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An unifying approach to solve a class of parametrically-convexifiable problems

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Abstract

The aim of this paper is to show how a wide class of generalized quadratic programs can be solved, in a unifying framework, by means of the so called optimal level solutions method. In other words, the problems are solved by analyzing, explicitly or implicitly, the optimal solutions of particular quadratic strictly convex parametric subproblems. In particular, it is pointed out that some of these problems share the same set of optimal level solutions. A solution algorithm is proposed and fully described. The obtained results are then deepened on in the particular case of box constrained problems.

Keywords generalized quadratic programming, fractional programming, optimal level solutions.
AMS - 2000 Math. Subj. Class. 90C20, 90C26, 90C31.
JEL - 1999 Class. Syst. C61, C63.

1 Introduction

The aim of this paper is to study and to propose a solution method for the following class of generalized quadratic problems:

$$P : \begin{cases} \inf \phi(x) = f\left(\frac{1}{2}x^T Q x + q^T x + q_0\right) g_1\left(d^T x + d_0\right) + g_2\left(d^T x + d_0\right) \\ x \in X = \{x \in \Re^n : Ax \geq b\} \end{cases}$$

where $A \in \Re^{m\times n}$, $q, d \in \Re^n$, $d \neq 0$, $b \in \Re^m$, $q_0, d_0 \in \Re$, $Q \in \Re^{n\times n}$ is symmetric and positive definite, $g_1, g_2 : \Omega_g \to \Re$, $f : \Omega_f \to \Re$, with g_1

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positive over Ω_g and f strictly increasing over Ω_f, where

$$\begin{aligned}\Omega_g &= \left\{y \in \Re : y = d^T x + d_0, x \in X\right\} \\ \Omega_f &= \left\{y \in \Re : y = \frac{1}{2} x^T Q x + q^T x + q_0, x \in X\right\}\end{aligned}$$

Various particular problems belonging to this class have been studied in the literature of mathematical programming and global optimization, from both a theoretic and an applicative point of view ([2, 12, 13, 14, 20]). In particular, it is worth noticing that this class covers several multiplicative, fractional, d.c. and generalized quadratic problems (see for all [4, 6, 7, 8, 11, 16, 18]) which are very used in applications, such as location models, tax programming models, portfolio theory, risk theory, Data Envelopment Analysis (see for all [1, 9, 11, 15, 16, 21]).

The solution method proposed to solve this class of problems is based on the so called "optimal level solutions" method (see [3, 4, 5, 6, 7, 8, 10, 17, 18, 19]). It is known that this is a parametric method, which finds the optimum of the problem by determining the minima of particular subproblems. In particular, the optimal solutions of these subproblems are obtained by means of a sensitivity analysis aimed to maintain the Karush-Kuhn-Tucker optimality conditions.

Applying the optimal level solutions method to problem P we obtain some strictly convex quadratic subproblems which result to be independent with respect to functions f, g_1 and g_2. In other words, different problems share the same set of optimal level solutions, and this allow us to propose an unifying method to solve all of them.

In Section 2 we describe how the optimal level solutions method can be applied to problem P; in Section 3 a solution algorithm is proposed and fully described; finally, in Section 4, the obtained results are deepened on for the particular case of box constrained problems.

2 Optimal level solutions approach

In this section we show how problem P can be solved by means of the optimal level solutions approach [3, 5, 6, 7, 10, 17]. With this aim, let $\xi \in \Re$ be a real parameter. The following parametric subproblem can be obtained just by adding to problem P the constraint $d^T x + d_0 = \xi$:

$$P_\xi : \begin{cases} \inf f\left(\frac{1}{2} x^T Q x + q^T x + q_0\right) g_1(\xi) + g_2(\xi) \\ x \in X_\xi = \{x \in \Re^n : Ax \geq b, d^T x + d_0 = \xi\} \end{cases}$$

The parameter ξ is said to be a *feasible level* if the set X_ξ is nonempty, that is if $\xi \in \Omega_g$. An optimal solution of problem P_ξ is called an *optimal level solution*. Since g_1 is positive over Ω_d and f is strictly increasing over Ω_q, then for any given $\xi \in \Omega_g$ the optimal solution of problem P_ξ coincides with the optimal solution of the following strictly convex quadratic problem $\overline{P}_\xi$:

$$\overline{P}_\xi : \begin{cases} \inf \frac{1}{2}x^TQx + q^Tx + q_0 \\ x \in X_\xi = \{x \in \Re^n : Ax \geq b, d^Tx + d_0 = \xi\} \end{cases}$$

In this light, we say that function ϕ is *parametrically-convexifiable*.

For the sake of completeness, let us now briefly recall the optimal level solutions approach (see for example [10]). Obviously, the optimal solution of problem P is also an optimal level solution and, in particular, it is the optimal level solution with the smallest value; the idea of this approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached or a feasible halfline carrying $\phi(x)$ down to its infimum value is found.

