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Riccardo Cambini
Claudio Sodini

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Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 2216231 Segr. Stud. 050 2216317 Fax 050 2216375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

# A sequential method for a class of box constrained quadratic programming problems

Riccardo Cambini - Claudio Sodini *
*Dept. of Statistics and Applied Mathematics, University of Pisa*
*Via Cosimo Ridolfi 10, 56124 Pisa, ITALY*
*E-mail: cambric@ec.unipi.it, csodini@ec.unipi.it*

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**Abstract**

The aim of this paper is to propose a solution algorithm, based on the optimal level solutions method, which solves a particular class of box constrained quadratic problems. The objective function is given by the sum of a quadratic strictly convex separable function and the square of an affine function multiplied by a real parameter. The convexity and the nonconvexity of the problem can be characterized by means of the value of the real parameter. Within the algorithm, some global optimality conditions are used as stopping criteria, even in the case of a nonconvex objective function. The results of a deep computational test of the algorithm are also provided.

**Keywords** Quadratic programming, optimal level solutions, d.c. optimization.
**AMS - 2000 Math. Subj. Class.** 90C20, 90C26, 90C31.
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## 1 Introduction

The aim of this paper is to study, from both a theoretical and an algorithmic point of view, the following class of box constrained problems:

$$P : \begin{cases} \min f(x) = \frac{1}{2}x^T Dx + c^T x + \frac{1}{2}k\left(h^T x + h_0\right)^2 \\ x \in B = \{x \in \Re^n : l \leqq x \leqq u\} \end{cases}$$

where $D = \text{diag}(d_1, \ldots, d_n) \in \Re^{n\times n}$ is a positive definite diagonal matrix, $c, h, l, u \in \Re^n$, with $h \geq 0$, $h \neq 0$, and $k, h_0 \in \Re$.

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The interest in studying this class of problems is witnessed by some papers appeared in the literature and related to the particular case $k \geq 0$ (see [1, 5, 6]). It is worth noticing that problems of this kind are a variation of the tax programming model studied in [7] (see also [1]).

The aim of this paper is to deepen on the properties of these problems and to propose, in an unifying approach, a solution algorithm able to solve them for any fixed value of $k \in \Re$. It is worth noticing that for $k < 0$ small enough the quadratic objective function is not convex.

The provided solution algorithm is based on the optimal level solutions method [2, 3, 4]. The algorithm, in both the convex and the nonconvex case, stops after no more than $2n-1$ iterations and its complexity is $O(n^2)$ from a CPU time point of view. The solution algorithm has been fully implemented and tested from a computational point of view. Some theoretical optimality conditions are also studied and used as stopping criteria; this improved the algorithm performance so much that nonconvex problems are solved in almost the same number of iterations than convex ones.

In Section 2, the convexity of the objective function is characterized and some further useful properties are stated. In Section 3, the optimal level solutions method is applied to the problem and some theoretical optimality conditions, useful to implement stopping criteria, are stated. In section 4, a solution algorithm is described in details, pointing out its correctness, its finiteness and its complexity. The algorithm has been implemented in a symbolic calculus environment and the results of a deep computational test are provided and discussed. Finally, in Section 5, we propose some enhancements to the initialization process of the solution algorithm which may improve its performance. The modified version of the algorithm has been implemented too and the results of a computational test are given.

## 2 Preliminary results

The aim of this section is to state some preliminary properties of problem $P$ which will be useful in the rest of the paper.

As regards to the definition of problem $P$, let us first notice that the semipositiveness of vector $h$ is not a restrictive assumption since each negative component $h_i < 0$ of $h$ can be trivially converted in a positive one by means of the following change of variables:

$$\tilde{x}_i := -x_i, \quad \tilde{h}_i := -h_i, \quad \tilde{c}_i := -c_i, \quad \tilde{l}_i := -u_i, \quad \tilde{u}_i := -l_i$$

### 2.1 Convexity property

First of all, it is worth noticing that the convexity of the problem depends on the value of parameter $k$. With this aim, let us prove the following preliminary result.

**Theorem 2.1** *Let $Q \in \Re^{n\times n}$ be a symmetric positive definite matrix, let $k \in \Re$ and let $h \in \Re^n$. Then, the symmetric matrix $A = \left(Q + khh^T\right)$ is positive semidefinite if and only if*

$$k \geq -\frac{1}{h^T Q^{-1} h}$$

*Proof* For the canonical form of symmetric real matrices, there exists a real unitary matrix $U \in \Re^{n\times n}$ (such as $UU^T = U^TU = I$) and a positive definite diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n) \in \Re^{n\times n}$ such that $Q = UDU^T$, where $\lambda_1, \dots, \lambda_n$ are the $n$ positive eigenvalues of $Q$. For the sake of convenience, let us define the positive definite diagonal matrix $D_R = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) \in \Re^{n\times n}$. It is worth noticing that $D = D_R^2$ and that $Q^{-1} = UD_R^{-2}U^T$. Matrix $A$ is positive semidefinite if and only if

$$x^T\left(Q + khh^T\right)x \geq 0 \qquad \forall x \in \Re^n \tag{2.1}$$

Let us now define the vector $y = D_R U^T x$. Since $D_R$ and $U$ are nonsingular, (2.1) holds if and only if:

$$(UD_R^{-1}y)^T(UD_R^2U^T + khh^T)(UD_R^{-1}y) \geq 0 \qquad \forall y \in \Re^n \tag{2.2}$$

that is to say, since $D_R^{-1}$ is diagonal and $U$ is a real unitary matrix, that:

$$y^T(I + k(D_R^{-1}U^Th)(D_R^{-1}U^Th)^T)y \geq 0 \qquad \forall y \in \Re^n \tag{2.3}$$

Condition (2.3) holds if and only if the eigenvalues of

$$C = \left(I + k(D_R^{-1}U^Th)(D_R^{-1}U^Th)^T\right)$$

are nonnegative. First note that $1 + kh^TQ^{-1}h$ is an eigenvalue of $C$ with $D_R^{-1}U^Th$ corresponding eigenvector:

$$\begin{aligned}
C\left(D_R^{-1}U^Th\right) &= \left(I + k(D_R^{-1}U^Th)(D_R^{-1}U^Th)^T\right)\left(D_R^{-1}U^Th\right) \\
&= \left(1 + k(D_R^{-1}U^Th)^T(D_R^{-1}U^Th)\right)\left(D_R^{-1}U^Th\right) \\
&= \left(1 + kh^TQ^{-1}h\right)\left(D_R^{-1}U^Th\right)
\end{aligned}$$

Note also that $C$ has $n-1$ eigenvalues equal to 1, with eigenvectors belonging to the set $(D_R^{-1}U^Th)^\perp = \{v \in \Re^n : h^TUD_R^{-1}v = 0\}$:

$$Cv = v + k(D_R^{-1}U^Th)(h^TUD_R^{-1}v) = v \quad \forall v \in (D_R^{-1}U^Th)^\perp$$

As a consequence, all the eigenvalues of $C$ are nonnegative if and only if $1 + kh^TQ^{-1}h \geq 0$ and the result is proved since $Q^{-1}$ is positive definite. □

By means of this theorem we can characterize the convexity of function $f$ in problem $P$. With this aim, let us define the following negative value:

$$k_0 = -\frac{1}{h^T D^{-1} h} = -\frac{1}{\sum_{i=1}^n \frac{1}{d_i} h_i^2} \tag{2.4}$$

Function $f$ is convex if and only if its hessian matrix $D + khh^T$ is positive semidefinite, and for Theorem 2.1 this holds if and only if:

$$k \geq k_0 \tag{2.5}$$

## 2.2 Initialization remark

It is worth noticing that the zero components $h_i = 0$ of vector $h$ provide a separable variables subproblem which can be solved explicitly *a priori*.

