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# An approach to discrete convexity and its use in an optimal fleet mix problem

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**Abstract**

A notion of convexity for discrete functions is first introduced, with the aim to guarantee both the increasing monotonicity of marginal increments and the convexity of the sum of convex functions. Global optimality of local minima is then studied both for single variable functions and for multi variables ones. Finally, a concrete optimal fleet mix problem is studied, pointing out its discrete convexity properties.

**Keywords** Discrete programming, discrete convexity, optimal fleet mix.
**AMS - 2000 Math. Subj. Class.** 90C10, 90C90, 90C31.
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## 1 Introduction

Concrete problems are often discrete, in the sense that the variables are defined over the set of integers. This happens, for instance, whenever the variables represent the number of units, such as workforce units, number of ambulances, number of vehicles, and so on.

Due to their importance in applications, discrete problems have been widely studied in the mathematical programming literature, especially from the algorithmic point of view. Some approaches to convexity properties of discrete functions have been proposed too (see for example [2, 3, 5]), pointing out the difficulty of this research field.

The aim of this paper is twofold. First, we propose an approach to the notion of convexity for discrete functions, with the aim to guarantee both the increasing monotonicity of marginal increments and the convexity of the sum of convex functions. Some properties of the defined class of functions are then studied, especially with respect to the global optimality of local minima. Then, a concrete problem of optimal fleet mix is analyzed. In particular, we consider a model involving both internal workforce units and external technicians; quality of service requirements and penalties for unfulfilled services are also considered.

The model is then studied from a theoretical point of view, pointing out that some of the variables can be parametrically fixed to their optimal value, thus obtaining a parametrical discrete convex objective function.

# 2 Discrete convex functions

Convexity property has been widely used in Mathematics and in Economics due to its usefulness in optimization problems (both critical points and local minima are global optimum points). As it is well known, such a concept regards to functions defined over convex sets. Unfortunately, many applicative problems arising in Operations Research and in Management Science deal with integer programming. As a consequence, some efforts have been done in the literature in order to determine a convexity concept suitable for discrete problems.

In this section, we aim to propose a new definition of discrete convexity by using an approach different from the ones already appeared in the literature. In particular, our aim is to guarantee two properties which results to be useful in Economics and in applicative problems, that are the increasing monotonicity of marginal increments and the discrete convexity of the sum of two discrete convex functions.

## 2.1 A brief overview

For the sake of completeness, let us now briefly recall some of the results already appeared in the literature.

Favati and Tardella in [2] introduced the concept of integer convexity extending a function $f$, defined over a discrete rectangle $X \subset Z^n$, to a piecewise-linear function $\overline{f}$ defined over the convex hull of $X$, denoted with $co(X) \subseteq \Re^n$.

**Definition 2.1** A set $X \subset Z^n$ is said to be a discrete rectangle if there exist $a, b \in Z^n$ such that:

$$X = \{x \in Z^n : a_i \leq x_i \leq b_i, \ i = 1, \ldots, n\}$$

Given a number $x \in \Re$ it is denoted with $N(x)$ the so called discrete neighborhood of $x$, defined as the set

$$N(x) = \{z \in Z^n : |x_i - z_i| < 1, i = 1, \ldots, n\}$$

**Definition 2.2** Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete rectangle. The so called *extension of* $f$ is the function $\overline{f} : co(X) \to \Re$ defined as follows:

$$\overline{f}(x) = \min\{\sum_{i=1}^{k} \alpha_i f(z^i) : z^i \in N(x), \sum_{i=1}^{k} \alpha_i z^i = x, \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0\},$$

where $k = card(N(x))$. Then, function $f$ is said to be *integrally convex* if its extension $\overline{f} : co(X) \to \Re$ is convex.

This discrete convexity property is not easy to be verified. In any case, the authors have been able to state some useful properties and a global optimality results, which deserve to be recalled for the sake of completeness.

**Proposition 2.1** *Let $f, g : X \to \Re$, where $X$ is a discrete rectangle, then*

$$\overline{f}(x) + \overline{g}(x) \leq \overline{(f+g)}(x), \ \forall x \in co(X) \tag{2.1}$$

*furthermore, if over any unit hypercube contained in $co(X)$ at least one of the functions $\overline{f}(x)$ and $\overline{g}(x)$ is linear, then*

$$\overline{f}(x) + \overline{g}(x) = \overline{(f+g)}(x), \ \forall x \in co(X) \tag{2.2}$$

**Proposition 2.2** *A point $x \in X$ is a local minimum point for $\overline{f}$ over $co(X)$ if and only if it is a local minimum point for $f$ over $X$.*

**Proposition 2.3** *Let $f$ be an integrally convex function on a discrete rectangle $X$. If $x$ is a local minimum point for $f$ over $X$, then $x$ is a global minimum point.*

Unfortunately, the class of integrally convex functions is not closed under addition (see Property 2.1). However, if $f$ and $g$ are integrally convex on $X$ and condition (2.2) holds, then $f + g$ is also integrally convex. This happens, for example, when $f$ and $g$ are submodular integrally convex functions.