Starting from an incumbent optimal level solution, this can be done by means of a sensitivity analysis on the parameter ξ, which allows us to move in the various steps through several optimal level solutions until the optimal solution is found.

Remark 2.1 Let us point out that problems $\overline{P}_\xi$ are independent with respect to the functions f, g_1 and g_2. This means that different parametrically-convexifiable problems, either multiplicative or fractional or d.c. quadratic ones, share the same set of optimal level solutions and can then be solved by means of the same algorithm iterations. In this light, it can be said that the solution method we propose in this paper represents an unifying framework for various classes of generalized quadratic problems.

2.1 Starting problem and sensitivity analysis

Let x' be the optimal solution of problem $\overline{P}_{\xi'}$, where $d^Tx' + d_0 = \xi'$, and let us consider the following Karush-Kuhn-Tucker conditions for $\overline{P}_{\xi'}$:

$$\begin{cases} Qx' + q = A^T\mu + d\lambda & \\ d^Tx' + d_0 = \xi' & \\ Ax' \geq b & \textit{feasibility} \\ \mu \geq 0 & \textit{optimality} \\ \mu^T(Ax' - b) = 0 & \textit{complementarity} \\ \lambda \in \Re, \ \mu \in \Re^m & \end{cases} \tag{2.1}$$

Since $\overline{P}_{\xi'}$ is a strictly convex problem, the previous system has at least one solution (μ', λ').

By means of a sort of sensitivity analysis, we now aim to study the optimal level solutions of problems $\overline{P}_{\xi'+\theta}$, $\theta \in (0, \epsilon)$ with $\epsilon > 0$ small enough. This can be done by maintaining the consistence of the Karush-Kuhn-Tucker systems corresponding to these problems.

Since the Karush-Kuhn-Tucker systems are linear whenever the complementarity conditions are implicitly handled, then the solution of the optimality conditions regarding to $\overline{P}_{\xi'+\theta}$ is of the kind:

$$x'(\theta) = x' + \theta\Delta_x \ , \ \lambda'(\theta) = \lambda' + \theta\Delta_\lambda \ , \ \mu'(\theta) = \mu' + \theta\Delta_\mu \qquad (2.2)$$

so that it results:

$$\begin{cases} Q(x' + \theta\Delta_x) + q = A^T(\mu' + \theta\Delta_\mu) + d(\lambda' + \theta\Delta_\lambda) \\ d^T(x' + \theta\Delta_x) + d_0 = \xi' + \theta, \\ A(x' + \theta\Delta_x) \geq b \\ (\mu' + \theta\Delta_\mu) \geq 0 \\ (\mu_i' + \theta\Delta_{\mu_i})\,(a_i(x' + \theta\Delta_x) - b_i) = 0 \ \forall i = 1, \dots, m \\ \Delta_\lambda \in \Re, \ \Delta_\mu \in \Re^m, \ \Delta_x \in \Re^n \end{cases} \qquad (2.3)$$

where a_i, $i = 1, \dots, m$, is the i-th row of A.

It is worth pointing out that the strict convexity of problem $\overline{P}_{\xi'+\theta}$ guarantees for any $\theta \in (0, \epsilon)$ the uniqueness of the optimal level solution $x'(\theta) = x' + \theta\Delta_x$; this implies also the following important property:

vector Δ_x is unique and different from 0.

Let us now provide an useful preliminary lemma which suggests how to study system (2.3). With this aim, let us define, the following sets of indices based on the binding and the nonbinding constraints:

$$B = \{i : \ a_i x' = b_i \ , \ i = 1, \dots, m\} \ , \ N = \{i : \ a_i x' > b_i \ , \ i = 1, \dots, m\}$$

Lemma 2.1 *Let (μ', λ') be a solution of (2.1). Then, for $\theta \in (0, \epsilon)$ system (2.3) is equivalent to:*

$$\begin{cases} Q\Delta_x = A^T\Delta_\mu + d\Delta_\lambda \\ d^T\Delta_x = 1, \\ Ax' + \theta A\Delta_x \geq b \\ \mu' + \theta\Delta_\mu \geq 0 \\ \mu_i' = \Delta_{\mu_i} = 0 \ \forall i \in N \\ \mu_i' a_i \Delta_x = 0 \ , \ \Delta_{\mu_i} a_i \Delta_x = 0 \quad \forall i \in B \end{cases} \qquad (2.4)$$

Proof The first and the second equations follow directly from (2.1) taking into account that $\theta \neq 0$. From (2.1) we have also that the complementarity conditions of (2.3) can be rewritten as:

$$\mu_i' a_i \Delta_x + \Delta_{\mu_i}(a_i x' - b_i) + \theta\Delta_{\mu_i} a_i \Delta_x = 0 \ \forall i = 1, \dots, m \qquad (2.5)$$