**Theorem 2.2** *Let $x^* \in B$ be the optimal solution of problem $P$. Then, for all indices $i = 1, \dots, n$ such that $d_i = 0$ it is:*

$$x_i^* = \begin{cases} l_i & \text{if } -\frac{c_i}{d_i} \leq l_i \\ u_i & \text{if } -\frac{c_i}{d_i} \geq u_i \\ -\frac{c_i}{d_i} & \text{if } l_i < -\frac{c_i}{d_i} < u_i \end{cases}$$

*Proof* Let $i \in \{1, \dots, n\}$ be such that $d_i = 0$. Then, function $f$ can be rewritten as follows:

$$\begin{aligned} f(x) &= \left(\frac{1}{2} d_i x_i^2 + c_i x_i\right) + \\ &+ \sum_{j \neq i} \left(\frac{1}{2} d_j x_j^2 + c_j x_j\right) + \frac{1}{2} k \left(h_0 + \sum_{j \neq i} h_j x_j\right)^2 \end{aligned}$$

and hence:

$$\begin{aligned} \min_{x \in B} \{f(x)\} &= \min_{l_i \leq x_i \leq u_i} \left\{\frac{1}{2} d_i x_i^2 + c_i x_i\right\} + \\ &+ \min_{x \in B} \left\{\sum_{j \neq i} \left(\frac{1}{2} d_j x_j^2 + c_j x_j\right) + \frac{1}{2} k \left(h_0 + \sum_{j \neq i} h_j x_j\right)^2\right\} \end{aligned}$$

Notice that the derivative of $\frac{1}{2} d_i x_i^2 + c_i x_i$ is $d_i x_i + c_i$ which vanishes in $-\frac{c_i}{d_i}$. The result then follows since $\frac{1}{2} d_i x_i^2 + c_i x_i$ is a convex parabola. □

As a consequence, the variables $x_i$ such that $h_i = 0$ can be fixed to their optimal value in an initialization process. This can be trivially implemented by reducing the feasible region as follows:

- if $-\frac{c_i}{d_i} \leq l_i$ then set $u_i := l_i$ ,
- if $-\frac{c_i}{d_i} \geq u_i$ then set $l_i := u_i$ ,
- if $l_i < -\frac{c_i}{d_i} < u_i$ then set $l_i := -\frac{c_i}{d_i}$ and $u_i := -\frac{c_i}{d_i}$ .

# 3 The optimal level solutions approach

In this section we show how problem $P$ can be solved by means of the optimal level solutions approach (see [2, 3, 4]).

## 3.1 Brief description of the approach

For any $\xi \in \Re$ let us define the following strictly convex parametric subproblem, which is obtained just by adding the constraint $h^T x + h_0 = \xi$:

$$P_\xi : \begin{cases} \min f(x) = \frac{1}{2}x^T Dx + c^T x + \frac{1}{2}k\xi^2 \\ x \in B_\xi = \{x \in \Re^n : l \leqq x \leqq u, \ h^T x + h_0 = \xi\} \end{cases}$$

Note that the feasible region $B_\xi$ is no more given by box constraints.

The parameter $\xi$ is said to be a feasible level if the set $B_\xi$ is nonempty. Clearly, since the feasible region $B$ is box constrained and since $h \geq 0$, then the set of feasible levels is given by the interval $[\xi_{min}, \xi_{max}]$ where:

$$\xi_{min} = h^T l + h_0 \quad \text{and} \quad \xi_{max} = h^T u + h_0$$

An optimal solution of problem $P_\xi$ is called an optimal level solution. For any given $\xi \in \Re$, the optimal solution for problem $P_\xi$ can be computed by means of any solution algorithm for strictly convex quadratic problems.

For the sake of completeness, let us now briefly recall the optimal level solutions approach (see for example [2, 3, 4]). Obviously, the optimal solution of problem $P$ is also an optimal level solution and, in particular, it is the optimal level solution with the smallest value; the idea of this approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached.

Starting from an incumbent optimal level solution, this can be done by means of a sensitivity analysis on the parameter $\xi$, which allows us to move in the various steps through several optimal level solutions until the minimizer is found.

Let us recall that Subsection 2.2 shows how some variables can be *a priori* fixed. As a consequence, from now on we can suppose that the optimal values of the variables $x_i$ such that $h_i = 0$ have been already fixed in an initialization process (see Subsection 4.1), that is to say that in the rest of this section we can assume:

$$u_i = l_i \text{ for all } i = 1, \ldots, n \text{ such that } h_i = 0. \tag{3.1}$$

Notice that assumption (3.1) implies $B_{\xi_{min}} = \{l\}$ and $B_{\xi_{max}} = \{u\}$.

## 3.2 A study of the optimal level solutions

Let $x'$ be the optimal solution of problem $P_{\xi'}$, with $\xi' = h^T x' + h_0$, and let us define, for the sake of convenience, the following partition $L \cup U \cup N \cup Z$ of the set of indices $\{1, \dots, n\}$:

$$L = \{i: \; l_i = x'_i < u_i\} \quad , \quad N = \{i: \; l_i < x'_i < u_i\}$$
$$U = \{i: \; l_i < x'_i = u_i\} \quad , \quad Z = \{i: \; l_i = x'_i = u_i\}$$

Since $P_{\xi'}$ is a strictly convex problem, $x'$ is its unique optimal solution if and only if the following Karush-Kuhn-Tucker conditions hold ([1]):

$$\begin{cases} Dx' + c = \lambda h + \alpha - \beta & \\ h^T x' + h_0 = \xi', & \\ l \leqq x' \leqq u & \textit{feasibility} \\ \alpha \geqq 0, \; \beta \geqq 0, & \textit{optimality} \\ \alpha^T(x' - l) = 0, \; \beta^T(u - x') = 0 & \textit{complementarity} \\ \lambda \in \Re, \; \alpha, \beta \in \Re^n & \end{cases} \tag{3.2}$$

These Karush-Kuhn-Tucker conditions can be rewritten in the following componentwise way:

$$\begin{cases} \alpha_i = 0, \; \beta_i = 0, \; \lambda = \frac{1}{h_i}(d_i x'_i + c_i) & \forall i \in N \\ \beta_i = 0, \; \alpha_i = d_i l_i + c_i - \lambda h_i \geq 0 & \forall i \in L \\ \alpha_i = 0, \; \beta_i = \lambda h_i - d_i u_i - c_i \geq 0 & \forall i \in U \\ \alpha_i = \max\{0, d_i l_i + c_i - \lambda h_i\} \geq 0 & \forall i \in Z \\ \beta_i = \max\{0, \lambda h_i - d_i u_i - c_i\} \geq 0 & \forall i \in Z \\ h^T x + h_0 = \xi \; , \; l \leqq x \leqq u & \end{cases}$$

Noticing that assumption (3.1) implies $h_i > 0$ for all $i \in L \cup U \cup N$, we get:

$$\begin{cases} \lambda = \frac{1}{h_i}(d_i x'_i + c_i) & \forall i \in N \\ \lambda \leq \frac{1}{h_i}(d_i l_i + c_i) & \forall i \in L \\ \lambda \geq \frac{1}{h_i}(d_i u_i + c_i) & \forall i \in U \end{cases}$$

As a consequence, given the optimal level solution $x'$ for problem $P_{\xi'}$, the value of $\lambda'$ can be determined as follows:

$$\lambda' = \begin{cases} \frac{d_i x'_i + c_i}{h_i}, \text{ for any } i \in N & \text{if } N \neq \emptyset \\ \min_{i \in L} \left\{ \frac{d_i l_i + c_i}{h_i} \right\} & \text{if } N = \emptyset \text{ and } L \neq \emptyset \\ \max_{i \in U} \left\{ \frac{d_i u_i + c_i}{h_i} \right\} & \text{if } N = \emptyset \text{ and } L = \emptyset \end{cases} \tag{3.3}$$

[1] If $l \not< u$, that is $l_i = u_i$ for some indices $i$, the Karush-Kuhn-Tucker conditions are sufficient but not necessary since no constraint qualification conditions are verified. These indices will be handled implicitly in the rest of the paper by properly choosing the values of the multipliers.

