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Some classes of pseudoconvex fractional functions via the Charnes-Cooper transformation

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Summary. Using a very recent approach based on the Charnes-Cooper trasformation we characterize the pseudoconvexity of the sum between a quadratic fractional function and a linear one. Furthemore we prove that the ratio between a quadratic fractional function and the cube of an affine one is pseudoconvex if and only if the product between a quadratic fractional function and an affine one is pseudoconvex and we provide a sort of canonical form for this latter class of functions. Benefiting by the new results we are able to characterize the pseudoconvexity of the ratio between a quadratic fractional function and the cube of an affine one.

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1 Introduction

Since the early sixties, the strict relationship between generalized convexity and fractional programming has been highlighted and from the beginning, fractional programming has benefited from advances in generalized convexity, and vice versa (see for instance [14, 16]). Generalized fractional programming and in particular quadratic and multiplicative fractional programming are extremely important even for their numerous applications such as Data Envelopment Analysis, tax programming, risk and portfolio theory, logistics and location theory (see for instance [2, 3, 11, 12, 15]). Among the different classes of generalized convex functions, the pseudoconvex one occupies a leading position in optimization for its good properties. Nevertheless pseudoconvex functions have no algebraic structure and this lack of structure causes many difficulties to establish whether a function is pseudoconvex or not.

There are several characterizations for continuously differentiable function and for twice differentiable functions [1, 13]. Since these conditions are not very easy to be checked, some more operative characterizations, dealing with quadratic fractional functions, have been recently proposed [4, 8]. According with a very recent approach the pseudoconvexity of fractional functions is studied by means of the generalized Charnes-Cooper transformation. It is shown [6, 7] that this transformation maintains the pseudoconvexity so that the fundamental idea behind this study is the following: if an unknown class of functions can be transformed in a class of pseudoconvex functions, it is possible to determine necessary and sufficient conditions guaranteeing the pseudoconvexity of the unknown class of functions.

Following this idea, we prove that the sum between a quadratic fractional function and a linear one is pseudoconvex if and only if a suitable quadratic fractional function is pseudoconvex. Therefore, using the known results for this latter class of functions we establish a new characterization and we give a simple algorithm in order to test the pseudoconvexity for the sum between a quadratic fractional function and a linear one.

Furthermore we address our attention to the pseudoconvexity of the ratio between a quadratic function and the power p of an affine one. Since the cases $p = 1$ and $p = 2$ have been handled in [8, 9] we deal with the case $p = 3$. Performing the Charnes-Cooper transformation, we prove that this class of functions is pseudoconvex if and only if the product between the quadratic function and a suitable affine one is pseudoconvex. As far as we know, even for this latter class of functions there are no easy to be checked conditions for testing the pseudoconvexity. Consequently, we first characterize the pseudoconvexity for the product between a quadratic function and an affine one: more precisely we prove that a function belonging to this class is pseudoconvex if and only if it has a suitable canonical form. The obtained result allows to provide a new characterization for the ratio between a quadratic function and the cube of an affine one.

2 Preliminary results and notations.

Throughout the paper we will use the following notations and properties.

- A is a $n \times n$ symmetric matrix such that $A \neq [0]$ where $[0]$ is the null matrix;
- $\nu_-(A)$ $(\nu_+(A))$ denotes the number of negative (positive) eigenvalues of a matrix A;
- $\ker A$ denotes the kernel of A i.e., $\ker A = \{v : Av = 0\}$;
- $\dim W$ denotes the dimension of the vector space W;
- $\mathrm{Im}A$ denotes the set $\mathrm{Im}A = \{z = Av,\ v \in \Re^s\}$;
- $v^\perp$ denotes the orthogonal space to a vector v i.e., $v^\perp = \{w : v^T w = 0\}$.

For the sake of completeness we recall the definition of pseudoconvex functions and the related properties we are going to use in the next section (for further details see for instance [1]).