For any index $i \in N$ and for $\theta > 0$ small enough it is $(a_i(x' + \theta\Delta_x) - b_i) \neq 0$, so that from (2.3) it results $\mu_i' + \theta\Delta_{\mu_i} = 0$. This last equation holds for any $\theta > 0$ small enough if and only if $\mu_i' = \Delta_{\mu_i} = 0$; in other words it is:

$$\mu_i' = \Delta_{\mu_i} = 0 \quad \forall i \in N$$

which also yields:

$$\Delta_{\mu_i}(a_i x' - b_i) = 0 \quad \forall i = 1, \ldots, m$$

This equality implies that condition (2.5) holds for any $\theta \in (0, \epsilon)$ if and only if for all $i = 1, \ldots, m$ it is:

$$\mu_i' a_i \Delta_x = 0 \ , \ \Delta_{\mu_i} a_i \Delta_x = 0$$

and the result is proved. □

Note that from the positivity of θ, the feasibility conditions and the optimality ones, we also have:

$$\begin{gathered} a_i \Delta_x \geq 0 \quad \forall i \in B \\ \Delta_{\mu_i} \geq 0 \quad \forall i \in B \ \text{ such that } \ \mu_i' = 0 \end{gathered}$$

As a conclusion, we can compute the values of the multipliers λ', μ', Δ_λ, Δ_μ, Δ_x by solving the following overall system (which has $2+2m+n$ variables):

$$\left\{ \begin{array}{c} Qx' + q = A^T \mu' + d\lambda' \\ Q\Delta_x = A^T \Delta_\mu + d\Delta_\lambda \\ d^T \Delta_x = 1, \\ \mu_i' = \Delta_{\mu_i} = 0 \quad \forall i \in N \\ \mu_i' \geq 0 \quad \forall i \in B \\ a_i \Delta_x \geq 0 \ \forall i \in B \\ \mu_i' a_i \Delta_x = 0 \ \forall i \in B \\ \Delta_{\mu_i} a_i \Delta_x = 0 \ \forall i \in B \\ \Delta_{\mu_i} \geq 0 \quad \forall i \in B \ \text{ s.t. } \ \mu_i' = 0 \\ \lambda', \Delta_\lambda \in \Re, \ \mu', \Delta_\mu \in \Re^m, \ \Delta_x \in \Re^n \end{array} \right. \tag{2.6}$$

This system is suitable for values of $\theta \geq 0$ verifying the following conditions:

$$\begin{array}{rcl} \textit{feasibility conditions} & : & Ax' + \theta A \Delta_x \geq b \\ \textit{optimality conditions} & : & \mu' + \theta \Delta_\mu \geq 0 \end{array}$$

Notice that system (2.6) is consistent if and only if the feasible regions $X_{\xi'+\theta}$ of problems $\overline{P}_{\xi'+\theta}$ are nonempty for $\theta > 0$ small enough.

In the case system (2.6) is consistent, we are finally able to determine the values of $\theta > 0$ which guarantee both the optimality and the feasibility of $x'(\theta)$. Let $N^- = \{i \in N : \ a_i \Delta_x < 0\}$ ([1]); since $Ax' \geq b$, from the feasibility conditions we have:

$$\theta \leq \hat{F} = \left\{ \begin{array}{ll} \min_{i \in N^-} \left\{ \frac{b_i - a_i x'}{a_i \Delta_x} \right\} & \text{if } N^- \neq \emptyset \\ +\infty & \text{if } N^- = \emptyset \end{array} \right.$$

[1]Since $\theta > 0$ then inequalities $a_i \Delta_x < 0$ and $a_i x' + \theta a_i \Delta_x \geq b_i$ imply $b_i - a_i x' < 0$, that is to say that $i \in N$.

where $\hat{F} > 0$. On the other hand, let $B^- = \{i \in B : \Delta_{\mu_i} < 0\}$ (recall that $\Delta_{\mu_i} = 0 \; \forall i \in N$); from the optimality conditions we have:

$$\theta \leq \hat{O} = \begin{cases} \min_{i \in B^-} \left\{ \frac{-\mu_i}{\Delta_{\mu'_i}} \right\} & \text{if } B^- \neq \emptyset \\ +\infty & \text{if } B^- = \emptyset \end{cases}$$

where $\hat{O} > 0$ (since $\theta > 0$ then inequalities $\Delta_{\mu_i} < 0$ and $\mu'_i + \theta\Delta_{\mu_i} \geq 0$ imply $\mu'_i > 0$). Hence, $x'(\theta)$ is an optimal level solution for all θ such that:

$$0 \leq \theta \leq \theta_m = \min \left\{ \hat{F}, \hat{O} \right\}$$

where $\theta_m > 0$ (obviously, when system (2.6) is consistent).

2.2 Solving the multipliers system

The aim of this subsection is to show how system (2.6) can be improved in order to determine its solutions. For the sake of convenience, from now on the rows of A and the components of b and μ' will be partitioned accordingly to the set of indices B and N.