A branch of the literature, then has concentrated its attention to this particular class of functions. Murota in [3] defines a concept of convexity for integer valued functions and investigates its relationship to submodularity. Yüceer in [5] establishes the equivalence of discrete convexity (in the sense of Yüceer) and increasing first forward differences of functions of a single variable.

**Definition 2.3** Let $S$ be a subspace of a discrete $n$-dimensional space. A function $f : S \to \Re$ is *discretely convex* (in the sense of Yüceer) if for all $x, y \in S$ and for all $\alpha \in (0, 1)$

$$\alpha f(x) + (1 - \alpha) f(y) \geq \min_{u \in N(z)} f(u)$$

where $N(z) = \{u \in S : \|u - z\| < 1\}, z = \alpha x + (1 - \alpha) y$ and $\|u\| = \max_{1 \leq i \leq n} \{|u_i|\}$

Then, Yüceer propose the concept of strong discrete convexity by imposing additional conditions on a discretely convex function such as submodularity.

## 2.2 A new approach

Let us now introduce a new notion of convexity for discrete functions by means of an approach not based neither on extended functions nor on submodular ones, hence different from the ones proposed in [2, 3, 5]. With this aim, let us first introduce the definition of discrete reticulum.

**Definition 2.4** Let $ret(x, y)$ be the set

$$ret(x,y) = \{z \in Z^n : \min\{x_i, y_i\} \leq z_i \leq \max\{x_i, y_i\}, \ i = 1, .., n\}$$

A set $X \subseteq Z^n$ is said to be a *discrete reticulum* if $ret(x, y) \subseteq X \ \forall x, y \in X$.

Obviously, any discrete rectangle is also a discrete reticulum; notice also that $Z^n_+$ is a discrete reticulum but not a discrete rectangle.

Let us now introduce the definition of discrete convex function. With this aim, from now on the infinite norm will be used, so that the norm of an $n$-dimensional vector $x$ will be denoted as follows:

$$\|x\| = \|x\|_\infty = \max_{i=1,\dots,n} |x_i|$$

As usual, if $\|x\| = 1$ then $x$ is said to be an unitary vector.

**Definition 2.5** Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum. Function $f$ is said to be a *discrete convex function* if for all $x \in X$, for all $v \in Z^n$, $\|v\| = 1$, such that $x + 2v \in X$, it is:

$$f(x + 2v) \geq 2f(x + v) - f(x) \tag{2.3}$$

Let us point out that any continuous convex function restricted over a discrete reticulum verifies the proposed definition.

**Remark 2.1** It is worth noticing that, by simply renaming the variables, if $x - 2v \in X$ then inequality (2.3) can be rewritten as:

$$f(x) \geq 2f(x - v) - f(x - 2v)$$

that is to say:

$$f(x - 2v) \geq 2f(x - v) - f(x)$$

In other words, if inequality (2.3) holds for a certain direction $v$ then it is necessarily verified also for the direction $-v$ (in the case $x - 2v \in X$ of course).

First of all, it is worth noticing that from Definition 2.5 it follows straightforward that the sum of two discrete convex functions is discrete convex too.

**Theorem 2.1** *Let $f, g : X \to \Re$, where $X$ is a discrete reticulum, be two discrete convex functions and let $\alpha \in \Re$, $\alpha > 0$. Then, $(f + g)(x)$ and $\alpha f(x)$ are discrete convex functions.*

Let us now prove the following characterization of discrete convex functions which points out that the proposed definition guarantees the increasing monotonicity of marginal increments.

**Theorem 2.2** *Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum. Function $f$ is a discrete convex function if and only if for all $x \in X$, for all $k, h \in Z$, with $h \geq 1$ and $k \geq h$, for all $v \in Z^n$, $\|v\| = 1$, such that $x + kv \in X$, it is:*

$$f(x + kv) - f(x + (k-1)v) \geq f(x + hv) - f(x + (h-1)v) \qquad (2.4)$$

*Proof* The sufficiency follows just assuming $h = 1$ and $k = 2$.