Definition 1. *Let f be a differentiable function on the open and convex set $C \subseteq \Re^n$. f is pseudoconvex if for $x, y \in C$*

$$f(y) < f(x) \text{ implies that } \nabla f(x)^T (y - x) < 0.$$

- f is pseudoconvex if and only if for every $x_0, v \in \Re^n$ the restriction of f on the line $x = x_0 + tv$, $t \in \Re$, is pseudoconvex.
- Let $C \subseteq \Re^n$ an open and convex set f is pseudoconvex if and only if $\forall x \in C, \forall v \in \Re^n \setminus \{0\}$, such that $\nabla f(x)^T v = 0$ the function $\varphi(t) = f(x+tv)$ attains a local minimum at $t = 0$.
- Let $C \subseteq \Re^n$ an open and convex set and let f be a twice continuously differentiable. f is pseudoconvex if and only if $\forall x_0 \in C$, $\forall v \in \Re^n \setminus \{0\}$, such that $\nabla f(x_0)^T v = 0$ either $v^T H(x_0) v > 0$ or $v^T H(x_0) v = 0$ and the function $\varphi(t) = f(x^0 + tv)$ attains a local minimum at $t = 0$.

Consider the Charnes-Cooper transformation [10]

$$y(x) = \frac{x}{b^T x + b_0} \tag{1}$$

defined on the set $S = \{x \in \Re^n : b^T x + b_0 > 0\}$ where $b \in \Re^n$ and $b_0 \in \Re$, $b_0 \neq 0$. It is well known that this map is a diffeomorphism and its inverse is

$$x(y) = \frac{b_0 y}{1 - b^T y} \tag{2}$$

defined on the set $S^* = \{y \in \Re^n : \frac{b_0}{1-b^T y} > 0\}$. As it is shown in [6, 7] the Charnes-Cooper transformation preserves the pseudoconvexity of f. More precisely the following theorem holds.

Theorem 1. *Let f be a differentiable function defined on $\Re^n$ and let $\psi(y)$ be the function obtained by applying the inverse of the Charnes-Cooper transformation (2) to $f(x)$.*
Function $f(x)$ is pseudoconvex on S if and only if function $\psi(y)$ is pseudoconvex on S^*.

In some cases, the study of the pseudoconvexity of the transformed function $\psi(y)$ may be easier than the study of the pseudoconvexity of f. Therefore, thanks to the previous theorem, by means of the results on $\psi(y)$ we can characterize the pseudoconvexity of f in terms of its initial data. Following this approach in the next section we aim to study the pseudoconvexity of some classes of generalized quadratic fractional functions.

The following Lemma will be also useful.

Lemma 1. *Consider a non-null symmetric matrix A of order n and a non-null vector $a \in \Re^n$. Then there exists $d \in \Re^n$ such that $d^T Ad \neq 0$ and $a^T d \neq 0$.*

Proof. Suppose on the contrary for every $d \in \Re^n$ $a^T d \neq 0$ implies $d^T Ad = 0$. Since $a \neq 0$, setting $d = a$ we get $\|a\|^2 \neq 0$ and hence $a^T Aa = 0$. Take $x = ta + w$, $w \in a^{\perp}, t \in \Re$; we have

$$\frac{1}{2}x^T Ax = \frac{1}{2}(ta+w)^T A(ta+w) = ta^T Aw + \frac{1}{2}w^T Aw.$$

Since $a^T x = t\|a\|^2 \neq 0$ for every $t \neq 0$ it results $ta^T Aw + \frac{1}{2}w^T Aw = 0$ for every $t \neq 0$. Then necessarily we have $a^T Aw = 0$ and $w^T Aw = 0$ for every $w \in a^{\perp}$. From the second equality it follows $Aw = ka$ and since $a^T Aw = k\|a\|^2 = 0$ we obtain $k = 0$, so that $Aw = 0$ for every $w \in a^{\perp}$. Taking into account that $A \neq [0]$, we get $A = \lambda aa^T$ and then $a^T Aa = \lambda \|a\|^2 \neq 0$ which is a contradiction.

3 New classes of pseudoconvex fractional functions

3.1 Pseudoconvexity of the sum between a quadratic fractional function and a linear one

Consider the following function

$$f(x) = \frac{\frac{1}{2}x^T Ax}{b^T x + b_0} + p^T x \tag{3}$$

on the halfspace $S = \{x \in \Re^n : b^T x + b_0 > 0\}$, $b_0 \neq 0$. Performing the Charnes-Cooper transformation (2) we obtain the following function defined on the halfspace $S^* = \{y \in \Re^n : \frac{b_0}{1-b^T y} > 0\}$

$$g(y) = \frac{\frac{b_0^2}{2(1-b^T y)^2}y^T Ay}{\frac{b_0}{1-b^T y}} + \frac{b_0}{1-b^T y}p^T y = \frac{b_0}{1-b^T y}\left(\frac{1}{2}y^T Ay + p^T y\right)$$

that is, setting $c = -\frac{b}{b_0}$, $c_0 = \frac{1}{b_0}$

$$g(y) = \frac{\frac{1}{2}y^T Ay + p^T y}{c^T y + c_0},\ y \in S^*. \tag{4}$$

From Theorem 1, the pseudoconvexity of f on S is equivalent to the pseudoconvexity of g on S^*. A characterization of the pseudoconvexity for such a class of functions is given in [4]. More precisely the following theorem holds.