Multiplying the first and the second equations of (2.6) by $\Delta_x \neq 0$ and taking into account that Q is positive definite, it follows:

$$\lambda' = (Qx' + q)^T \Delta_x \quad \text{and} \quad \Delta_\lambda = \Delta_x^T Q \Delta_x > 0 \tag{2.7}$$

Multiplying the first equation of (2.6) by $d \neq 0$ and after simple calculations we also get:

$$\lambda' = \frac{1}{d^T d} d^T \left(Qx' + q - A^T \mu' \right)$$

Let us now define the matrix $\hat{D} = \left(I - \frac{1}{d^T d} dd^T \right)$; note that $\hat{D}$ is symmetric, singular (since $\hat{D}d = 0$) and positive semidefinite (the $n-1$ nonzero eigenvalues are all equal to 1 since $\hat{D}y = y \; \forall y \in d^\perp$). Noticing that $d\lambda' = \left(I - \hat{D} \right) \left(Qx' + q - A^T \mu' \right)$ and that $\mu'_N = 0$, we can rewrite the first equation of (2.6) as follows:

$$\hat{D} A_B^T \mu'_B = \hat{D} \left(Qx' + q \right)$$

The solution of this system is not unique in general; in particular note that:

$$\text{rank}(\hat{D} A_B^T) \leq \min\{n-1, \text{rank}(A_B)\}$$

For the sake of convenience, let us now define the scalar $\delta = \frac{1}{d^T Q^{-1} d} > 0$ and the symmetric matrix $\hat{Q}_d = (Q^{-1} - \delta Q^{-1} dd^T Q^{-1})$ which results to be singular (since $\hat{Q}_d d = 0$) and positive semidefinite (for Theorem 2.1 in [8]

([2])). Since Q is nonsingular then, from the second and the third equations of (2.6), we get:

$$\begin{aligned}\Delta_\lambda &= \frac{1-d^TQ^{-1}A^T\Delta_\mu}{d^TQ^{-1}d} = \delta - \delta d^TQ^{-1}A^T\Delta_\mu \\ \Delta_x &= Q^{-1}A^T\Delta_\mu + Q^{-1}d\Delta_\lambda = \delta Q^{-1}d + \hat{Q}_dA^T\Delta_\mu\end{aligned}$$

As a conclusion, we have the following explicit solutions of system (2.6), some of them depending on Δ_{μ_B}:

$$\begin{aligned}\mu'_N &= 0 \\ \Delta_{\mu_N} &= 0 \\ \Delta_x &= \delta Q^{-1}d + \hat{Q}_dA_B^T\Delta_{\mu_B} \\ \lambda' &= (Qx'+q)^T\Delta_x \\ \Delta_\lambda &= \Delta_x^TQ\Delta_x\end{aligned}$$

Note that the uniqueness of vector Δ_x implies the uniqueness of λ' and Δ_λ.

We are now left to compute the values of vectors μ_B and Δ_{μ_B}. With this aim, for the sake of convenience, let $v_B = A_BQ^{-1}d$ and $R_B = A_B\hat{Q}_dA_B^T = (A_BQ^{-1}A_B^T - \delta v_Bv_B^T)$. Matrix R_B is symmetric and positive semidefinite (due to the semipositiveness of $\hat{Q}_d$) with:

$$\text{rank}(R_B) \leq \min\{n-1, \text{rank}(A_B)\}$$

notice also that the i-th component of v_B is $v_i = a_iQ^{-1}d$ while the i-th row of R_B is given by $r_i = (a_iQ^{-1}A_B^T - \delta v_iv_B^T)$, so that $a_i\Delta_x = r_i\Delta_{\mu_B} + \delta v_i$. Vectors μ_B and Δ_{μ_B} are then solutions of the following system:

$$\begin{cases} \hat{D}A_B^T\mu_B = \hat{D}\,(Qx'+q) \\ \mu_B \geqq 0 \\ R_B\Delta_{\mu_B} + \delta v_B \geqq 0 \\ \mu_i(r_i\Delta_{\mu_B} + \delta v_i) = 0\ \forall i \in B \\ \Delta_{\mu_i}(r_i\Delta_{\mu_B} + \delta v_i) = 0\ \forall i \in B \\ \Delta_{\mu_i} \geq 0\ \forall i \in B \text{ s.t. } \mu_i = 0 \end{cases} \tag{2.8}$$

Notice that the number of variables in system (2.8) is just $2card(B)$.

2.3 Optimal level solutions comparison

The optimal level solutions $x'(\theta)$ obtained by means of the sensitivity analysis can be compared just by evaluating the function $z(\theta) = \phi\,(x'(\theta))$. Defining $z' = \frac{1}{2}x'^TQx' + q^Tx' + q_0$ and recalling equations (2.7) it then results:

$$\frac{1}{2}x'(\theta)^TQx'(\theta) + q^Tx'(\theta) + q_0 = \frac{1}{2}\Delta_\lambda\theta^2 + \lambda'\theta + z'$$

[2]**Theorem 2.1** [8] Let $Q \in \Re^{n\times n}$ be a symmetric positive definite matrix, let $k \in \Re$ and let $h \in \Re^n$. Then, the symmetric matrix $A = \left(Q + khh^T\right)$ is positive semidefinite if and only if $k \geq -\frac{1}{h^TQ^{-1}h}$.