The necessity is proved by induction on $k$. Let $h \geq 1$; if $k = h$ the inequality is trivially verified. Let us now assume the inequality true for $k \geq h$ and let us verify it for $k + 1$. For the discrete convexity of $f$ it is:

$$\begin{aligned} f(x + (k+1)v) - f(x + kv) &\geq 2f(x + kv) - f(x + (k-1)v) - f(x + kv) \\ &= f(x + kv) - f(x + (k-1)v) \\ &\geq f(x + hv) - f(x + (h-1)v), \end{aligned}$$

so that the whole result is proved. □

The following further result will be useful in the next section in order to prove some global optimality conditions.

**Theorem 2.3** *Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum, be a discrete convex function. Then, for all $x \in X$, for all $k \in Z$, $k \geq 1$, for all $v \in Z^n$, $\|v\| = 1$, such that $x + kv \in X$, it is:*

$$f(x + kv) - f(x) \geq k\,[f(x + v) - f(x)] \qquad (2.5)$$

*Proof* By induction on $k$. For $k = 1$ the inequality is trivial; let us now suppose the inequality true for $k \geq 1$ and let us verify it for $k + 1$. From the induction assumption it yields:

$$\begin{aligned} f(x + (k+1)v) - f(x) &= [f(x + (k+1)v) - f(x + kv)] + [f(x + kv) - f(x)] \\ &\geq [f(x + (k+1)v) - f(x + kv)] + k\,[f(x + v) - f(x)] \end{aligned}$$

The result then follows noticing that for Theorem 2.2 it is:

$$f(x + (k+1)v) - f(x + kv) \geq f(x + v) - f(x)$$

The whole theorem is then proved. □

# 3 Local and global optimality

In this section we aim to study the global optimality properties of discrete convex functions; in particular we are going to deepen on the behaviour of local minima.

## 3.1 Definitions and preliminary results

For the sake of convenience, let us first introduce the following notations and definitions.

**Definition 3.1** Given a point $x \in Z^n$ the following sets are defined:

$$H(x) = \{y \in Z^n : \ y = x + v, \ v \in Z^n, \ \|v\| = 1\}$$
$$S(x) = \{y \in Z^n : \ y = x + kv, \ k \in Z, \ v \in Z^n, \ \|v\| = 1\}$$

The set $H(x)$ represents the surface of a sort of discrete unitary hypercube around point $x$, so that it may be intended as a sort of neighbourhood of $x$ itself; $S(x)$ is a discrete star shaped set centered in $x$ and generated by the discrete unitary directions. Obviously, it is $H(x) \subset S(x)$.

**Definition 3.2** Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum. A point $x \in X$ is said to be a local minimum if:

$$f(x) \leq f(y) \qquad \forall y \in X \cap H(x)$$

while it is said to be a global minimum if:

$$f(x) \leq f(y) \qquad \forall y \in X$$

The next preliminary result follows straightforward from Theorem 2.3.

**Corollary 3.1** *Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum, be a discrete convex function. If $x \in X$ is a local minimum then $f(x) \leq f(y)$ for all $y \in X \cap S(x)$.*

*Proof* The result follows from Theorem 2.3 noticing that the local optimality assumption implies that $f(x + v) - f(x) \geq 0$. □

## 3.2 Convexity and optimality in $Z$

It is worth focusing on the attention to single variable discrete functions, due to their usefulness in applicative problems. First of all, let us show that single variable discrete convex functions can be characterized with properties which result to be easier to be verified with respect of the general definition.

**Theorem 3.1** *Let $f : X \rightarrow \Re$, where $X \subset Z$ is a discrete reticulum. Function $f$ is discrete convex if and only if for all $x \in X$ such that $x + 2 \in X$, it is:*

$$f(x+2) \geq 2f(x+1) - f(x) \tag{3.1}$$

*Proof* The result follows directly from Definition 2.5 and Remark 2.1. □

**Corollary 3.2** *Let $f : X \rightarrow \Re$, where $X \subset Z$ is a discrete reticulum. Function $f$ is discrete convex if and only if for all $x, y \in X$ such that $y \geq x$, it is:*

$$f(y+1) - f(y) \geq f(x+1) - f(x)$$

*Proof* The sufficiency follows trivially assuming $y = x + 1$. The necessity follows from Theorem 2.2 by assuming $v = 1$ and $y = x + k$. □

Let us finally point out that for single variable functions the discrete convexity property guarantees the global optimality of local optima.

**Corollary 3.3** *Let $f : X \rightarrow \Re$, where $X \subset Z$ is a discrete reticulum, be a discrete convex function. If $x \in X$ is a local minimum then it is also a global one.*

*Proof* Follows directly from Corollary 3.1 since in the single variable case it is $S(x) = Z$. □

As a conclusion, it is worth noticing that in the case of single variable functions the proposed definition of discrete convexity verifies all the typical properties of continuous convexity, such as the increasing monotonicity of the marginal increments, the global optimality of local optima, the discrete convexity of the sum of discrete convex functions.