Theorem 2. *Consider function $g(y) = \frac{\frac{1}{2}y^T Ay + p^T y}{c^T y + c_0}$ on the halfspace $S^* = \{y \in \Re^n : c^T y + c_0 > 0\}$, $c_0 \neq 0$. g is pseudoconvex if and only if one of the following conditions holds:*

a) $\nu_-(A) = 0$,
b) $\nu_-(A) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ *such that* $A\overline{x} = p$ *and* $A\overline{y} = c$, $c^T\overline{y} = 0$, $c^T\overline{x} = c_0$ *and* $p^T\overline{x} \leq 0$;
c) $\nu_-(A) = 1$, $\exists \overline{x}, \overline{y} \in \Re^n$ *such that* $A\overline{x} = p$ *and* $A\overline{y} = c$, $c^T\overline{y} < 0$ *and* $\frac{\Delta}{4} = (c_0 - c^T\overline{x})^2 - c^T\overline{y}(p^T\overline{x}) \leq 0$.

Thanks to the Charnes-Cooper transformation, Theorem 2 allows us to characterize the pseudoconvexity of f in term of its initial data.

Theorem 3. *Consider function* $f(x) = \frac{\frac{1}{2}x^T Ax}{b^T x + b_0} + p^T x$ *on the set* $S = \{x \in \Re^n : b^T x + b_0 > 0\}$, $b_0 \neq 0$. *f is pseudoconvex if and only if one of the following conditions holds:*

a) $\nu_-(A) = 0$,
b) $\nu_-(A) = 1$, $\exists \overline{x}, \overline{z} \in \Re^n$ *such that* $A\overline{x} = p$ *and* $A\overline{z} = b$, $b^T\overline{z} = 0$, $b^T\overline{x} = -1$ *and* $p^T\overline{x} \leq 0$;
c) $\nu_-(A) = 1$, $\exists \overline{x}, \overline{z} \in \Re^n$ *such that* $A\overline{x} = p$ *and* $A\overline{z} = b$, $b^T\overline{z} < 0$ *and* $\frac{\Delta}{4} = (1 + b^T\overline{x})^2 - b^T\overline{z}(p^T\overline{x}) \leq 0$.

Proof. From Theorem 1 f is pseudoconvex on S if and only if g is pseudoconvex on S^* and so if and only if one of conditions a), b), c) in Theorem 2 holds. Recalling that $b = -\frac{c}{c_0}$, $b_0 = \frac{1}{c_0}$, by means of simple calculations it can be proved that conditions a), b), c) are equivalent to the corresponding ones given in Theorem 2.

The following example shows that function $f(x)$ in (3) can be pseudoconvex even if the fractional quadratic function is not pseudoconvex.

Example 1. Consider function $f(x, y) = \frac{x^2 - y^2}{-x + y + 2} + x + y$, that is $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. It is easy to verify that $\frac{x^2 - y^2}{-x + y + 2}$ is not pseudoconvex on S. On the other hand $v_-(A) = v_+(A) = 1$, $\bar{z} = \left(-\frac{1}{2}-, \frac{1}{2}\right)^T$, $\bar{x} = \left(\frac{1}{2}, -\frac{1}{2}\right)^T$ and so $b^T\bar{z} = 0$, $b^T\bar{x} = -1$ and $p^T\bar{x} = 0$. Hence condition b) in Theorem 3 holds and f is pseudoconvex on S.

According with the previous result, we suggest the following algorithm for testing the pseudoconvexity of the sum between a quadratic fractional function and an affine one.

ALGORITHM

STEP 1

Calculate the eigenvalues of A. If $\nu_-(A) > 1$, STOP : f is not pseudoconvex. If $\nu_-(A) = 0$, STOP : f is pseudoconvex; otherwise go to STEP 2.