Hence, since $d^T x'(\theta) + d_0 = \xi' + \theta$, we get:

$$\begin{aligned}
z(\theta) &= \phi\left(x'(\theta)\right) = f\left(\frac{1}{2}\Delta_\lambda \theta^2 + \lambda'\theta + z'\right) g_1\left(\xi' + \theta\right) + g_2\left(\xi' + \theta\right) \\
\frac{dz}{d\theta}(\theta) &= \frac{df}{d\theta}\left(\frac{1}{2}\Delta_\lambda \theta^2 + \lambda'\theta + z'\right)\left(\Delta_\lambda \theta + \lambda'\right) g_1\left(\xi' + \theta\right) + \\
&\quad + f\left(\frac{1}{2}\Delta_\lambda \theta^2 + \lambda'\theta + z'\right) \frac{dg_1}{d\theta}\left(\xi' + \theta\right) + \frac{dg_2}{d\theta}\left(\xi' + \theta\right)
\end{aligned}$$

so that, in particular:

$$\frac{dz}{d\theta}(0) = \lambda' \frac{df}{d\theta}\left(z'\right) g_1\left(\xi'\right) + f\left(z'\right) \frac{dg_1}{d\theta}\left(\xi'\right) + \frac{dg_2}{d\theta}\left(\xi'\right)$$

As it is very well known, the derivative $\frac{dz}{d\theta}(0)$ can be useful since its sign implies the local decreasing or increasing behaviour of $z(\theta)$.

Level optimality is helpful also in studying local optimality, since a local minimum point in a segment of optimal level solutions is a local minimizer of the problem. This fundamental property allows to prove the following global optimality conditions in the case of a convex objective function $\phi(x)$.

Theorem 2.1 *Consider problem P, assume $\phi(x)$ convex and let $x'(\theta)$ be the optimal solution of problem $\overline{P}_{\xi'+\theta}$.*

i) if $\frac{dz}{d\theta}(0) > 0$ then $\phi(x') \leq \phi(x)$ for all $x \in B$ such that $d^T x \geq d^T x'$

ii) if $\theta_m < +\infty$ and $\overline{\theta} = \arg\min_{\theta \in [0,\theta_m]} \{z(\theta)\}$ is such that $0 < \overline{\theta} < \theta_m$, then $x'(\overline{\theta})$ is the optimal solution of problem P.

Proof Since $\phi(x)$ is convex any local optimum is also global. The results then follow since a local minimum point in a segment of optimal level solutions is also a local minimizer. □

3 A solution algorithm

In order to find a global minimum (or just the infimum) it would be necessary to solve problems $\overline{P}_\xi$ for all the feasible levels. In this section we will show that, by means of the results stated so far, this can be done algorithmically in a finite number of iterations.

The solution algorithm starts from a certain minimal level and then scans all the greater ones looking for the optimal solution, as it is pointed out in the next initialization process.

Initialization Steps

Compute, by means of two linear programs, the values ([3]):

$$\xi_{min} := \inf_{x \in X} d^T x + d_0 \quad , \quad \xi_{max} := \sup_{x \in X} d^T x + d_0$$

[3] Obviously, it may be $\xi_{min} = -\infty$ and/or $\xi_{max} = +\infty$.

One of the following cases occurs:

1) if $\xi_{min} > -\infty$ then solve problem P from the starting feasible level $\xi_{start} = \xi_{min}$ up to the level $\xi_{end} = \xi_{max}$;

2) if $\xi_{min} = -\infty$ and $\xi_{max} < +\infty$ then let $\tilde{g}_1(\xi) = g_1(-\xi)$ and $\tilde{g}_2(\xi) = g_2(-\xi)$, so that the objective function of P can be rewritten as:

$$\phi(x) = f\left(\frac{1}{2}x^T Qx + q^T x + q_0\right) \tilde{g}_1\left(-d^T x - d_0\right) + \tilde{g}_2\left(-d^T x - d_0\right)$$

We can then solve problem P using $\tilde{g}_1$ and $\tilde{g}_2$ and scanning the feasible levels from the starting value $\xi_{start} = -\xi_{max} > -\infty$ up to $\xi_{end} = +\infty$;

3) if $\xi_{min} = -\infty$ and $\xi_{max} = +\infty$ then solve sequentially the next two problems from the starting level $\xi_{start} = 0$ up to the level $\xi_{end} = +\infty$:

$$P_+ : \begin{cases} \inf f(x) \\ d^T x + d_0 \geq 0 \\ x \in X \end{cases} \quad \text{and} \quad P_- : \begin{cases} \inf f(x) \\ d^T x + d_0 \leq 0 \\ x \in X \end{cases}$$

where P_- is defined using $\tilde{g}_1$ and $\tilde{g}_2$.

□

Once the starting feasible level ξ_{start} is found, the optimal solution can be searched iteratively by means of the following algorithm.