## 3.3 Convexity and optimality in $Z^n$, $n \geq 2$

Unlike the single variable case, when two or more discrete variables are involved then the discrete convexity of the function is not sufficient to guarantee the global optimality of a local optima. With this regard, it is worth noticing that Corollary 3.1 is not a complete global optimality result, since it states the global optimality of a local optimum only with respect to the set $X \cap S(x)$. This behaviour is pointed out in the next example.

**Example 3.1** Let us consider the following function defined over $X = Z^2$:

$$f(x_1, x_2) = (x_2 - 2x_1)^2 + \frac{1}{2}\left|x_2 + \frac{1}{2}x_1\right|$$

This is clearly a strictly convex function over $\Re^2$ and hence it is also discrete convex over $Z^2$. Point $x = (0,0)$ is the unique global minimum, but by means of simple calculations it can be seen that, for example, the points $(1,2)$, $(2,4)$, $(3,6)$, are local optima (with respect of Definition 3.2) but not global ones.

As a consequence, some additional regularity assumptions are required to extend the optimality range of a local optimum. A first tentative regularity assumption is proposed in the next definition.

**Definition 3.3** Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum. Let also be $W = \{w^{(1)}, \ldots, w^{(n)}\} \subset Z^n$ be a set of $n$ linearly independent unitary vectors. The following regularity condition is then defined:

(R1) for all $x \in X$, for all $i, j = 1, \ldots, n$, $i \neq j$, such that $x + w^{(i)} + w^{(j)} \in X$, it is $f(x + w^{(j)} + w^{(i)}) - f(x + w^{(j)}) \geq f(x + w^{(i)}) - f(x)$

In the case of discrete convex functions property (R1) can be characterized as follows.

**Theorem 3.2** *Let $f : X \to \Re$, where $X \subset Z^n$ is a discrete reticulum, be a discrete convex function. Let also be $W = \{w^{(1)}, \ldots, w^{(n)}\} \subset Z^n$ be a set of $n$ linearly independent unitary vectors. The regularity condition (R1) holds if and only if for all $x \in X$, for all $i = 1, \ldots, n$, for all $y \in Z^n \cap cone\{W\}$, such that $x + y + w^{(i)} \in X$, it is:*

$$f(x + y + w^{(i)}) - f(x + y) \geq f(x + w^{(i)}) - f(x)$$

*Proof* Let us first prove the result for $y = \beta w^{(j)}$, $\beta \in \aleph$; in other words let us first prove that:

$$f(x + \beta w^{(j)} + w^{(i)}) - f(x + \beta w^{(j)}) \geq f(x + w^{(i)}) - f(x) \qquad (3.2)$$

For $\beta = 0$ the result is trivial; let now be $\beta > 0$ and assume by induction that the inequality holds for $\beta - 1$. By applying the induction assumption and the regularity condition (R1) it yields:

$$\begin{aligned} f(x + \beta w^{(j)} + w^{(i)}) - f(x + \beta w^{(j)}) &= f(x + (\beta - 1)w^{(j)} + w^{(j)} + w^{(i)}) - f(x + \beta w^{(j)}) \\ &\geq f(x + (\beta - 1)w^{(j)} + w^{(j)}) + f(x + (\beta - 1)w^{(j)} + w^{(i)}) \\ &\quad - f(x + (\beta - 1)w^{(j)}) - f(x + \beta w^{(j)}) \\ &= f(x + (\beta - 1)w^{(j)} + w^{(i)}) - f(x + (\beta - 1)w^{(j)}) \\ &\geq f(x + w^{(i)}) - f(x) \end{aligned}$$

Let now $y$ be any vector in $W$; then, it can be expressed in the form

$$y = \sum_{j=1}^{n} \beta^{(j)} w^{(j)}$$

As a consequence, we have to prove that

$$f(x+\beta^{(1)}w^{(1)}+\cdots+\beta^{(n)}w^{(n)}+w^{(i)})-f(x+\beta^{(1)}w^{(1)}+\cdots+\beta^{(n)}w^{(n)}) \geq f(x+w^{(i)})-f(x)$$

The result follows directly by applying $n$ times, one for every component $\beta^{(j)}w^{(j)}$ of $y$, the preliminary result (3.2). □

The previous result allows us to improve the range of optimality of a local minimum.