Then, $\alpha'$ and $\beta'$ can be obtained as described below:

$$\alpha'_i = \begin{cases} 0 & \forall i \in N \cup U \\ d_i l_i + c_i - \lambda' h_i & \forall i \in L \\ \max\{0, d_i l_i + c_i - \lambda' h_i\} & \forall i \in Z \end{cases} \tag{3.4}$$

$$\beta'_i = \begin{cases} 0 & \forall i \in L \cup N \\ \lambda' h_i - d_i u_i - c_i & \forall i \in U \\ \max\{0, \lambda' h_i - d_i u_i - c_i\} & \forall i \in Z \end{cases} \tag{3.5}$$

## 3.3 Sensitivity analysis

In the light of the optimal level solutions parametrical approach we now have to study the solution of problem $P_{\xi'+\theta}$, with $\theta > 0$. Since the Karush-Kuhn-Tucker system is linear whenever the complementarity conditions are implicitly handled, then the solution of the optimality conditions regarding to $P_{\xi'+\theta}$ is of the kind:

$$x'(\theta) = x' + \theta\Delta_x \ , \ \lambda'(\theta) = \lambda' + \theta\Delta_\lambda \tag{3.6}$$

$$\alpha'(\theta) = \alpha' + \theta\Delta_\alpha \ , \ \beta'(\theta) = \beta' + \theta\Delta_\beta \tag{3.7}$$

Since $x'$, $\lambda'$, $\alpha'$ and $\beta'$ are known, we are left to compute $\Delta_x, \Delta_\lambda, \Delta_\alpha, \Delta_\beta$. From (3.6) and (3.7) we obtain the following Karush-Kuhn-Tucker conditions corresponding to $P_{\xi'+\theta}$:

$$\begin{cases} D(x' + \theta\Delta_x) + c = (\lambda' + \theta\Delta_\lambda)h + (\alpha' + \theta\Delta_\alpha) - (\beta' + \theta\Delta_\beta) \\ h^T(x' + \theta\Delta_x) + h_0 = \xi' + \theta \\ l \leqq x' + \theta\Delta_x \leqq u \\ \alpha' + \theta\Delta_\alpha \geqq 0, \ \beta' + \theta\Delta_\beta \geqq 0 \\ (\alpha' + \theta\Delta_\alpha)^T(x' + \theta\Delta_x - l) = 0, \ (\beta' + \theta\Delta_\beta)^T(u - x' - \theta\Delta_x) = 0 \end{cases} \tag{3.8}$$

This system can be simplified as described in the following lemma.

**Lemma 3.1** *Let $(\lambda', \alpha', \beta')$ be a solution of (3.2) and let $\theta$ be any value in the interval $(0, \epsilon)$, $\epsilon > 0$ small enough. Then, system (3.8) is equivalent to:*

$$\begin{cases} D\Delta_x = \Delta_\lambda h + \Delta_\alpha - \Delta_\beta \\ h^T\Delta_x = 1 \\ l \leqq x' + \theta\Delta_x \leqq u \\ \alpha' + \theta\Delta_\alpha \geqq 0, \ \beta' + \theta\Delta_\beta \geqq 0 \\ \Delta_{x_i} = 0 \ \forall i \in Z \\ \Delta_{\alpha_i} = 0 \ \forall i \in N \cup U \\ \Delta_{\beta_i} = 0 \ \forall i \in L \cup N \\ \alpha'_i\Delta_{x_i} = 0 \ , \ \Delta_{\alpha_i}\Delta_{x_i} = 0 \quad \forall i \in L \\ \beta'_i\Delta_{x_i} = 0 \ , \ \Delta_{\beta_i}\Delta_{x_i} = 0 \quad \forall i \in U \end{cases} \tag{3.9}$$

*Proof* The first and the second equations follow directly from (3.2) taking into account that $\theta \neq 0$, while $\Delta_{x_i} = 0 \ \forall i \in Z$ follows directly from the definition of $Z$. From (3.2) we have also that the complementarity conditions of (3.8) can be rewritten as:

$$\Delta_{\alpha_i}(x'_i - l_i) + \alpha'_i\Delta_{x_i} + \theta\Delta_{\alpha_i}\Delta_{x_i} = 0 \quad \forall i = 1,\dots,n$$
$$\Delta_{\beta_i}(u_i - x'_i) - \beta'_i\Delta_{x_i} - \theta\Delta_{\beta_i}\Delta_{x_i} = 0 \quad \forall i = 1,\dots,n$$

These conditions hold for all $\theta \in (0,\epsilon)$ if and only if for all $i = 1,\dots,n$:

$$\Delta_{\alpha_i}\Delta_{x_i} = 0 \quad , \quad \Delta_{\beta_i}\Delta_{x_i} = 0 \tag{3.10}$$
$$\Delta_{\alpha_i}(x'_i - l_i) + \alpha'_i\Delta_{x_i} = 0 \quad , \quad \Delta_{\beta_i}(u_i - x'_i) - \beta'_i\Delta_{x_i} = 0 \tag{3.11}$$

Noticing that $x'_i + \theta\Delta_{x_i} < u_i$ for all $i \in L \cup N$ and for $\theta > 0$ small enough, from the complementarity conditions $(\beta'_i + \theta\Delta_{\beta_i})(u_i - x'_i - \theta\Delta_{x_i}) = 0$ it yields $\beta'_i + \theta\Delta_{\beta_i} = 0$; analogously, we also have $\alpha'_i + \theta\Delta_{\alpha_i} = 0$ for all $i \in U \cup N$. Since $\theta > 0$, for (3.4) and (3.5) these conditions imply:

$$\Delta_{\alpha_i} = 0 \; \forall i \in N \cup U \quad , \quad \Delta_{\beta_i} = 0 \; \forall i \in L \cup N$$

so that:

$$\Delta_{\alpha_i}(x'_i - l_i) = \Delta_{\beta_i}(u_i - x'_i) = 0 \quad \forall i = 1,\dots,n$$

and the result is proved. □

Let us point out that the first and the second equations of (3.9) and the positive definiteness of $D$ imply:

$$\Delta_\lambda = \Delta_x^T D \Delta_x > 0$$

We are now able to prove the following further preliminary lemma.

**Lemma 3.2** *The following properties hold for all indices $i = 1,\dots,n$:*

*i) it is $\Delta_\lambda = \frac{1}{\sum_{i:\Delta_{x_i}\neq 0} \frac{1}{d_i} h_i^2} > 0$;*

*ii) if $\Delta_{x_i} \neq 0$ then $h_i > 0$, $\Delta_{x_i} = \Delta_\lambda \frac{h_i}{d_i} > 0$ and $\Delta_{\alpha_i} = \Delta_{\beta_i} = 0$;*

*iii) if $\Delta_{x_i} = 0$ then $i \notin N$, that is to say that either $x'_i = l_i$ or $x'_i = u_i$;*

*iv) if $\Delta_{x_i} = 0$ and $i \in L$ then $\Delta_{\beta_i} = 0$ and $\Delta_{\alpha_i} = -\Delta_\lambda h_i < 0$;*

*v) if $\Delta_{x_i} = 0$ and $i \in U$ then $\Delta_{\alpha_i} = 0$ and $\Delta_{\beta_i} = \Delta_\lambda h_i > 0$,*

*vi) if $\Delta_{x_i} = 0$ and $i \in Z$ we can assume $\Delta_{\alpha_i} = 0$ and $\Delta_{\beta_i} = \Delta_\lambda h_i \geq 0$,*

*vii) it is $\Delta_{x_i} \neq 0$ if and only if either $i \in N$ or $i \in L$ with $\alpha'_i = 0$,*

*viii) it is $\Delta_x \geq 0$, $\Delta_\alpha \leq 0$, $\Delta_\beta \geq 0$.*

*Proof i),ii)* If $\Delta_{x_i} \neq 0$ then from the two last conditions in (3.9) we have that $\Delta_{\alpha_i} = \Delta_{\beta_i} = 0$, hence from the first equation of (3.9) it yields $\Delta_{x_i} = \Delta_\lambda \frac{h_i}{d_i}$;

this implies $h_i > 0$ and $\Delta_{x_i} > 0$ since $\Delta_\lambda > 0$, $d_i > 0$, $h_i \geq 0$ and $\Delta_{x_i} \neq 0$. From the second equation of (3.9) it then follows:

$$1 = h^T \Delta_x = \sum_{i=1,\ldots,n} h_i \Delta_{x_i} = \sum_{i:\Delta_{x_i} \neq 0} h_i \Delta_{x_i} = \Delta_\lambda \sum_{i:\Delta_{x_i} \neq 0} \frac{1}{d_i} h_i^2$$

and the results are proved.

*iii)* Let $\Delta_{x_i} = 0$; in the case $h_i = 0$ for assumption (3.1) it is $i \in Z$ and the result is proved. Let now be $h_i \neq 0$ and assume by contradiction that $i \in N$. From (3.9) it follows $\Delta_{\alpha_i} = \Delta_{\beta_i} = 0$ so that from the first equation of (3.9) we get $0 = \Delta_\lambda h_i > 0$ which is a contradiction.

*iv),v),vi)* Follow directly from (3.9).

*vii)* The necessity follows from (3.9) taking into account that for *ii)* it is $\Delta_{x_i} > 0$, so that $i \notin U \cup Z$. For the sufficiency assume by contradiction that $\Delta_{x_i} = 0$; from *iii)* it yields that $i \notin N$ so that it is $i \in L$ with $\alpha'_i = 0$; from *iv)* it then results $\Delta_{\alpha_i} < 0$ and since $\theta > 0$ this contradicts $\alpha'_i + \theta \Delta_{\alpha_i} \geq 0$.

