a^* = 2 = b_0$. Since $a^T a^* = 2$, f is pseudoconvex for every $a_0 \geq 1$ while is not pseudoconvex for every $a_0 < 1$.

Example 5.5 *Consider the function*

$$f(x, y, z) = \frac{\delta(\frac{1}{2}x^2 + 2y^2 + \frac{1}{2}z^2 + 2xy - xz - 2yz + 2x + 4y - 2z + 1)}{(x + 2y - z + b_0)^2}$$

We have

$$A = \delta \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}, \; \delta \neq 0, \; a = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \; a_0 = 1, \; b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \; b_0 \neq 0.$$

It is easy to verify that $A = \delta b b^T$ and $a = 2b$; we go to step 9. It results $P(b_0, \delta) = \delta b_0^2 - 2\gamma b_0 + 2a_0 \geq 0$ in the following cases:
a) $\delta \geq 2, \; b_0 \neq 0$;
b) $0 < \delta < 2, \; b_0 \in (-\infty, \frac{2-\sqrt{4-2\delta}}{\delta}) \cup (\frac{2+\sqrt{4-2\delta}}{\delta}, +\infty)$;
c) $\delta < 0, \; b_0 \in \left[\frac{2-\sqrt{4-2\delta}}{\delta}, \frac{2+\sqrt{4-2\delta}}{\delta}\right]$.
As a consequence, f is pseudoconvex when a), b), c) occur.
It results $P(b_0, \delta) < 0$ and $2 \leq \delta b_0$ in the following cases:
d) $0 < \delta < 2, \; b_0 \in [\frac{2}{\delta}, \frac{2+\sqrt{4-2\delta}}{\delta})$;
e) $\delta < 0, \; b_0 \in \left(-\infty, \frac{2-\sqrt{4-2\delta}}{\delta}\right]$.
As a consequence, f is pseudoconvex when d), e) occur.
In all other cases f is not pseudoconvex.
Let us note that when $\delta < 0$, the matrix A is negative semidefinite, nevertheless f may be pseudoconvex (see cases c) and e)).

The following example points out that f is not necessary pseudoconvex even if the matrix A is positive definite.

Example 5.6 *Consider the function*

$$f(x_1, x_2) = \frac{x_1^2 + 2x_2^2 + 2x_1x_2 + a_1x_1 + 2x_2 + 1}{(x_1 + x_2 + 1)^2}$$

We have

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, a = \begin{bmatrix} a_1 \\ 2 \end{bmatrix}, \; a_1 \in \Re, \; a_0 = 1, \; b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; b_0 = 1.$$

Since A is positive definite, for Corollary 4.3, f is pseudoconvex if and only if $2a_0 - a^T A^{-1} a + \frac{\left(b_0 - b^T A^{-1} a\right)^2}{b^T A^{-1} b} \geq 0$. This last inequality is equivalent to $-\frac{1}{2}a_1^2 + 2 \geq 0$, so that f is pseudoconvex if and only if $a_1 \in [-2, 2]$.

6 Special cases

In this section we characterize the pseudoconvexity of some classes of fractional functions for which the conditions given in Theorems 4.2, 4.3 assume a simple form.

Theorem 6.1 *Consider the function*

$$h(x) = \frac{a^T x + a_0}{(b^T x + b_0)^2}$$

on the halfspace $S = \{x : b^T x + b_0 > 0\}$.
Then h is pseudoconvex on S if and only if $a = \gamma b$ with $a_0 - \gamma b_0 \geq 0$ or with $a_0 - \gamma b_0 < 0$ and $\gamma \leq 0$.

Proof. It follows immediately from Theorem 4.2 by noting that A is the null matrix. ■

Corollary 6.1 *Consider the function*

$$h(x) = \frac{a_0}{(b^T x + b_0)^2}$$

on the halfspace S.
Then h is pseudoconvex on S for every $a_0 \in \Re$.

Theorem 6.2 *Consider the function*

$$h(x) = \frac{\frac{1}{2}x^T A x + a_0}{(b^T x + b_0)^2}$$

on the halfspace S, where A is not singular. Then f is pseudoconvex if and only if A is positive semidefinite on $b^\perp$ and one of the following conditions holds:
i) $b^T A^{-1} b = 0$ and $a_0 \geq 0$;
ii) $b^T A^{-1} b \neq 0$ and $2a_0 \geq -\frac{b_0^2}{b^T A^{-1} b}$.