**Theorem 3.3** *Let $f : X \rightarrow \Re$, where $X \subset Z^n$ is a discrete reticulum, be a discrete convex function. Assume also that the regularity condition (R1) holds. If $x \in X$ is a local minimum, then $x$ is a global minimum with respect to the sets $x + cone\{W\}$ and $x - cone\{W\}$.*

*Proof* Assume by contradiction that $x$ is not a global minimum with respect to $x + cone\{W\}$, that is to say that there exists $z \in X \cap (x + cone\{W\})$ such that $f(z) < f(x)$. It is now possible to construct a finite sequence of $k$ elements $\{z^{(j)}\} \in (x + cone\{W\}) \cap (z - cone\{W\})$ such that $z^{(0)} = x$, $z^{(k)} = z$ and $z^{(j+1)} - z^{(j)} \in W$ for all $j = 0, \ldots, k-1$. Since $f(z) < f(x)$ there exists $\bar{k} \in [0, k-1]$ such that $f(z^{(\bar{k})}) > f(z^{(\bar{k}+1)})$. Let us define $y = z^{(\bar{k})} - x$ and let $i$ be such that $w^{(i)} = z^{(\bar{k}+1)} - z^{(\bar{k})} \in W$; then we have $f(x+y) > f(x+y+w^{(i)})$ which implies, for Theorem 3.2, $f(x) > f(x+w^{(i)})$ which contradicts the local optimality of $x$.

Analogously, it can be proved that $x$ is a global minimum with respect to $x - cone\{W\}$. □

# 4 Convexity in an optimal fleet mix problem

Discrete optimization has many applications in everyday life and for this reason it has been widely studied in the literature.

This kind of problems are algorithmically difficult to be solved from a complexity point of view and are usually approached with integer programming techniques, branch and bound algorithms, local search, genetic algorithms.

In this section we aim to study a concrete optimal fleet mix problem, which is a discrete variables model related to the management of internal and external workforce units.

A theoretical study will points out that this problem can be solved with a polynomial complexity by means of a sort of parametrical approach. It will be also proved that this approach will provide a discrete convex parametrical objective function. This property allows to solve the problem very efficiently, that is with a very small CPU time, so that it could be used in a real time environment, such as in connection with real time routing problems.

## 4.1 Optimal fleet mix: an integer programming problem

This concrete problem is referred to routing of maintenance units (see for example [1, 4]). The firm employes internal and external technicians for repairing ATMs. Customers signal technical malfunctions to the call center. After the signalling the company has a contractual time window to repair the machine. If the time elapses the firm has to pay a penalty. Main targets are: to minimize call rates, repair time, travel time, and penalty costs. Call rates depend on product reliability, repair times on service diagnostic and service tools, while the travel time is dependent on transportation methods and environmental conditions. The first three aspects concern internal politics of renovating machines and personal training. The last one is the one we treat in this work.

We introduce a suitable objective function that takes into account both fixed and variable costs. The aim is to minimize this objective function subject to quality of service (QoS) constraints. Let us study the problem with respect to a particular geographic area and within a period of one year and let us denote by $I$ the set of days of the year. The variables represent the number of internal and external technicians to be employed. The input data are:

- the daily cost of the technicians
- the penalty costs
- the minimum service level the firm wants to guarantee.

First of all we examine the available historical series of calls for failures (without distinguishing among different types of failures) and we establish two benchmarks: the minimum and the maximum number of calls per day. From these parameters we can extrapolate the range of workforce necessary to reply to the failure calls. The graph in the appendix represents the need of technicians of a particular city in a week. Two measures appears in the graph: $M_i$ and $m_i$ are, respectively, the maximum number of calls that the firm's call center receives the day $i$ according to the data of the historical series and the minimum number.

These two values determine the unique constraint of the model; in fact, the total number of calls that an employee is able to fulfill can not be less than the minimum $m_i$ for each $i$, that is $\beta_x x \geq m_i \; \forall i = 1, \ldots, I$. Actually, in order to guarantee a sort of quality of service, the firm may want to guarantee a higher minimum level of calls fulfilled; this can be represented by means of a parameter $\rho \in [0, 1]$. In order to define more in detail the model structure, let us introduce the following definition.

**Definition 4.1** Let us consider the following data and parameters:
$M \in \aleph^I$ : *estimated maximum number of calls*
$m \in \aleph^I$ : *estimated minimum number of calls*
$I \in \aleph$ : *number of working days under consideration*
$x \in \aleph$ : *the number of employees of the firm*