STEP 2

Solve the linear systems $Ax = p$ and $Az = b$. If one of these systems has no solutions STOP: f is not pseudoconvex; otherwise go to STEP 3.

STEP 3
Let $\bar{z}$ such that $A\bar{z} = b$. Calculate $b^T\bar{z}$. If $b^T\bar{z} > 0$ STOP : f is not pseudoconvex. If $b^T\bar{z} = 0$ go to STEP 4, otherwise go to STEP 5.
STEP 4
Let $\bar{x}$ such that $A\bar{x} = p$. Calculate $b^T\bar{x}$. If $b^T\bar{x} \neq -1$ STOP: f is not pseudoconvex, otherwise calculate $p^T\bar{x}$. If $p^T\bar{x} > 0$ STOP: f is not pseudoconvex otherwise STOP: f is pseudoconvex.
STEP 5
Let $\bar{x}$ such that $A\bar{x} = p$. Calculate $\frac{\Delta}{4} = (1 + b^T\bar{x})^2 - b^T\bar{z}(p^T\bar{x})$. If $\Delta > 0$ STOP : f is not pseudoconvex otherwise f is pseudoconvex.

3.2 Pseudoconvexity of the ratio between a quadratic function and the cube of an affine one.

Consider now the following function

$$h(x) = \frac{1}{2}\frac{x^TAx}{(b^Tx + b_0)^p} \quad x \in S$$

where $p \in \mathbf{N}\backslash\{0\}$, $b_0 \neq 0$.
Performing the transformation (2) we get

$$g(y) = \frac{1}{2}\frac{b_0^2}{(1 - b^Ty)^2}y^TAy\frac{1}{\left(\frac{b_0}{1-b^Ty}b^Ty + b_0\right)^p} = \frac{1}{2}y^TAy\frac{(1 - b^Ty)^{p-2}}{b_0^{p-2}}. \quad (5)$$

When $p = 1$ and $p = 2$, the pseudoconvexity of the function $h(x)$ has been completely characterized in [7, 9]. In this section we aim to study the case $p = 3$, that is

$$h(x) = \frac{1}{2}\frac{x^TAx}{(b^Tx + b_0)^3}. \quad (6)$$

Setting $p = 3$, $a = -\frac{b}{b_0}$ and $a_0 = -\frac{1}{b_0}$ in (5) we obtain

$$g(y) = \frac{1}{2}y^TAy\left(a^Ty - a_0\right). \quad (7)$$

In order to study the pseudoconvexity of h, we first deal with the pseudoconvexity of its transformed function g. In this light, the next subsection is devoted to the study of the pseudoconvexity of the product between a quadratic function and a linear one. The obtained results will allow us to characterize the pseudoconvexity of function (6).

Pseudoconvexity of the product between a quadratic function and a linear one.

Let us consider the following function

$$f(x) = \frac{1}{2}x^T Ax\left(a^T x - a_0\right). \tag{8}$$

Taking into account Lemma 1, we can easily prove that f in (8) is not pseudoconvex on $\Re^n$. More precisely take a vector $d \in \Re^n$ such that $d^T Ad \neq 0$ and $a^T d \neq 0$; the restriction of f along the line $x = td$ is $\varphi(t) = f(td) = \frac{1}{2}\left(t^2 d^T Ad\right)\left(ta^T d - a_0\right)$ and $\varphi'(t) = \frac{1}{2}3t^2\left(a^T d\right)\left(d^T Ad\right) - 2a_0 d^T Adt$. $\varphi(t)$ has two distinct critical points so that it is not pseudoconvex and hence f is not pseudoconvex on $\Re^n$. Due to this, we study the pseudoconvexity of f on the halfspace $S^* = \{x \in \Re^n : a^T x - a_0 > 0\}$.