*viii)* Follows directly from *ii),iv),v)* and *vi)*. □

The following important monotonicity properties follow straightforward from *viii)* of the previous Lemma 3.2:

i) since $\Delta_x \geq 0$ then $x'(\theta)$ is componentwise nondecreasing,

ii) since $\Delta_\alpha \leq 0$ then $\alpha'(\theta)$ is componentwise nonincreasing,

iii) since $\Delta_\beta \geq 0$ then $\beta'(\theta)$ is componentwise nondecreasing.

We are now able to fully determine the values of $\Delta_x, \Delta_\lambda, \Delta_\alpha, \Delta_\beta$. With this aim, the following further sets of indices are needed:

- $L^+ = \{i \in L : \alpha'_i > 0\}$,
- $L^0 = \{i \in L : \alpha'_i = 0\}$,
- $M = L^0 \cup N = \{i : x'_i \neq u_i \text{ and } \alpha'_i = 0\}$

thus obtaining the partition $L^+ \cup M \cup U \cup Z$ of the set of indices $\{1, \ldots, n\}$.

**Theorem 3.1** *For all indices $i = 1, \ldots, n$ it is:*

$$\Delta_\lambda = \frac{1}{\sum_{i \in M} \frac{1}{d_i} h_i^2} > 0 \tag{3.12}$$

$$\Delta_{x_i} = \begin{cases} 0 & \text{if } i \notin M \\ \Delta_\lambda \frac{h_i}{d_i} > 0 & \text{if } i \in M \end{cases} \tag{3.13}$$

$$\Delta_{\alpha_i} = \begin{cases} 0 & \text{if } i \notin L^+ \\ -\Delta_\lambda h_i < 0 & \text{if } i \in L^+ \end{cases} \tag{3.14}$$

$$\Delta_{\beta_i} = \begin{cases} 0 & \text{if } i \in L^+ \cup M \\ \Delta_\lambda h_i \geq 0 & \text{if } i \in U \cup Z \end{cases} \tag{3.15}$$

*Proof* Follows straightforward from Lemma 3.2 □

We are finally able to determine the values of $\theta$ which guarantee both the optimality and the feasibility of $x'(\theta)$. From the feasibility conditions $l \leq x' + \theta\Delta_x \leq u$ and (3.13) we have:

$$\theta \leq \hat{F} = \begin{cases} \min_{i \in M} \left\{ \frac{u_i - x'_i}{\Delta_{x_i}} \right\} & \text{if } M \neq \emptyset \\ 0 & \text{if } M = \emptyset \end{cases}$$

For the way $\lambda'$ is computed, it yields $M = \emptyset$ if and only if $x' = u$, so that $\hat{F} > 0$ if and only if $x' \neq u$. On the other hand, from the optimality conditions $\alpha' + \theta\Delta_\alpha \geq 0$ and (3.14) we have:

$$\theta \leq \hat{O} = \begin{cases} \min_{i \in L^+} \left\{ \frac{-\alpha'_i}{\Delta_{\alpha_i}} \right\} & \text{if } L^+ \neq \emptyset \\ +\infty & \text{if } L^+ = \emptyset \end{cases}$$

so that $\hat{O} > 0$. As a consequence, $x'(\theta)$ is an optimal level solution for all $\theta$ such that:

$$0 \leq \theta \leq \theta_m = \min\left\{\hat{F}, \hat{O}\right\}$$

Let us notice that $\theta_m > 0$ if and only if $x' \neq u$.

## 3.4 Some optimality results

Since $\Delta_\lambda = \Delta_x^T D \Delta_x > 0$, $\Delta_x^T \alpha' = \Delta_x^T \beta' = 0$ and $h^T \Delta_x = 1$, the following explicit formula for the function $z(\theta)$ can be stated:

$$\begin{aligned} z(\theta) &= \frac{1}{2} x'(\theta)^T D x'(\theta) + c^T x'(\theta) + \frac{1}{2} k \left(\xi' + \theta\right)^2 \\ &= \frac{1}{2}(\Delta_x^T D \Delta_x + k)\theta^2 + (\Delta_x^T (Dx' + c) + k\xi')\theta + f(x') \\ &= \frac{1}{2}(\Delta_\lambda + k)\theta^2 + (\lambda' + k\xi')\theta + f(x') \end{aligned} \quad (3.16)$$

As a consequence, we have that $\frac{dz}{d\theta}(\theta) = (\Delta_\lambda + k)\theta + (\lambda' + k\xi')$ and hence:

*If $\lambda' + k\xi' > 0$ [$\lambda' + k\xi' < 0$] then $z(\theta)$ is locally increasing [decreasing] at $\theta = 0$.*

Level optimality can be helpful also in studying local optimality, since a minimum point in a segment of optimal level solutions is a local minimizer of the problem. This fundamental property, together with (3.16), allows to prove the following conditions.

**Theorem 3.2** *Let $x'$ be an optimal solution of problem $P_{\xi'}$. The following properties hold:*

*i) if $\lambda' + k\xi' \geq 0$ and $\Delta_\lambda + k \geq 0$ then $x'$ is a minimizer for $f(x)$ on the region*

$$\left\{x \in B: \ \xi' \leq h^T x + h_0 \leq \xi' + \hat{O}\right\};$$

*ii) if $\lambda'+k\xi' < 0$, $\Delta_\lambda+k > 0$ and $\theta' = -\frac{\lambda'+k\xi'}{\Delta_\lambda+k} \leq \min\left\{\hat{F}, \hat{O}\right\}$ with $\theta' \neq \hat{O}$ then $x'(\theta') = x' + \theta'\Delta_x$ is a local minimizer for problem $P$.*

Some more optimality conditions, which will be very useful in stating efficient stopping criteria in the solution algorithm, can be obtained by analyzing the following parametric problem which differs from $P_\xi$ problem only in the absence of the box constraints:

$$\begin{cases} \min \ f(x) = \frac{1}{2}x^T Dx + c^T x + \frac{1}{2}k\xi^2 \\ \qquad h^T x + h_0 = \xi \end{cases} \tag{3.17}$$

For the sake of convenience, let us also define the following notations:

$$\xi_h = h_0 - h^T D^{-1} c \tag{3.18}$$

$$\xi_0 = \frac{-k_0 \xi_h}{k - k_0} = \xi_h - \frac{k\xi_h}{k - k_0} \qquad \text{if } k \neq k_0 \tag{3.19}$$

**Lemma 3.3** *The quadratic problem (3.17) attains its minimum at*

$$x(\xi) = -k_0(\xi - \xi_h)D^{-1}h - D^{-1}c$$

*with minimum value*

$$\phi(\xi) = \frac{1}{2}\xi^2 (k - k_0) + \xi k_0 \xi_h - \frac{1}{2}k_0\xi_h^2 - \frac{1}{2}c^T D^{-1} c.$$

*Proof* The minimum point of the strictly convex problem (3.17) verifies the following necessary and sufficient optimality condition:

$$\begin{cases} Dx + c = \lambda h \\ h^T x + h_0 = \xi \end{cases}$$

Hence $x(\xi) = \lambda(\xi)D^{-1}h - D^{-1}c$ and, by means of simple calculations, it is:

$$\begin{aligned} \lambda(\xi) &= -k_0(\xi - \xi_h) \\ x(\xi) &= -k_0(\xi - \xi_h)D^{-1}h - D^{-1}c \\ \phi(\xi) &= \frac{1}{2}x(\xi)^T Dx(\xi) + c^T x(\xi) + \frac{1}{2}k\xi^2 = \\ &= \frac{1}{2}\lambda(\xi)^2 h^T D^{-1} h - \frac{1}{2}c^T D^{-1} c + \frac{1}{2}k\xi^2 = \\ &= \frac{1}{2}\xi^2 (k - k_0) + \xi k_0 \xi_h - \frac{1}{2}k_0 \xi_h^2 - \frac{1}{2}c^T D^{-1} c. \end{aligned}$$

□

It is worth noticing that, since $k_0 < 0$ and $h \geq 0$, the optimal solutions $x(\xi)$ of problems (3.17) are componentwise either strictly monotone or constant with respect to the level $\xi$.