Algorithm Structure

1) Let $\xi' := \xi_{start}$; $x' := \arg\min\{\overline{P}_{\xi_{start}}\}$; $UB := \phi(x')$; $x^* := x'$; unbounded:= $false$; stop:= $false$;

2) While not stop do

   2a) With respect to ξ' and x' determine μ', λ', Δ_x, Δ_μ, Δ_λ, $\hat{F}$, $\hat{O}$; $\theta_m := \min\{\hat{F}, \hat{O}\}$;

   2b) If $\inf_{\theta\in[0,\theta_m]} \{z(\theta)\} = -\infty$ then unbounded:= $true$ else $\overline{\theta} = \arg\min_{\theta\in[0,\theta_m]} \{z(\theta)\}$;

   2c) If unbounded= $true$ or $\{\phi(x)$ is convex and $\frac{dz}{d\theta}(0) > 0\}$ then stop:= $true$ else begin

   - If $z\left(\overline{\theta}\right) < UB$ then $x^* := x'\left(\overline{\theta}\right)$ and $UB := z\left(\overline{\theta}\right)$;
   - If $\xi' + \theta_m \geq \xi_{end}$ or $\{\phi(x)$ is convex and $0 < \overline{\theta} < \theta_m\}$ then stop:= $true$ else $x' := x' + \theta_m\Delta_x$; $\xi' := \xi' + \theta_m$;

end;

3) If unbounded= *true* then $\inf_{x \in X} \phi(x) = -\infty$ else x^* is the optimal solution for problem P.

Variable UB gives in the various iterations an upper bound for the optimal value with respect to the levels $\xi > \xi'$, while x^* is the best optimal level solution with respect to the levels $\xi \leq \xi'$. Let us also point out that:

- in *Step 1)* we have to determine the optimal solution of the strictly convex quadratic problem $\overline{P}_{\xi_{start}}$; actually, this is the only quadratic problem which needs to be solved within the solution algorithm;

- in *Step 2a)* the multipliers have to be determined by solving a system whose dimension has been reduced as much as possible (see Subsection 2.2 and system (2.8)); notice that these multipliers do not depend on the chosen functions f, g_1 and g_2; in the next section we will show that this step can be improved in the case of box constrained problems;

- in *Step 2b)* we have to determine the minimum of $z(\theta)$ for $\theta \in [0, \theta_m]$; notice that $z(\theta)$ is a single variable function and that its minimum over the segment $[0, \theta_m]$ can be computed with various numerical methods; notice also that *Step 2b)* is the only step which depends on the chosen functions f, g_1 and g_2;

- finally, it is worth noticing that for particular classes of functions this solution algorithm can be improved and detailed; in other words, for particular functions f, g_1 and g_2, the algorithm can be optimized for convex functions ϕ, and/or the multipliers in *Step 2a)* and the value of $\overline{\theta}$ in *Step 2b)* can be determined analytically (see for all [4, 6, 7, 8, 18]).

Once *Step 2b)* is implemented, the correctness of the proposed algorithm follows since all the feasible levels are scanned and the optimal solution, if it exists, is also an optimal level solution. As regards to the convergence (finiteness) of the procedure, note that in every iteration the set of binding constraints B changes; note also that the level is increased from ξ' to $\xi' + \theta_m$ so that it is not possible to obtain again an already used set of binding constraints B; the convergence then follows since we have a finite number of sets of binding constraints.

In particular, if $\theta_m = +\infty$ an halfline of optimal level solutions is found and the algorithm stops. Consider now the case $\theta_m < +\infty$; if $\theta_m = \hat{F}$ then at least one non binding constraint enters the set B; if $\theta_m = \hat{O}$ then at least one of the positive multipliers corresponding to a binding constraints vanishes, so that the related constraint will leave the set B in the following iteration.

4 Box constrained case

The aim of this section is to deepen on the results stated so far in the particular case of box constrained problems:

$$P : \begin{cases} \inf \phi(x) = f\left(\frac{1}{2}x^T Qx + q^T x + q_0\right) g_1\left(d^T x + d_0\right) + g_2\left(d^T x + d_0\right) \\ x \in X^B = \{x \in \Re^n : l \leqq x \leqq u\} \end{cases}$$

where $l, u, d \in \Re^n$, $d \geq 0$ (4). Obviously, all the other hypotheses required in Section 1 are assumed too. By means of the general approach described in Section 2 we have:

$$\overline{P}_\xi : \begin{cases} \min \frac{1}{2}x^T Qx + q^T x + q_0 \\ x \in X^B_\xi = \{x \in \Re^n : l \leqq x \leqq u, d^T x + d_0 = \xi\} \end{cases}$$

Note that the feasible region X^B_ξ is no more given by box constraints.