Preliminary and useful computations are the following

$$\begin{aligned} \nabla f(x) &= Ax\left(a^T x - a_0\right) + \frac{1}{2}x^T Axa && (9)\\ H(x) &= A\left(a^T x - a_0\right) + 2Axa^T \\ \varphi(t) &= f(x_0 + td) = \frac{1}{2}\left({x_0}^T Ax_0 + 2x_0^T Adt + t^2 d^T Ad\right)\left(\alpha + ta^T d\right) && (10)\\ \varphi''(t) &= 3t\left(a^T d\right)\left(d^T Ad\right) + 2\left(\alpha d^T Ad + 2\left(a^T d\right)\left(d^T Ax_0\right)\right). && (11) \end{aligned}$$

Moreover for every $d \in (\nabla f(x))^\perp$ we get

$$\varphi'(t) = \frac{3}{2}t^2\left(a^T d\right)\left(d^T Ad\right) + \left(\alpha d^T Ad + 2\left(d^T Ax_0\right)\left(a^T d\right)\right)t \tag{12}$$

where $\alpha = a^T x_0 - a_0$. Before presenting a complete characterization of the pseudoconvexity of f, we state the following necessary conditions.

Theorem 4. *Consider function f in (8). If f is pseudoconvex on $S^* = \{x \in \Re^n : a^T x - a_0 > 0\}$ then*
i) $a_0 \geq 0$.
ii) A is not indefinite.

Proof. i) Suppose $a_0 < 0$. From Lemma 1 there exists $u \in \Re^n$ such that $u^T Au \neq 0$ and $a^T u \neq 0$. Consider the line $x = tu$, $t \in \Re$. It results $\varphi(t) = f(tu) = \frac{1}{2}t^2 u^T Au\left(ta^T u - a_0\right)$, $\varphi'(t) = \frac{1}{2}u^T Au\left(3ta^T u - 2a_0\right)$, $\varphi''(t) = u^T Au\left(3ta^T u - a_0\right)$. $\varphi(t)$ has two distinct critical points $t_1 = 0$ and $t_2 = \frac{2}{3}\frac{a_0}{a^T u}$ with $\varphi''(t_1) = -u^T Aua_0$, $\varphi''(t_2) = u^T Aua_0$. Since t_1 and t_2 are both feasible, $\varphi(t)$ has a feasible maximum point and so it is not pseudoconvex. Consequently f is not pseudoconvex and this is a contradiction.
ii) By contradiction suppose that A is indefinite and take a unit norm eigenvector u associated with a negative eigenvalue λ. We first show that $a^T u \neq 0$; suppose on the contrary that $a^T u = 0$ and take $x = ka + tu$, $k, t \in \Re$. Since $\left(a^T x - a_0\right) = \left(k\|a\|^2 - a_0\right)$, for a sufficiently big k we get $x \in S^*$ for every $t \in \Re$. The restriction of f along the line $x = ka + tu$, $t \in \Re$ is the following

$$\varphi(t) = \left(\frac{1}{2}\lambda t^2 + \frac{1}{2}k^2 a^T Aa\right)\left(k\|a\|^2 - a_0\right).$$

Since $\lambda < 0$, $\varphi(t)$ has a feasible maximum point and so it is not pseudoconvex. Therefore f is not pseudoconvex and this is a contradiction.
Without any loss of generality we can assume $a^T u > 0$. Let v be an eigenvector associated with a positive eigenvalue μ, such that $\|v\| = 1$, $u^T v = 0$. We are going to prove that $a^T v = 0$. Suppose on the contrary that $a^T v \neq 0$; take $k \in \Re$ such that $x_0 = kv \in S^*$, that is $\alpha = ka^T v - a_0 > 0$ and consider $x = x_0 + tu = kv + tu$. Observe that $x \in S^*$ for every $t > -\frac{\alpha}{a^T u}$ and that the restriction of f along the line $x = kv + tu$, $t \in \Re$ is the following

$$\varphi(t) = \left(\frac{1}{2}\lambda t^2 + \frac{1}{2}\mu k^2\right)\left(ta^T u + \alpha\right)$$

so that

$$\varphi'(t) = \frac{3}{2}\lambda a^T u t^2 + \lambda \alpha t + \frac{1}{2}\mu k^2 a^T u\,.$$