The study of the minima line $x(\xi)$ allows us to propose the following stopping criteria which will greatly improve the performance of the solution algorithm in the case of nonconvex problems. With this aim, notice that the first derivative of $\phi(\xi)$ is:

$$\phi'(\xi) = \xi\,(k - k_0) + k_0\xi_h$$

Notice also that in the case $k \neq k_0$ it is $\phi'(\xi) = (k - k_0)\,(\xi - \xi_0)$ so that $\phi'(\xi) = 0$ if and only if $\xi = \xi_0$.

**Theorem 3.3** *Consider problem $P$ and let $x^* \in B$ and $\xi' \in [\xi_{min}, \xi_{max}]$ be such that $f(x^*) \leq \phi(\xi')$. If one of the following conditions holds:*

*i) $k \leq k_0$ and $\phi(\xi_{max}) \geq f(x^*)$,*

*ii) $k > k_0$ and $\phi'(\xi') = \xi'\,(k - k_0) + k_0\xi_h \geq 0$,*

*then $f(x^*) \leq f(x)$ $\forall x \in B$ such that $h^T x + h_0 \geq \xi'$.*

*Proof* *i)* Noticing that function $\phi(\xi)$ is concave for $k \leq k_0$ we get

$$\phi(\xi) \geq \min\{\phi(\xi'), \phi(\xi_{max})\} \geq f(x^*) \quad \forall \xi \in [\xi', \xi_{max}]$$

hence the result is proved since $f(x) \geq \phi(h^T x + h_0)$ $\forall x \in B$.

*ii)* Noticing that $\phi'(\xi') \geq 0$ and that function $\phi(\xi)$ is strictly convex for $k > k_0$ we get

$$\phi(\xi) \geq \phi(\xi') \geq f(x^*) \quad \forall \xi \geq \xi'$$

and the result is proved. □

# 4 Solution algorithm

In order to find a global minimum it is necessary to solve $P_\xi$, either explicitly or implicitly, for all the feasible levels $\xi$ (obviously, if $f(x)$ is convex we can stop as soon as a local minimizer is reached). In this subsection we will show that this can be done in a finite number of iterations, by using the results stated so far.

## 4.1 Initialization steps

The algorithm starts from the minimum level and then scans all the greater ones, either explicitly or implicitly, looking for the optimal solution.

**Initialization**

1) For all $i$ such that $h_i = 0$ and $l_i < u_i$ do
    $\eta := -\frac{c_i}{d_i}$;
    if $\eta \leq l_i$ then $u_i := l_i$
        else if $\eta \geq u_i$ then $l_i := u_i$
            else $l_i := \eta$ and $u_i := \eta$
            end if
    end if
  end do

2) Compute the values $\xi_{min} := h^T l + h_0$ and $\xi_{max} := h^T u + h_0$;

3) Determine the starting optimal level solution and the starting incumbent optimal (level) solution $x^*$, that is:

$$\text{if } f(u) < f(l) \text{ then } x^* := u \text{ else } x^* := l \text{ end if}$$
$$\xi' := \xi_{min}; \quad x' := l; \quad UB := f(x^*)$$

□

The optimal solution can now be searched iteratively by means of the following algorithms.

## 4.2 Convex case $k \geq k_0$

First note that when $f(x)$ is a convex function, that is to say that $k \geq k_0$, then $\Delta_\lambda + k \geq \Delta_\lambda + k_0 \geq 0$.

**Algorithm-CX**

1) local:= $false$;

2) While not local and $\xi' < \xi_{max}$ do

    2a) With respect to $\xi'$ and $x'$ determine $\lambda'$, $\alpha'$, $\beta'$, $\Delta_x$, $\Delta_\lambda$, $\Delta_\alpha$, $\Delta_\beta$, $\hat{F}$, $\hat{O}$; $\theta_m := \min\{\hat{F}, \hat{O}\}$;

    2b) If $\lambda' + k\xi' \geq 0$ then local:= $true$
        else if $\Delta_\lambda + k = 0$ then $\xi' := \xi' + \theta_m$; $x' := x' + \theta_m \Delta_x$; $\overline{x} := x'$;
            else begin
            - Let $\theta_1 := -(\lambda' + k\xi')/(\Delta_\lambda + k)$;

- If $\theta_1 < \theta_m$ then $\overline{x} := x' + \theta_1 \Delta_x$; local:= *true*;
else $\xi' := \xi' + \theta_m$; $x' := x' + \theta_m \Delta_x$; $\overline{x} := x'$;
- If $f(\overline{x}) < UB$ then $x^* := \overline{x}$ and $UB := f(\overline{x})$;

end

3) $x^*$ is the optimal solution for problem $P$.

□

This procedure looks for the minimizer by visiting segments of optimal level solutions where the objective function is decreasing. The procedure stops when the region has been fully scanned or a local minimizer (which is also global for the convexity of the objective function) is found. Note that the stopping criterion *ii)* of Theorem 3.3 is implicitly implemented in Algorithm-CX by means of the use of the flag "local".

## 4.3 Nonconvex case $k < k_0$

Let us notice that in this case function $f$ is nonconvex while the convexity of $z(\theta)$ depends on the value of $\Delta_\lambda + k$. As a consequence, all the feasible levels have to be examined (either explicitly or implicitly).

**Algorithm-NC**

1) *stop*:= $false$;

2) While not *stop* and $\xi' < \xi_{max}$ and $UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$ do

   2a) With respect to $\xi'$ and $x'$ determine $\lambda'$, $\alpha'$, $\beta'$, $\Delta_x$, $\Delta_\lambda$, $\Delta_\alpha$, $\Delta_\beta$, $\hat{F}$, $\hat{O}$; $\theta_m := \min\{\hat{F}, \hat{O}\}$;

   2b) Determine the best optimal level solution $\overline{x}$ for the levels $\xi \in [\xi', \xi' + \theta_m]$ and check the variable *stop*;

   2c) If $f(\overline{x}) < UB$ then $x^* := \overline{x}$ and $UB := f(\overline{x})$;

   2d) $\xi' := \xi' + \theta_m$ and $x' := x' + \theta_m \Delta_x$;

3) $x^*$ is the optimal solution for problem $P$.

□

Note that in all the iterations the variable $UB$ gives an upper bound for the optimal value with respect to the levels $\xi > \xi'$, while $x^*$ is the best optimal level solution with respect to the levels $\xi \leq \xi'$. Note also that the stopping criterion *i)* of Theorem 3.3 has been implemented in Algorithm-NC by means of the condition "$UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$" in line 2).

It remains to show how to implement step *2b)* in the previous procedure. With this aim, first note that for all $\theta \in [0, \hat{O}]$, the value $z(\theta)$ is a lower bound for the parametric problem $P_{\xi'+\theta}$; in fact if $\theta \leq \hat{F}$ then $x'(\theta)$ is an optimal level solution, otherwise (if $\theta > \hat{F}$) $x'(\theta)$ is unfeasible for $P_{\xi'+\theta}$ but is an optimal solution of a problem with the same objective function as $P_{\xi'+\theta}$ and a feasible region containing $B_{\xi'+\theta}$.

**Moving Steps 2b)**
One of the following exhaustive cases occurs:

1) ($\Delta_\lambda + k > 0$ and $\lambda' + k\xi' < 0$), that is $z(\theta)$ is strictly convex and locally decreasing at $\theta = 0$. Let $\theta_1 := -(\lambda' + k\xi')/(\Delta_\lambda + k)$; two subcases have to be considered:

   1a) $\theta_1 \leq \theta_m$: $\overline{x} := x' + \theta_1 \Delta_x$; if $\xi' + \hat{O} > \xi_{max}$ then $stop := true$;

   1b) $\theta_1 > \theta_m$: $\overline{x} := x' + \theta_m \Delta_x$;

2) ($\Delta_\lambda + k \geq 0$ and $\lambda' + k\xi' \geq 0$), that is $z(\theta)$ is convex and locally non-decreasing at $\theta = 0$. Then $\overline{x} := x'$; if $\xi' + \hat{O} > \xi_{max}$ then $stop := true$;

3) ($\Delta_\lambda + k = 0$ and $\lambda' + k\xi' < 0$) or ($\Delta_\lambda + k < 0$ and $\lambda' + k\xi' \leq 0$), that is $z(\theta)$ is concave and locally decreasing at $\theta = 0$. Then $\overline{x} := x' + \theta_m \Delta_x$;

4) ($\Delta_\lambda + k < 0$ and $\lambda' + k\xi' > 0$), that is $z(\theta)$ is strictly concave and locally increasing at $\theta = 0$. Let $\theta_r$ be the positive root of the second order equation $z(\theta) = UB$, that is:

$$\theta_r = \frac{-(\lambda' + k\xi') - \sqrt{(\lambda' + k\xi')^2 - 2(\Delta_\lambda + k)(f(x') - UB)}}{(\Delta_\lambda + k)};$$

   three subcases have to be considered:

   4a) $\hat{O} \leq \theta_r$: $\overline{x} := x'$; if $\xi' + \hat{O} > \xi_{max}$ then $stop := true$;

   4b) $\hat{F} \leq \theta_r < \hat{O}$: $\overline{x} := x'$; if $\xi' + \theta_r > \xi_{max}$ then $stop := true$;

   4c) $\theta_r < \theta_m$: $\overline{x} := x' + \theta_m \Delta_x$;

□

The previously described Moving Steps 2b) use some global optimality conditions which allow us to implicitly examine some of the feasible levels thus reducing the number of iterations needed to solve the problem.