Clearly, this class of box constrained problems can be solved by means of the solution algorithm described in Section 3. With this aim, notice that it results $\xi_{start} = \xi_{min} = d^T l + d_0$ and $\xi_{end} = \xi_{max} = d^T u + d_0$, and that the only strictly convex quadratic problem which has to be explicitly solved in *Step 1)* is:

$$\overline{P}_{\xi_{start}} : \begin{cases} \min \frac{1}{2}x^T Qx + q^T x + q_0 \\ x_i = l_i \quad \forall i = 1, \ldots, n \ \text{ such that } \ d_i > 0 \\ l_i \leq x_i \leq u_i \quad \forall i = 1, \ldots, n \ \text{ such that } \ d_i = 0 \end{cases}$$

In the rest of this section we will point out how the solution method can be improved in the case of box constrained problems, in particular with respect to the calculus of the multipliers in *Step 2a)*.

4.1 Incumbent problem

Let x' be the optimal solution of problem $\overline{P}_{\xi'}$, let $\xi' = d^T x' + d_0 \in [\xi_{min}, \xi_{max}]$, and let us define, for the sake of convenience, the following partition $L \cup U \cup N \cup Z$ of the set of indices $\{1, \ldots, n\}$:

$$L = \{i : \ l_i = x'_i < u_i\} \quad , \quad N = \{i : \ l_i < x'_i < u_i\}$$
$$U = \{i : \ l_i < x'_i = u_i\} \quad , \quad E = \{i : \ l_i = x'_i = u_i\}$$

Since $\overline{P}_{\xi'}$ is a strictly convex problem, x' is its unique optimal solution if and only if the following Karush-Kuhn-Tucker conditions hold (5):

4Notice that the $d \geq 0$ assumption is not restrictive, since it can be obtained by means of a trivial change of the variables x_i corresponding to the components $d_i < 0$.

5If $l \nleq u$, that is $l_i = u_i$ for some indices i, the Karush-Kuhn-Tucker conditions are sufficient but not necessary since no constraint qualification conditions are verified. These indices will be handled implicitly in the rest of the paper by properly choosing the values of the multipliers.

$$\begin{cases} Qx' + q = \lambda d + \alpha - \beta & \\ d^T x' + d_0 = \xi', & \\ l \leqq x' \leqq u & \textit{feasibility} \\ \alpha \geqq 0,\ \beta \geqq 0, & \textit{optimality} \\ \alpha^T(x' - l) = 0,\ \beta^T(u - x') = 0 & \textit{complementarity} \\ \lambda \in \Re,\ \alpha, \beta \in \Re^n & \end{cases} \tag{4.1}$$

Denoting with Q_i the i-th row of Q, we can rewrite these Karush-Kuhn-Tucker conditions as follows:

$$\begin{cases} \alpha_i = 0,\ \beta_i = 0,\ Q_i x' + q_i = 0 & \forall i \in N \text{ s.t. } d_i = 0 \\ \alpha_i = 0,\ \beta_i = 0,\ \lambda = \frac{1}{d_i}(Q_i x' + q_i) & \forall i \in N \text{ s.t. } d_i \neq 0 \\ \beta_i = 0,\ \alpha_i = Q_i x' + q_i - \lambda d_i \geq 0 & \forall i \in L \\ \alpha_i = 0,\ \beta_i = \lambda d_i - Q_i x' - q_i \geq 0 & \forall i \in U \\ \alpha_i = \max\{0, Q_i x' + q_i - \lambda d_i\} \geq 0 & \forall i \in E \\ \beta_i = \max\{0, \lambda d_i - Q_i x' - q_i\} \geq 0 & \forall i \in E \\ d^T x + d_0 = \xi',\ l \leqq x \leqq u & \end{cases}$$

Let $Z = \{i :\ d_i \neq 0\}$. Since $d \geq 0$ it results:

$$\begin{cases} \lambda = \frac{1}{d_i}(Q_i x' + q_i) & \forall i \in N \cap Z \\ \lambda \leq \frac{1}{d_i}(Q_i x' + q_i) & \forall i \in L \cap Z \\ \lambda \geq \frac{1}{d_i}(Q_i x' + q_i) & \forall i \in U \cap Z \end{cases}$$

Given the optimal level solution x' for problem $P_{\xi'}$ the multipliers $\lambda', \alpha', \beta'$ can then be computed as follows. First, notice that when $(L \cup N \cup U) \cap Z = \emptyset$ then the linear function $d^T x + d_0$ is constant on the box feasible region, that is to say that the problem admits one unique feasible level and is then trivial.