Since $\Delta = \alpha^2\lambda^2 - 3\lambda\left(a^T u\right)^2 \mu k^2 > 0$ and $\frac{3}{2}\lambda a^T u < 0$, then $\varphi(t)$ has a feasible maximum point at $t_1 = -\frac{\alpha}{3a^T u} - \frac{\sqrt{\Delta}}{3\lambda a^T u}$ and so φ and f are not pseudoconvex, which is a contradiction. Consequently $a^T v = 0$.
At last consider $x = t\left(u - \sqrt{\frac{|\lambda|}{\mu}}v\right) + kv$, $k, t \in \Re$. It results $x \in S^*$ for $t > \frac{a_0}{a^T u}$ and for every $k \in \Re$; the restriction of f along the line $x = t\left(u - \sqrt{\frac{|\lambda|}{\mu}}v\right) + kv$ is the following

$$\varphi(t) = -ka^T u\mu\sqrt{\frac{|\lambda|}{\mu}}t^2 + \left(\frac{1}{2}k^2\mu a^T u + a_0 k\sqrt{\frac{|\lambda|}{\mu}}\mu\right)t - \frac{1}{2}k^2\mu a_0$$

and hence

$$\varphi'(t) = ka^T u\mu\left(-2\sqrt{\frac{|\lambda|}{\mu}}t + \frac{1}{2}k + \frac{a_0}{a^T u}\sqrt{\frac{|\lambda|}{\mu}}\right).$$

Consequently $\varphi(t)$ as a critical point at $t_1 = \frac{k}{4\sqrt{\frac{|\lambda|}{\mu}}} + \frac{a_0}{2a^T u}$. For $k > \frac{2a_0}{a^T u}\sqrt{\frac{|\lambda|}{\mu}}$, $t_1 \in S^*$ and it is a feasible maximum point for $\varphi(t)$. This implies φ and f are not pseudoconvex, which is a contradiction.

Theorem 5. *Consider function f in (8). If f is pseudoconvex on S^* then*
i) $x^T Ax \geq 0$ for every $x \in a^\perp$.
ii) $v_-(A) \leq 1$.

Proof. i) Suppose there exists $d \in a^\perp$ such that $d^T Ad < 0$. Take $x_0 \in S^*$ and the line $x = x_0 + td$, $t \in \Re$. Observe that $x \in S^*$ for every $t \in \Re$ and since $d^T Ad < 0$ the restriction $\varphi(t) = f(x_0 + td) = \frac{1}{2}\left(t^2 d^T Ad + 2d^T Ax_0 t + x_0^T Ax_0\right)\left(a^T x_0 - a_0\right)$ has a feasible maximum point. Therefore $\varphi(t)$ is not pseudoconvex and this is a contradiction.

ii) Suppose by contradiction that $v_-(A) > 1$ and let u, v be two orthogonal eigenvectors of A associated with two distinct negative eigenvalues λ_1, λ_2. Since $\dim\{u,v\} = 2$ and $\dim a^\perp = n-1$, there exists $d = \alpha u + \beta v$ such that $d \in a^\perp$. Consider $x = x_0 + td$ $t \in \Re$ and the corresponding restriction $\varphi(t)$ of f. By means of simple calculations we get

$$\varphi(t) = \frac{1}{2}\left(\lambda_1 t^2 \alpha^2 \|u\|^2 + \lambda_2 t^2 \beta^2 \|u\|^2 + 2(\alpha u + \beta v)^T A x_0 t + x_0^T A x_0\right)\left(a^T x_0 - a_0\right)$$

Since $\lambda_1, \lambda_2 < 0$, $\varphi(t)$ is not pseudoconvex and this is a contradiction.

The following theorem presents a complete characterization of the pseudoconvexity of f.

Theorem 6. *Consider function f in (8). f is pseudoconvex on S^* if and only if f is of the following form*

$$f(x) = \frac{1}{2}\lambda\left(a^T x\right)^2\left(a^T x - a_0\right) \quad \text{where } a_0 \geq 0. \tag{13}$$

Proof. $\Longrightarrow$From Theorem 4, $a_0 \geq 0$ and A can not be indefinite. We are left to deal with the case A is semidefinite. We first assume that A is negative semidefinite. From ii) of Theorem 5, if follows that A has exactly one negative eigenvalues and so A can be rewritten as $A = \mu u u^T$ with $\mu < 0$. From i) of Theorem 5 $d^T A d = 0$ for every $d \in a^\perp$ and so we necessarily have that $u = ka$, i.e., a is an eigenvector of A associated with the negative eigenvalue μ. Therefore $f(x) = \frac{1}{2}\lambda\left(a^T x\right)^2\left(a^T x - a_0\right)$ where $\lambda = k^2\mu < 0$.
Finally consider the case A positive semidefinite. Let be $x_0 \in S^*$ such that $\nabla f(x_0) \neq 0$. Since A is semidefinite positive, $Ax_0 = 0$ if and only if $x_0^T A x_0 = 0$ and so from (9) it follows that $x_0^T A x_0 \neq 0$. We are going to prove that $a^T d = 0$ for every $d \in (\nabla f(x_0))^\perp$. Suppose on the contrary there exists $d \in (\nabla f(x_0))^\perp$ such that $a^T d \neq 0$. Without any loss of generality we can assume $a^T d > 0$. It results $d^T \nabla f(x_0) = 0$ if and only if $d^T A x_0 \alpha + \frac{1}{2} x_0^T A x_0 d^T a = 0$ where $\alpha = a^T x_0 - a_0$. Consider $x = x_0 + td$ where $t > -\frac{\alpha}{a^T d}$, i.e., $x \in S^*$; the corresponding restriction of f is