$z \in \aleph^I$ : *number of external technicians employed at the days* $i = 1, ..., I$
$\beta_x \in \aleph$ : *number of calls fulfilled per single technician in a working day*
$p \in \Re_+$ : *daily cost of the single internal technician*
$c_w \in \Re_+$ : *penalty cost, proportional to the lack of technicians to repair the faults*
$c_z \in \Re_+$ : *cost per call of the external technician.*
$\rho \in [0,1]$ : *penalty coefficient.*
The optimization problem can be modelled as follows:

$$P : \begin{cases} \min f(x,z) \\ (x,z) \in S \end{cases}$$

where the cost objective function is:

$$f(x,z) = Ixp + c_z \sum_{i=1}^{I} z_i + c_w w(x,z) \tag{4.1}$$

and the number of not fulfilled calls is:

$$w(x,z) = \sum_{i=1}^{I} \max\{0; M_i - \beta_x x - z_i\} = \frac{1}{2} \sum_{i=1}^{I} (M_i - \beta_x x - z_i + |M_i - \beta_x x - z_i|)$$

while the feasible region is given by the following daily constraints:

$$S = \left\{ x \in \aleph, z \in \aleph^I \, | M_i - \rho(M_i - m_i) \le \beta_x x + z_i \quad \forall i = 1....I \right\} \tag{4.2}$$

External technicians are employed not every day. In the days during which the call center receives many calls, the firm can decide to employ an unlimited number of external technicians and it pays them for the whole day. On the other hand, if internal employees can cover all the demand peaks, $z_i$ will be equal to zero. This kind of mixed fleet is usually employed in firms with an high volatility of demand and a stochastic trend.

**Remark 4.1** Since $(x,z) \in S$, i.e. $M_i - \beta_x x - z_i \le \rho(M_i - m_i)$ $\forall i = 1, \ldots, I$, then $w(x,z) \le \rho \sum_{i=1}^{I}(M_i - m_i)$. In this light, $\rho \sum_{i=1}^{I}(M_i - m_i)$ is the maximum number of calls which might be left unfulfilled. Note also that, since the objective function has to be minimized, we can restrict the study of the problem to the following interval of variable $x$:

$$0 \le x \le \bar{M} = \max_{i=1..I} \{ \left\lceil \frac{M_i}{\beta_x} \right\rceil \}. \tag{4.3}$$

## 4.2 Fundamental properties of the problem

Problem $P$ is a discrete variable minimum problem, and can be solved with any of the known discrete programming algorithms. Clearly, due to the great number of variables ($I+1$ with $I$ equal to the number of working days in the year), the complexity of such algorithms could make impossible the use of this problem in real time environments.

Actually, deepening the study of the problem, we can state properties which will allow to solve it with just a linear complexity and a very small CPU time requirement. First of all, let us notice that the objective function of problem $P$ can be rewritten as

$$f(x,z) = Ipx + \sum_{i=1}^{I} \psi(x, z_i)$$

where for all $i = 1, \ldots, I$ it is:

$$\psi(x, z_i) = c_z z_i + c_w \max\{0; M_i - \beta_x x - z_i\}$$

In other words, the $z_i$ variables are independent one each other, so that whenever $x$ is considered as a parameter then problem $P$ can be solved separately with respect to each variable $z_i$. This suggest us to state the following result.

**Theorem 4.1** *Let us consider problem $P$ and assume $x$ to be a fixed parameter. For any $i \in \{1, ..., I\}$ the optimal solution of the following problem:*

$$\begin{cases} \min\ g(z_i) = c_z z_i + c_w \max\{0; M_i - \beta_x x - z_i\} \\ \quad z_i \geq M_i - \rho(M_i - m_i) - \beta_x x \end{cases}$$

*is given by:*

$$\hat{z}_i(x) = \begin{cases} \max\{0, M_i - \beta_x x\} & \text{if } c_z < c_w \\ \max\{0, M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor\} & \text{if } c_z \geq c_w \end{cases} \tag{4.4}$$

*Proof* *(Case $c_z \geq c_w$)* We just need to prove that $g(z_i)$ is monotone increasing for $z_i \geq 0$, that is to say that $g(z_i + 1) - g(z_i) \geq 0$ for all $z_i \geq 0$. Noticing that

$$g(z_i+1) - g(z_i) = c_z - c_w(\max\{0; M_i - \beta_x x - z_i\} - \max\{0; M_i - \beta_x x - z_i - 1\})$$

and taking into account that $M_i - \beta_x x - z_i$ is an integer value, it results:

$$g(z_i+1) - g(z_i) = \begin{cases} c_z & \text{if } M_i - \beta_x x - z_i \leq 0 \\ c_z - c_w & \text{if } M_i - \beta_x x - z_i \geq 1 \end{cases}$$

and the result is proved since $c_z \geq 0$ and $c_z \geq c_w$, taking into account that $\lceil M_i - \rho(M_i - m_i) - \beta_x x \rceil = M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor$.