Corollary 6.2 *Consider the function*

$$h(x) = \frac{\frac{1}{2}x^T Ax + a_0}{(b^T x + b_0)^2}$$

on the halfspace S, where A is positive definite. Then f is pseudoconvex if and only if $2a_0 \geq -\frac{b_0^2}{b^T A^{-1} b}$.

Corollary 6.3 *Consider the function*

$$h(x) = \frac{\frac{1}{2}x^T Ax}{(b^T x + b_0)^2}$$

on the halfspace S. Then h is pseudoconvex if and only if A is positive semidefinite or $A = \delta bb^T$ with $\delta < 0$ and $b_0 < 0$.

Proof. It follows immediately from Corollary 3.1 taking into account that $Q = \frac{1}{2}A$. ■

7 Pseudolinearity of the function $f(x)$.

It is well known that a function is pseudolinear if and only if it is both pseudoconvex and pseudoconcave. Taking into account that a function is pseudoconcave if and only if its opposite is pseudoconvex, from Theorem 3.1 we obtain

Theorem 7.1 *The function $Q(y)$ is pseudoconcave on H if and only if one of the following conditions holds:*
i) $\nu_+(Q) = 0$;
ii) $\nu_+(Q) = 1,\ \ker Q = c^\perp,\ q = \beta c,\ c_0 \leq \frac{\|c\|^4 \beta}{2c^T Qc}$.

Combining *i)* and *ii)* of Theorem 3.1 with *i)* and *ii)* of Theorem 7.1 and taking into account that *ii)* of Theorem 3.1 and *ii)* of Theorem 7.1 cannot occur simultaneously, we reach the following result:

Theorem 7.2 *The function $Q(y)$ is pseudolinear on H if and only if one of the following conditions hold:*
i) $Q = 0$;
ii) $Q = \mu cc^T,\quad \mu \neq 0,\quad q = \beta c,\quad \beta \in \Re,\quad c_0 \leq \frac{\beta}{2\mu}$.

In terms of tha data A, a, a_0, b, b_0, taking into account that the function $f(x)$ is pseudolinear on S if and only if $Q(y)$ is pseudolinear on H (see Theorem 2.2), we have the following theorem:

Theorem 7.3 *The function $f(x)$ is pseudolinear on S if and only if one of the following conditions holds:*
i) $A = \frac{ab^T+ba^T}{b_0} - \frac{2a_0}{b_0^2}bb^T$;
ii) $A = \delta bb^T$, $a = \gamma b$, $\delta, \gamma \in \Re$ *with* $\delta b_0^2 - 2\gamma b_0 + 2a_0 > 0$ *and* $\gamma \geq \delta b_0$ *or* $\delta b_0^2 - 2\gamma b_0 + 2a_0 < 0$ *and* $\gamma \leq \delta b_0$.

Proof. Condition *i)* is equivalent to *i)* of Theorem 7.2 taking into account relation (2), while *ii)* is equivalent to *ii)* of Theorem 7.2 taking into account the following relationships: $\mu = \frac{1}{2}\delta b_0^2 - \gamma b_0 + a_0$, $\beta = \frac{2a_0}{b_0} - \gamma$, $c_0 = \frac{1}{b_0}$. ∎

Corollary 7.1 *The function $f(x)$ is pseudolinear on S if and only if it can be reduced to a linear fractional function or to the following canonical form*

$$f(x) = \frac{B}{b^Tx + b_0} + \frac{C}{(b^Tx + b_0)^2} + D \tag{27}$$

where $C > 0$ and $B \geq 0$ or $C < 0$ and $B \leq 0$.

Proof. Corresponding to case *i)* of Theorem 7.3, it results $f(x) = \frac{b_0a^Tx - a_0b^Tx + a_0b_0}{b_0^2(b^Tx+b_0)}$ so that $f(x)$ is a linear fractional function; the canonical form (27) follows by *ii)* of Theorem 7.3 taking into account Corollary 4.1. ∎

Remark 7.1 *Since* $\ker A \subset a^\perp \cap b^\perp$, *the pseudolinearity of $f(x)$ implies the singularity of the matrix A.*
The function $h(x)$ in Corollary 6.1 is pseudolinear on S for every $a_0 \in \Re$ (set $B = 0$ in (27)) and the function $h(x)$ in Theorem 6.1 is pseudolinear if and only if $a = \gamma b$ with $\gamma \geq 0$ and $a_0 - \gamma b_0 > 0$ or $\gamma \leq 0$ and $a_0 - \gamma b_0 < 0$.

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