## 4.4 Correctness and finiteness

The correctness of the proposed procedures follows just noticing that all the optimal level solutions are examined, either explicitly or implicitly.

As regards to the finiteness of the procedures, it can be proved that the optimal solution is found in no more that $2n - 1$ iterations. With this aim, let us determine the maximum number of iterations that the algorithm needs in order to reach the maximum feasible level $\xi_{max}$ starting from a certain optimal level solution $x'$. Let $L$, $U$, $N$, $Z$, be the sets of indices corresponding to $x'$ and computed as described in Subsection 3.1; let also $n_L$, $n_U$, $n_N$ and $n_Z$ be the number of indices in $L$, $U$, $N$ and $Z$, respectively.

First notice that in every iteration of the while cycle at least one of the two following situations occurs:

- at least one of the variables $x_i$, with $i \in L$, has a corresponding multiplier $\alpha_i > 0$ which is reduced to zero;
- at least one of the variables $x_i$, with $i \in L \cup N$, is incremented up to its upper bound.

Notice also that no more than $n_L + n_N$ variables $x_i$ in $L \cup N$ have to be moved to their upper bound $u_i$, while no more than $n_L - 1$ variables $x_i$ in $L$ have a corresponding positive multiplier to be reduced to zero. As a consequence, since the optimal level solutions $x'(\theta)$ are componentwise nondecreasing (as it has been pointed out by Theorem 3.1), the maximum number of iterations needed to complete the algorithm results to be:

$$(2n_L - 1) + n_N$$

In this light, we can say that the variables $x_i$ in $L \cup N$ are "active", while the variables in $U$ and $Z$ do not affect the complexity of the algorithm with respect to the number of iterations. As a conclusion, considering the starting case $x' = l$ (hence $N = \emptyset$) we can say that the maximum number of iterations needed to solve the problem is $2n_L - 1$; in particular, the maximum number of iterations is $2n - 1$ in the worst case $l < u$.

In other words, we have just pointed out that the problem is solved in $O(n)$ iterations. With respect to the time spent to find the optimal solutions, in each of the $O(n)$ iterations a linear number of additions and multiplications are needed to compute the required scalar products, hence the total number of additions and multiplications is $O(n^2)$ and this provides a quadratic complexity with respect to the CPU time spent by the algorithm.

### 4.5 Computational results

The proposed solution algorithm has been fully implemented by means of a symbolic calculus software (Maple$^{TM}$ in an Apple$^{TM}$ MacOSX environment).

Both the average number of iterations and the average number of seconds needed to obtain the optimal solution have been evaluated.

The following different implementations and classes of problems have been tested:

*C*) convex problems solved with *Algorithm-CX*,

| n | Num Prob | Average number of iterations | | | Average CPU time (seconds) | | |
|---|---|---|---|---|---|---|---|
| | | $C$ | $N_1$ | $N_2$ | $C$ | $N_1$ | $N_2$ |
| 2 | 100000 | 1.8016 | 2.2074 | 1.8326 | 0.0047254 | 0.0067800 | 0.0062559 |
| 3 | 100000 | 2.7112 | 3.6621 | 2.8560 | 0.0069679 | 0.010932 | 0.0095064 |
| 4 | 100000 | 3.6714 | 5.2691 | 3.9333 | 0.0096195 | 0.016146 | 0.013296 |
| 5 | 100000 | 4.6655 | 6.9418 | 5.0104 | 0.012718 | 0.022302 | 0.017594 |
| 6 | 100000 | 5.6472 | 8.6696 | 6.1137 | 0.016332 | 0.029338 | 0.022542 |
| 7 | 100000 | 6.6675 | 10.426 | 7.1884 | 0.020400 | 0.037329 | 0.027871 |
| 8 | 100000 | 7.6653 | 12.211 | 8.2448 | 0.024888 | 0.046460 | 0.033982 |
| 9 | 100000 | 8.6596 | 14.012 | 9.3038 | 0.030113 | 0.056750 | 0.040885 |
| 10 | 100000 | 9.6438 | 15.831 | 10.387 | 0.035491 | 0.067485 | 0.047785 |
| 15 | 50000 | 14.593 | 25.248 | 15.616 | 0.070148 | 0.13737 | 0.092317 |
| 20 | 50000 | 19.554 | 35.135 | 20.907 | 0.11804 | 0.23520 | 0.15253 |
| 25 | 50000 | 24.501 | 45.229 | 26.030 | 0.17769 | 0.35771 | 0.22666 |
| 35 | 25000 | 34.426 | 65.706 | 36.427 | 0.33151 | 0.67896 | 0.41625 |
| 50 | 25000 | 49.314 | 96.366 | 51.756 | 0.65825 | 1.3607 | 0.80886 |

Table 1: Computational results

$N_1$) nonconvex problems solved with *Algorithm-NC* without the use of the stopping criteria "$UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$" in line *2)*,

$N_2$) nonconvex problems solved with *Algorithm-NC* and with the use of the stopping criteria "$UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$" in line *2)*,

The obtained results are summarized in Table 1 and graphically represented in Figure 1. In particular, the first column provides the dimension of the solved problems (number of variables), the second column provides the number of different randomly created problems solved in each of the tested implementations.

The obtained results confirm the $O(n)$ theoretical number of iterations and the $O(n^2)$ theoretical number of seconds needed to find the optimal solution. It is also worth noticing that:

- convex problems are solved in an average number of iterations very close to the number of the variables, pointing out the enhancement given to the performance of the solution algorithm by the use of the "local minimum" stopping criterion
- nonconvex problems without the use of the "$UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$" stopping criterion are solved in an average number of iterations very close to the upper limit $2n - 1$, pointing out the inherent difficulties of nonconvex problems,
- nonconvex problems using the "$UB > \min\{\phi(\xi'), \phi(\xi_{max})\}$" stopping criterion are solved in an average number of iterations very close to the number of the variables (just like convex problems), pointing out

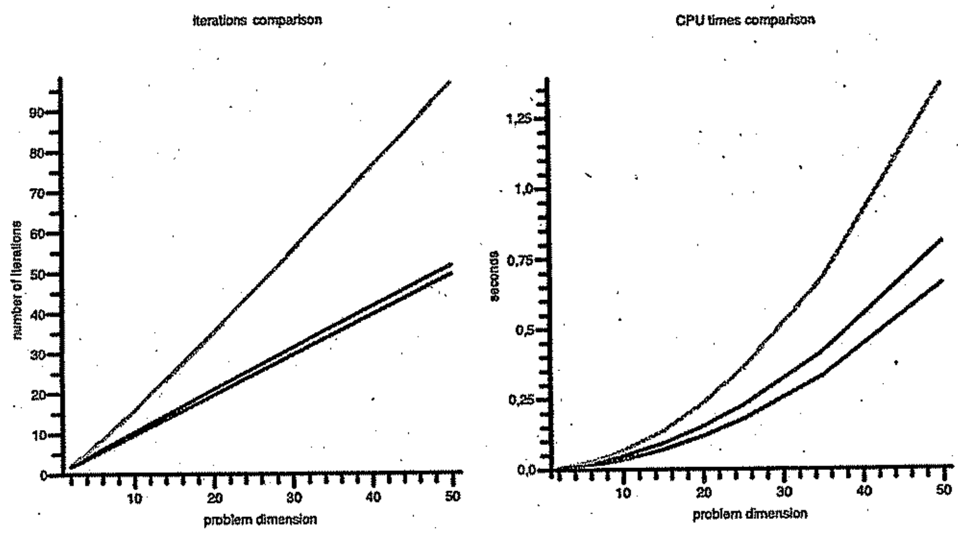


Figure 1: Performance representation

the great enhancement given by such a stopping criterion to the performance of the solution algorithm.