$$\varphi(t) = f(x_0 + td) = \frac{1}{2}\left(t^2 d^T A d + 2 d^T A x_0 t + x_0^T A x_0\right)\left(t a^T d + \alpha\right)$$

and from (12)

$$\varphi'(t) = \frac{3}{2}t^2\left(a^T d\right)\left(d^T A d\right) + \left(\alpha d^T A d + 2\left(d^T A x_0\right)\left(a^T d\right)\right)t\,.$$

Observe that $d^T A d \neq 0$; in fact if $d^T A d = 0$ then $Ad = 0$ and hence $d^T A x_0 = 0$. This can not be true since $d^T \nabla f(x_0) = 0$ and $x_0^T A x_0 \neq 0$.
Since $d^T A d > 0$, $\varphi'(t) = 0$ for $t_1 = 0$ and $t_2 = -\frac{\alpha d^T A d + 2\left(d^T A x_0\right)\left(a^T d\right)}{\frac{3}{2} d^T A d \left(a^T d\right)}$.
Obviously $t_1 \in S^*$ and t_2 is feasible if and only if

$$-\frac{\alpha d^T A d + 2\left(d^T A x_0\right)\left(a^T d\right)}{\frac{3}{2} d^T A d\left(a^T d\right)} > -\frac{\alpha}{a^T d}$$

that is

$$\frac{1}{2}\alpha d^T A d - 2\left(d^T A x_0\right)\left(a^T d\right) > 0. \tag{14}$$

Since $d^T \nabla f(x_0) = 0$ we have $d^T A x_0 = -\frac{1}{2\alpha} x_0^T A x_0 d^T a$ and so condition (14) becomes

$$\frac{1}{2}\alpha d^T A d + \frac{x_0^T A x_0}{\alpha}\left(a^T d\right)^2 > 0$$

which is always verified because A is positive semidefinite. Therefore $\varphi(t)$ has a feasible maximum point and this contradicts the pseudoconvexity of f.
Since $a^T d = 0$ for every $d \in (\nabla f(x_0))^{\perp}$, $\nabla f(x_0)$ is proportional to a ; from (9) it results that for every $x \in S^*$ with $\nabla f(x_0) \neq 0$ we have $Ax_0 = ha$, for some $h \in \Re$. We are going to prove that a is an eigenvector of A associated with a positive eigenvalue λ and that λ is the unique positive eigenvalue of A. Consider $x_0 = k_1 a$ and observe that $x_0 \in S^*$ if and only if $k_1 > \frac{a_0}{\|a\|^2}$. It follows that $Ax_0 = k_1 ha$ and hence $Aa = A\frac{x_0}{k_1} = \frac{h}{k_1} a = \lambda a$. Let $\mu > 0$ be a positive eigenvalue of A, with $\mu \neq \lambda$ and let u be a corresponding eigenvector such that $u \in a^{\perp}$. Take $x_0 = k_1 a + u$ with $k_1 > \frac{a_0}{\|a\|^2}$, i.e. $x_0 \in S^*$. It is easy to verify that $\nabla f(x_0) \neq 0$, so that there exists $\bar{h}$ such that $Ax_0 = \bar{h}a$. On the other hand, $Ax_0 = \lambda k_1 a + \mu u$ and therefore $\bar{h}a = \lambda k_1 a + \mu u$ that is $\left(\bar{h} - \lambda k_1\right) a = \mu u$ which contradicts $u \in a^{\perp}$. Consequently a is an eigenvector associated with the unique positive eigenvalue λ and hence f is of the form in (13).
$\Longleftarrow$ It results $\nabla f(x) = \lambda\left(\frac{3}{2}\left(a^T x\right)^2 - \left(a^T x\right) a_0\right) a$, $H(x) = \lambda\left(3a^T x - a_0\right) a a^T$. Since $a_0 \geq 0$, then the critical points of f do not belong to S^* and so it remains to prove that for every $d \in (\nabla f(x))^{\perp}$ we get $d^T H(x) d \geq 0$. Since $d \in (\nabla f(x))^{\perp}$ if and only if $d \in a^{\perp}$ we get $d^T H(x) d = \lambda\left(3a^T x - a_0\right) d^T a a^T d = 0$ and the proof is complete.