*(Case $c_z < c_w$)* First notice that:

$$\max\{0; M_i - \beta_x x - z_i\} = \begin{cases} 0 & \text{if } z_i \geq \max\{0, M_i - \beta_x x\} \\ M_i - \beta_x x - z_i & \text{if } 0 \leq z_i < \max\{0, M_i - \beta_x x\} \end{cases}$$

so that it yields:

$$g(z_i) = \begin{cases} c_z z_i & \text{if } z_i \geq \max\{0, M_i - \beta_x x\} \\ z_i(c_z - c_w) + c_w(M_i - \beta_x x) & \text{if } 0 \leq z_i < \max\{0, M_i - \beta_x x\} \end{cases}$$

The result then follows since $c_z > 0$ and $c_z - c_w < 0$, taking into account that $\max\{0, M_i - \beta_x x\}$ is a feasible value. □

**Remark 4.2** It is worth pointing out an economic interpretation of the previously obtained results.

In the case $c_z \geq c_w$ the cost of an additional external technician is greater than the cost of the penalty. This means that, from the firm's point of view, it is better to pay the penalty than to employ an additional external technician; as a consequence the optimal value of $z_i$ corresponds to the lower value it can assume, that is $\max\{0, M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor\}$.

On the other hand, in the case $c_z < c_w$ it is better for the firm to avoid penalties fulfilling all of the daily calls. In this light the firm employs all the necessary external technicians, given by $\max\{0, M_i - \beta_x x\}$.

Theorem 4.1 and Remark 4.1 allow to rewrite problem $P$ as follows:

$$P : \begin{cases} \min \ \varphi(x) = f(x, \hat{z}(x)) \\ 0 \leq x \leq \bar{M} \end{cases} \tag{4.5}$$

where $\hat{z}(x) = (\hat{z}_1(x), \ldots, \hat{z}_I(x))$ as given in (4.4). Just notice also that

$$\varphi(x) = Ixp + c_z \sum_{i=1}^{I} \hat{z}_i(x) + c_w w(x, \hat{z}(x)) \tag{4.6}$$

$$= Ixp + \sum_{i=1}^{I} \psi(x, \hat{z}_i(x)) \tag{4.7}$$

and that in the case $c_z < c_w$ it is $w(x, \hat{z}(x)) = 0$ for all $x \in [0, \bar{M}]$, while in the case $c_z \geq c_w$ it is

$$\begin{aligned} w(x, \hat{z}(x)) &= \sum_{i=1}^{I} \max\{0; M_i - \beta_x x - \max\{0, M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor\}\} \\ &= \sum_{i=1}^{I} \max\{0; \min\{M_i - \beta_x x; \lfloor \rho(M_i - m_i) \rfloor\}\} \end{aligned}$$

As a conclusion, problem $P$ has become a single variable one and can be solved by simply comparing the values of $\varphi(x)$ for all $x \in [0, \bar{M}]$.

## 4.3 Discrete convexity of the objective function $\varphi(\mathbf{x})$

In the previous subsection we have shown that problem $P$ can be easily solved, from a mathematical point of view, with a single variable discrete problem.

In order to improve the use of this problem as part of a real time system, it is important to determine the optimal solution with a CPU time as small as possible.

In this light, we now aim to study the discrete convexity of function $\varphi(x)$, in order to use the global optimality of local minima (see Corollary 3.3) as an efficient stopping criterion.

**Theorem 4.2** *Consider problem $P$ and function $\varphi(x)$ as defined in (4.5) and (4.6). Then, function $\varphi(x)$ is discrete convex.*

*Proof* For the sake of convenience, let us define $\Delta^2\varphi(x)$ as follows:

$$\Delta^2\varphi(x) = \varphi(x+2) + \varphi(x) - 2\varphi(x+1)$$

By means of Theorem 3.1 function $\varphi(x)$ is discrete convex if and only if $\Delta^2\varphi(x) \geq 0$ for all $x \in [0, \bar{M}]$. Two exhaustive cases are now going to be considered.

*(Case $c_z < c_w$)* Since $w(x, \hat{z}(x)) = 0$ for all $x \in [0, \bar{M}]$ it results

$$\begin{aligned} \Delta^2\varphi(x) &= c_z \sum_{i=1}^{I} [\hat{z}_i(x+2) + \hat{z}_i(x) - 2\hat{z}_i(x+1)] \\ &= c_z \sum_{i=1}^{I} \Delta^2 \hat{z}_i(x) \end{aligned}$$

By means of simple calculation, for all $i = 1, \dots, I$ we get:

$$\Delta^2 \hat{z}_i(x) = \begin{cases} 0 & \text{if } M_i - \beta_x x \geq 2\beta_x \\ 2\beta_x - M_i + \beta_x x & \text{if } \beta_x \leq M_i - \beta_x x < 2\beta_x \\ M_i - \beta_x x & \text{if } 0 \leq M_i - \beta_x x < \beta_x \\ 0 & \text{if } M_i - \beta_x x < 0 \end{cases}$$

so that $\Delta^2 \hat{z}_i(x) \geq 0$ for all $i = 1, \dots, I$ which implies $\Delta^2\varphi(x) \geq 0$ too.