For the sake of completeness, we finally provide some computational results related to problems having 100-300 variables. These results have been obtained by solving about 500 randomly created problems for each dimension. It is worth noticing that these further results confirm the previously described behaviour.

| n | Average number of iterations | | | Average CPU time (seconds) | | |
|---|---|---|---|---|---|---|
| | $C$ | $N_1$ | $N_2$ | $C$ | $N_1$ | $N_2$ |
| 100 | 98.620 | 196.45 | 113.61 | 2.5491 | 5.2490 | 3.3579 |
| 150 | 148.61 | 295.25 | 164.52 | 5.8038 | 11.803 | 7.2609 |
| 200 | 197.10 | 394.04 | 217.46 | 10.248 | 20.786 | 12.653 |
| 250 | 248.08 | 493.18 | 244.44 | 16.325 | 32.750 | 17.886 |
| 300 | 297.08 | 591.49 | 306.18 | 23.871 | 47.648 | 27.148 |

Table 2: Problems with 100-300 variables

## 5 Algorithm initialization enhancements

The aim of this section is to deepen on the study of the considered class of problems in order to state some improvements for the initialization process of the solution algorithm.

Let us recall that the maximum number of iterations needed to solve the algorithm is $2n_L - 1$. As a consequence, a feasible region reduction might improve the algorithm performance whenever the set of not active indices $U \cup Z$ is augmented, that is whenever at least one variable $x_i$ is fixed to its optimal value $x_i^*$ by setting $l_i = u_i = x_i^*$.

With this aim, two different kinds of results are proposed.

## 5.1 Nonempty intersection with the minima line

Let us first see which conditions guarantee the existence of a nonempty intersection between the minima line

$$x(\xi) = -k_0(\xi - \xi_h)D^{-1}h - D^{-1}c$$

and the feasible region $B$. Such an intersection, which provides a segment of optimal level solutions, exists if for all $i \in \{1, \dots, n\}$ it is:

$$l_i \leq -k_0(\xi - \xi_h)\frac{h_i}{d_i} - \frac{c_i}{d_i} \leq u_i$$

In other words, this intersection exists if both the two following conditions are verified:

$$l_i \leq -\frac{c_i}{d_i} \leq u_i \quad \forall i: \ h_i = 0 \tag{5.1}$$

$$-\frac{1}{k_0} \cdot \max_{i:\ h_i \neq 0} \left\{\frac{d_i l_i + c_i}{h_i}\right\} + \xi_h \leq \xi \leq -\frac{1}{k_0} \cdot \min_{i:\ h_i \neq 0} \left\{\frac{d_i u_i + c_i}{h_i}\right\} + \xi_h \tag{5.2}$$

As a consequence, by using the following notations:

$$\mu_1 = \max_{i:\ h_i \neq 0} \left\{\frac{d_i l_i + c_i}{h_i}\right\} \qquad \mu_2 = \min_{i:\ h_i \neq 0} \left\{\frac{d_i u_i + c_i}{h_i}\right\}$$

$$\xi_1 = -\frac{1}{k_0} \cdot \mu_1 + \xi_h \qquad \xi_2 = -\frac{1}{k_0} \cdot \mu_2 + \xi_h$$

we can say that the itersection exists (involving the feasible levels $\xi \in [\xi_1, \xi_2]$) when both $\mu_1 \leq \mu_2$ and condition (5.1) holds. Note finally that for Lemma 3.3 it is:

$$x(\xi_1) = \mu_1 D^{-1}h - D^{-1}c\,, \quad x(\xi_2) = \mu_2 D^{-1}h - D^{-1}c \tag{5.3}$$

and that if there exists at least one indices $i \in Z$ such that $h_i \neq 0$ then $\mu_1 \geq \mu_2$.

The existence of the intersection described so far suggests a reduction of the feasible region in the light of the following result.

**Theorem 5.1** *Let us consider problem P and let $\hat{\xi} \in [\xi_{min}, \xi_{max}]$ be a feasible level with $\hat{x}$ corresponding optimal level solution.*

*i) If $\hat{x}$ is the best incumbent solution for all $\xi \leq \hat{\xi}$ then*

$$\min_{l \leq x \leq u} \{f(x)\} = \min_{\hat{x} \leq x \leq u} \{f(x)\}$$

*ii) If $\hat{x}$ is the best incumbent solution for all $\xi \geq \hat{\xi}$ then*

$$\min_{l \leq x \leq u} \{f(x)\} = \min_{l \leq x \leq \hat{x}} \{f(x)\}$$

*Proof* Follows from the assumptions taking into account that for Theorem 3.1 the optimal level solutions $x'(\theta)$ are componentwise nondecreasing. □

In the case the intersection exists, the initialization process can then be improved by adding some optional ***Steps 1-bis)*** as described below. With this aim, some exhaustive subcases have to be considered. Let us notice that the use of the minima line guarantees in the next subcases that at least one variable is fixed to its optimal value whenever the bounds $l$ or $u$ are redefined.

***Subcase*** $k > k_0$
Noticing that $\phi(\xi)$ is a convex parabola with vertex in $\xi_0$, the following possibilities occur:

i) if $\xi_0 > \xi_2$ then $x(\xi_2)$ is the best incumbent solution for all $\xi \leq \xi_2$, hence we can reduce the feasible region as follows:

$$l := x(\xi_2)$$

ii) if $\xi_1 \leq \xi_0 \leq \xi_2$ then the problem is solved since the global unconstrained minimum $x(\xi_0)$ is feasible.

iii) if $\xi_0 < \xi_1$ then $x(\xi_1)$ is the best incumbent solution for all $\xi \geq \xi_1$ and the region can be reduced as follows:

$$u := x(\xi_1)$$

***Subcase*** $k = k_0$
Since $\phi(\xi)$ is a linear function which results to be decreasing if $\xi_h > 0$, increasing if $\xi_h < 0$, constant if $\xi_h = 0$, then the next possibilities occur:

i) if $\xi_h > 0$ then $x(\xi_2)$ is the best incumbent solution for all $\xi \leq \xi_2$, hence $B$ can be reduced as follows:

$$l := x(\xi_2)$$

ii) if $\xi_h = 0$ then the problem is solved since the global uncostrained minima $x(\xi)$ with $\xi \in [\xi_1, \xi_2]$ are feasible.

iii) if $\xi_h < 0$ then $x(\xi_1)$ is the best incumbent solution for all $\xi \geq \xi_1$ and $B$ can be reduced as follows:

$$u := x(\xi_1)$$

***Subcase*** $k < k_0$

Noticing that $\phi(\xi)$ is a concave parabola with vertex in $\xi_0$, the following possibilities occur:

i) if $\xi_0 \geq \frac{\xi_1+\xi_2}{2}$ then $x(\xi_1)$ is the best incumbent solution for all $\xi \in [\xi_1, 2\xi_0 - \xi_1]$. As a consequence, problems $P_\xi$ can be studied only for

$$\begin{cases} \xi \in [\xi_{min}, \xi_1] & \text{if } \xi_{max} \leq 2\xi_0 - \xi_1 \\ \xi \in [\xi_{min}, \xi_1] \cup [2\xi_0 - \xi_1, \xi_{max}] & \text{if } \xi_{max} > 2\xi_0 - \xi_1 \end{cases}$$

ii) if $\xi_0 < \frac{\xi_1+\xi_2}{2}$ then $x(\xi_2)$ is the best incumbent solution for all $\xi \in [2\xi_0 - \xi_2, \xi_2]$. As a consequence, problems $P_\xi$ can be studied only for

$$\begin{cases} \xi \in [\xi_2, \xi_{max}] & \text{if } \xi_{min} \geq 2\xi_0 - \xi_2 \\ \xi \in [\xi_{min}, 2\xi_0 - \xi_2] \cup [\xi_2, \xi_{max}] & \text{if } \xi_{min} < 2\xi_0 - \xi_2 \end{cases}$$

The following improvements can then be obtained:

- if $\xi_0 \geq \frac{\xi_1+\xi_2}{2}$ and $\xi_{max} \leq 2\xi_0 - \xi_1$ then $u := x(\xi_1)$
- if $\xi_0 < \frac{\xi_1+\xi_2}{2}$ and $\xi_{min} \geq 2\xi_0 - \xi_2$ then $l := x(\xi_2)$
- otherwise, in the solution subprocedure we can just take into account that whenever $\xi' \geq \min\{\xi_1, 2\xi_0 - \xi_2\}$ and $\xi' < \xi_2$ it is possible to skip directly to $\xi' := \xi_2$ with $x' := x(\xi_2)$.