Remark 1. It is worth noticing that f in (13) is also pseudoconcave on S^*; in fact the critical points do not belong to S^* and for every $d \in (\nabla f(x))^{\perp}$ we get $d^T H(x) d = 0$. Therefore, the second order characterization for the pseudoconcave function is verified and so f is pseudolinear.

Recalling that function f is the Charnes-Cooper transformed function of h, by means of the previous result we can characterize the pseudoconvexity of function h.

Theorem 7. *Consider function $h(x) = \frac{1}{2}\frac{x^T A x}{(b^T x + b_0)^3}$. h is pseudoconvex on $S = \{x \in \Re^n : b^T x + b_0 > 0\}$ if and only if h is of the following form*

$$h(x) = \frac{1}{2}\frac{\mu\left(b^T x\right)^2}{\left(b^T x + b_0\right)^3} \quad \textit{where } b_0 < 0.$$

References

1. Avriel M, Diewert WE, Schaible S, Zang I (1988) Generalized Concavity. Plenum Press, New York
2. Bajalinov EB (2004) Linear-Fractional Programming: Theory, Methods, Applications, and Software. Kluwer Academic Publishers, Boston MA
3. Barros AI (1998) Discrete and Fractional Programming Techniques for Location Models. Kluwer Academic Publishers, Dordrecht
4. Cambini A, Crouzeix JP, Martein L (2002) On the pseudoconvexity of a quadratic fractional function. Optimization 51:667-687
5. Cambini A, Martein L (1986) A modified version of Martos' algorithm. Methods of Operation Research 53:33-44
6. Cambini A, Martein L, Schaible S. (2005.a) On the pseudoconvexity of the sum of two linear fractional functions. In: Eberherd A, Hadjisavvas N, Luc DT (eds) Generalized convexity, generalized monotonicity and applications:161-172, Springer, New York NY
7. Cambini A, Martein L, Schaible S. (2005.b) Pseudoconvexity under the Charnes-Cooper transformation. Pacific Journal of Optimization 2:1-11
8. Cambini R, Carosi L (2002) On generalized convexity of quadratic fractional functions. Rendiconti del Circolo Matematico di Palermo Series II Suppl.70:155-176
9. Carosi L, Martein L (2004) On the pseudoconvexity and pseudolinearity of some classes of fractional functions. Report n. 247, Department of Statistics and Applied Mathematics, University of Pisa
10. Charnes A, Cooper WW (1962) Programming with linear fractional functionals. Naval Research Logistics Quarterly 9:181-186
11. Charnes A, Cooper WW, Rhodes E (1978) Measuring the efficiency of decision making units. European Journal of Operational Research 2:429-444.
12. Charnes A, Cooper WW, Lewin AY, Seiford LM (eds) (1995) Data Envelopment Analysis: Theory, Methodology and Applications. Kluwer Academic Publishers, Boston MA
13. Crouzeix JP (1998) Characterizations of Generalized Convexity and Monotonicity, a survey. In: Crouzeix JP, Martinez-Legaz JE, Volle M (eds) Generalized Convexity, Generalized Monotonicity:237-256. Kluwer Academic Publishers, London
14. Frenk JBG, Schaible S (2005) Fractional programming. In: Hadjisavvas N, Komlòsi S, Schaible S (eds) Handbook of generalized convexity and generalized monotonicity:335-386. Springer, New York
15. Mjelde KM (1978) Allocation of resources according to a fractional objective. European Journal of Operational Research 2:116-124
16. Schaible S (1995) Fractional Programming. In Horst R, PM Pardalos (eds) Handbook of Global Optimization:495-608. Kluwer Academic Publishers, Dordrecht