*(Case $c_z \geq c_w$)* For the sake of convenience, let us introduce the following notation:

$$\hat{h}_i(x) = c_z \hat{z}_i(x) + c_w \max\{0; \min\{M_i - \beta_x x; \lfloor \rho(M_i - m_i) \rfloor\}\}$$

so that $\varphi(x) = Ixp + \sum_{i=1}^{I} \hat{h}_i(x)$ and hence $\Delta^2\varphi(x) = \sum_{i=1}^{I} \Delta^2 \hat{h}_i(x)$. Some exhaustive subcases have now to be considered for any $i = 1, \dots, I$.

Assume $M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor \geq 2\beta_x$. Then, it results $\Delta^2 \hat{h}_i(x) = 0$.

Assume $\beta_x \leq M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor < 2\beta_x$. Then, by means of simple calculations and taking into account that $c_z \geq c_w$, we have:

$$\begin{aligned} \Delta^2 \hat{h}_i(x) &= c_z [-M_i + \beta_x x + 2\beta_x + \lfloor \rho(M_i - m_i) \rfloor] \\ &\quad + c_w [\max\{0; M_i - \beta_x x - 2\beta_x\} - \lfloor \rho(M_i - m_i) \rfloor] \\ &\geq c_w [-M_i + \beta_x x + 2\beta_x + \max\{0; M_i - \beta_x x - 2\beta_x\}] \\ &= c_w \max\{0; -M_i + \beta_x x + 2\beta_x\} \geq 0 \end{aligned}$$

Assume $0 \leq M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor < \beta_x$. Then, by means of simple calculations and taking into account that $c_z \geq c_w$, we have:

$$\begin{aligned}\Delta^2 \hat{h}_i(x) &= c_z \left[M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor\right] + c_w \lfloor \rho(M_i - m_i) \rfloor \\ &\quad + c_w \left[\max\{0; M_i - \beta_x x - 2\beta_x\} - 2\max\{0; M_i - \beta_x x - \beta_x\}\right] \\ &\geq c_w \left[M_i - \beta_x x + \max\{0; M_i - \beta_x x - 2\beta_x\} - 2\max\{0; M_i - \beta_x x - \beta_x\}\right]\end{aligned}$$

By means of the exhaustive cases $M_i - \beta_x x \geq 2\beta_x$, $\beta_x \leq M_i - \beta_x x < 2\beta_x$ and $M_i - \beta_x x < \beta_x$, and recalling that $M_i - \beta_x x \geq \lfloor \rho(M_i - m_i) \rfloor \geq 0$, it can then be easily verified that $\Delta^2 \hat{h}_i(x) \geq 0$.

Assume $M_i - \beta_x x - \lfloor \rho(M_i - m_i) \rfloor < 0$. Then, it results

$$\Delta^2 \hat{h}_i(x) = c_w \left[ \begin{array}{c} \max\{0; M_i - \beta_x x - 2\beta_x\} + \max\{0; M_i - \beta_x x\} \\ -2\max\{0; M_i - \beta_x x - \beta_x\} \end{array} \right]$$

so that

$$\Delta^2 \hat{h}_i(x) = \begin{cases} 0 & \text{if } M_i - \beta_x x \geq 2\beta_x \\ 2\beta_x - M_i + \beta_x x & \text{if } \beta_x \leq M_i - \beta_x x < 2\beta_x \\ M_i - \beta_x x & \text{if } 0 \leq M_i - \beta_x x < \beta_x \\ 0 & \text{if } M_i - \beta_x x < 0 \end{cases}$$

which implies the nonnegativity of $\Delta^2 \hat{h}_i(x)$.

As a conclusion, we have stated that $\Delta^2 \hat{h}_i(x) \geq 0$ for all $i = 1, \ldots, I$, and this implies $\Delta^2 \varphi(x) \geq 0$ too. The result is then proved. □

Finally, let us conclude our study pointing out how the optimal solution can be efficiently found.

**Algorithm Structure**

1) Determine $\bar{M}$ and let $x^* := 0$, $x' := 0$ and *local* := *false*;

2) While not *local* and $x' < \bar{M}$ do

   2a) $x' := x' + 1$;

   2b) if $\varphi(x') < \varphi(x^*)$ then $x^* := x'$ else *local* := *true*

3) The optimal solution of problem $P$ is $(x^*, \hat{z}(x^*))$ with optimal value $\varphi(x^*)$.

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