**Remark 5.1** In order to avoid the need of solving problem $P_{(2\xi_0-\xi_1)}$, in the case $\xi_0 > \frac{\xi_1+\xi_2}{2}$ with $\xi_{max} > 2\xi_0 - \xi_1$ we suggest to skip to the level $\xi_2$, even if it is smaller than $2\xi_0 - \xi_1$, since the optimal level solution $x(\xi_2)$ is analytically known.

## 5.2 Componentwise monotonicity

A variable $x_i$ can be a priori fixed to its optimal value if the objective function is monotone on the whole region with respect to $x_i$ itself. With this aim, notice that for any variable $x_i$ it is:

$$\frac{\partial f}{\partial x_i}(x) = x_i d_i + c_i + k h_i \left(h^T x + h_0\right)$$

Hence, since $h \geq 0$ and $D$ is positive diagonal, in $B$ we have:

$$\gamma_{min_i} = \min_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\} = \begin{cases} l_i d_i + c_i + k h_i (h^T l + h_0) & \text{if} \quad k \geq 0 \\ l_i d_i + c_i + k h_i (h^T u + h_0) & \text{if} \quad k < 0 \end{cases} \quad (5.4)$$

and

$$\gamma_{max_i} = \max_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\} = \begin{cases} u_i d_i + c_i + k h_i (h^T u + h_0) & \text{if} \quad k \geq 0 \\ u_i d_i + c_i + k h_i (h^T l + h_0) & \text{if} \quad k < 0 \end{cases} \quad (5.5)$$

It is now clear that:

- if $\gamma_{min_i} = \min_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\} \geq 0$ then $f(x)$ is always increasing in the feasible region with respect to variable $x_i$ and hence the optimal value will be reached for the value $x_i^* = l_i$,
- if $\gamma_{max_i} = \max_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\} \leq 0$ then $f(x)$ is always decreasing in the feasible region with respect to variable $x_i$ and hence the optimal value will be reached for the value $x_i^* = u_i$.

Summarizing these results, we can say that the feasible region can be reduced, without affecting the optimal solutions of the problem, by simply applying the following optional ***Steps 1-ter)***.

**Box Reduction Steps 1-ter)**

$\xi_{min} := h^T l + h_0$; $\xi_{max} := h^T u + h_0$; *found* := *true*;
while *found* do
  *found* := *false*;
  for all $i$ such that $l_i < u_i$ while not *found* do
    compute $\gamma_{min_i}$ as in (5.4);
    if $\gamma_{min_i} \geq 0$
      then *found* := *true*; $\xi_{max} := \xi_{max} - h_i(u_i - l_i)$; $u_i := l_i$;
      else compute $\gamma_{max_i}$ as in (5.5);
        if $\gamma_{max_i} \leq 0$
          then *found* := *true*; $\xi_{min} := \xi_{min} + h_i(u_i - l_i)$; $l_i := u_i$;
        end if;
    end if;
  end do;
end do. □

**Remark 5.2** Note that at the end of the previous steps the following implication holds:

$$l_i < u_i \quad \Rightarrow \quad \min_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\} < 0 < \max_{x \in B} \left\{ \frac{\partial f}{\partial x_i}(x) \right\}$$

Note also that the external "while" cycle in the previously described steps produces at most $n$ iterations, so that the complexity of the previous procedure is $O(n^2)$.

| n | Average number of iterations | | | Average CPU time (seconds) | | |
|---|---|---|---|---|---|---|
| | $C$ | $N_1$ | $N_2$ | $C$ | $N_1$ | $N_2$ |
| 2 | 0.51440 | 0.69478 | 0.59354 | 0.0038832 | 0.0046186 | 0.0048749 |
| 3 | 1.3606 | 2.0509 | 1.5619 | 0.0059829 | 0.0085522 | 0.0080690 |
| 4 | 2.3962 | 3.6760 | 2.6840 | 0.0086411 | 0.013725 | 0.011956 |
| 5 | 3.4877 | 5.4163 | 3.8543 | 0.011885 | 0.019920 | 0.016503 |
| 6 | 4.5992 | 7.2032 | 5.0364 | 0.015773 | 0.027031 | 0.021676 |
| 7 | 5.7189 | 9.0349 | 6.1907 | 0.020064 | 0.035126 | 0.027227 |
| 8 | 6.7965 | 10.891 | 7.3232 | 0.024797 | 0.044467 | 0.033625 |
| 9 | 7.8705 | 12.766 | 8.4431 | 0.030175 | 0.054714 | 0.040593 |
| 10 | 8.9406 | 14.673 | 9.5865 | 0.035803 | 0.065696 | 0.047997 |
| 15 | 14.148 | 24.476 | 15.076 | 0.071410 | 0.13661 | 0.093246 |
| 20 | 19.249 | 34.633 | 20.557 | 0.12014 | 0.23627 | 0.15601 |
| 25 | 24.306 | 44.915 | 25.789 | 0.18148 | 0.36198 | 0.23204 |
| 35 | 34.279 | 65.463 | 36.210 | 0.33818 | 0.68659 | 0.42428 |
| 50 | 49.161 | 96.092 | 51.494 | 0.66815 | 1.3725 | 0.82021 |

Table 3: Performance with initialization enhancements

### 5.3 Computational remarks

The enhancements to the initialization process proposed in this section have been fully implemented and tested from a computational point of view. In particular, the same problems used in the computational test summarized in Table 1 have been solved with implementations based on the enhanced initialization. The obtained results are given in Table 3.

These results have been compared with the ones of Table 1 in order to verify the usefulness of the proposed enhancements. The obtained percentage improvements are provided in Table 4.

It is worth noticing that:

- with respect to the average number of iterations, the improvement is decreasing with respect to the number of variables; for example, for problems having 5 variables the number of iterations are reduced of a 22-25%, while for problems having 25 variables the number of iterations are reduced of less than 1%;
- with respect to the average CPU time needed to solve the problems, the improvement is decreasing with respect to the number of variables; in particular, for problems up to 8 variables the enhanced initialization reduces so much the number of iterations that the global CPU time is reduced too. On the other hand, for problems having more than 15-20 variables the improvements provided by the enhanced initialization result to be not sufficient to decrease the average CPU time needed to solve the problem; nevertheless, notice that less than 2.4% of CPU time is wasted in these cases.

| n | Average number of iterations | | | Average CPU time (seconds) | | |
|---|---|---|---|---|---|---|
| | $C$ | $N_1$ | $N_2$ | $C$ | $N_1$ | $N_2$ |
| 2 | 71.447 | 68.525 | 67.612 | 17.823 | 31.879 | 22.076 |
| 3 | 49.816 | 43.997 | 45.310 | 14.136 | 21.767 | 15.120 |
| 4 | 34.734 | 30.235 | 31.762 | 10.170 | 14.993 | 10.077 |
| 5 | 25.245 | 21.975 | 23.075 | 6.5478 | 10.677 | 6.2056 |
| 6 | 18.558 | 16.914 | 17.622 | 3.4182 | 7.8630 | 3.8439 |
| 7 | 14.227 | 13.339 | 13.879 | 1.6466 | 5.9025 | 2.3130 |
| 8 | 11.334 | 10.811 | 11.178 | 0.36773 | 4.2909 | 1.0498 |
| 9 | 9.1131 | 8.8941 | 9.2508 | -0.20595 | 3.5880 | 0.71434 |
| 10 | 7.2910 | 7.3196 | 7.7032 | -0.87834 | 2.6510 | -0.44382 |
| 15 | 3.0483 | 3.0588 | 3.4587 | -1.7985 | 0.55507 | -1.0068 |
| 20 | 1.5606 | 1.4282 | 1.6734 | -1.7854 | -0.45449 | -2.2815 |
| 25 | 0.79784 | 0.69380 | 0.92632 | -2.1322 | -1.1940 | -2.3721 |
| 35 | 0.42701 | 0.37092 | 0.59670 | -2.0108 | -1.1234 | -1.9295 |
| 50 | 0.30888 | 0.28367 | 0.50606 | -1.5041 | -0.86906 | -1.4027 |

Table 4: Average improvements (percentages)

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