Assuming $(L \cup N \cup U) \cap Z \neq \emptyset$, we can determine the value of λ' as described below:

$$\lambda' = \begin{cases} \frac{Q_i x' + q_i}{d_i}, \text{ for any } i \in N \cap Z & \text{if } N \cap Z \neq \emptyset \\ \min_{i \in L \cap Z} \left\{ \frac{Q_i x' + q_i}{d_i} \right\} & \text{if } N \cap Z = \emptyset \text{ and } L \cap Z \neq \emptyset \\ \max_{i \in U \cap Z} \left\{ \frac{Q_i x' + q_i}{d_i} \right\} & \text{if } N \cap Z = \emptyset \text{ and } L \cap Z = \emptyset \end{cases} \tag{4.2}$$

Then, the components of α' and β' can be obtained as follows:

$$\alpha'_i = \begin{cases} 0 & \forall i \in N \cup U \\ Q_i x' + q_i - \lambda' d_i & \forall i \in L \\ \max\{0, Q_i x' + q_i - \lambda' d_i\} & \forall i \in E \end{cases} \tag{4.3}$$

$$\beta'_i = \begin{cases} 0 & \forall i \in L \cup N \\ \lambda' d_i - Q_i x' - q_i & \forall i \in U \\ \max\{0, \lambda' d_i - Q_i x' - q_i\} & \forall i \in E \end{cases} \tag{4.4}$$

Let us remark that, unlike the general case of Subsection 2.1, we have been able to determine explicitly the values of all the multipliers of the Karush-Kuhn-Tucker conditions regarding to $\overline{P}_{\xi}$.

4.2 Sensitivity analysis

In the light of the optimal level solution parametrical approach we now have to study the optimal solution of problem $\overline{P}_{\xi'+\theta}$, with $\theta > 0$. In order to avoid trivialities, we can assume $\xi' < \xi_{max}$. Since the Karush-Kuhn-Tucker system is linear whenever the complementarity conditions are implicitly handled, then the solution of the optimality conditions regarding to $\overline{P}_{\xi'+\theta}$ results:

$$x'(\theta) = x' + \theta\Delta_x \ , \ \lambda'(\theta) = \lambda' + \theta\Delta_\lambda$$
$$\alpha'(\theta) = \alpha' + \theta\Delta_\alpha \ , \ \beta'(\theta) = \beta' + \theta\Delta_\beta$$

so that it follows:

$$\begin{cases} Q(x' + \theta\Delta_x) + q = (\alpha' + \theta\Delta_\alpha) - (\beta' + \theta\Delta_\beta) + d(\lambda' + \theta\Delta_\lambda) \\ d^T(x' + \theta\Delta_x) + d_0 = \xi' + \theta \\ l \leqq x' + \theta\Delta_x \leqq u \\ \alpha' + \theta\Delta_\alpha \geqq 0, \ \beta' + \theta\Delta_\beta \geqq 0 \\ (\alpha' + \theta\Delta_\alpha)^T(x' + \theta\Delta_x - l) = 0, \ (\beta' + \theta\Delta_\beta)^T(u - x' - \theta\Delta_x) = 0 \end{cases} \tag{4.5}$$

Since x', λ', α' and β' are known, we are left to compute $\Delta_x, \Delta_\lambda, \Delta_\alpha, \Delta_\beta$. With this aim, let us provide the following lemma.

Lemma 4.1 *Let $(\lambda', \alpha', \beta')$ be a solution of (4.1). Then, for $\theta \in (0, \epsilon)$ system (4.5) is equivalent to:*

$$\begin{cases} Q\Delta_x = \Delta_\alpha - \Delta_\beta + d\Delta_\lambda \\ d^T\Delta_x = 1 \\ l \leqq x' + \theta\Delta_x \leqq u \\ \alpha' + \theta\Delta_\alpha \geqq 0, \ \beta' + \theta\Delta_\beta \geqq 0 \\ \Delta_{x_i} = 0 \ \forall i \in E \\ \Delta_{\alpha_i} = 0 \ \forall i \in N \cup U \\ \Delta_{\beta_i} = 0 \ \forall i \in L \cup N \\ \alpha'_i\Delta_{x_i} = 0 \ , \ \Delta_{\alpha_i}\Delta_{x_i} = 0 \quad \forall i \in L \\ \beta'_i\Delta_{x_i} = 0 \ , \ \Delta_{\beta_i}\Delta_{x_i} = 0 \quad \forall i \in U \end{cases} \tag{4.6}$$

Proof The first and the second equations follow directly from (4.1) taking into account that $\theta \neq 0$, while $\Delta_{x_i} = 0 \ \forall i \in E$ follows directly from the definition of E. From (4.1) we have also that the complementarity conditions of (4.5) can be rewritten as:

$$\Delta_{\alpha_i}(x'_i - l_i) + \alpha'_i\Delta_{x_i} + \theta\Delta_{\alpha_i}\Delta_{x_i} = 0 \quad \forall i = 1, \dots, n$$
$$\Delta_{\beta_i}(u_i - x'_i) - \beta'_i\Delta_{x_i} - \theta\Delta_{\beta_i}\Delta_{x_i} = 0 \quad \forall i = 1, \dots, n$$

Since $\theta \in (0, \epsilon)$ these conditions hold if and only if for all $i = 1, \dots, n$:

$$\Delta_{\alpha_i}\Delta_{x_i} = 0 \quad , \quad \Delta_{\beta_i}\Delta_{x_i} = 0 \tag{4.7}$$
$$\Delta_{\alpha_i}(x'_i - l_i) + \alpha'_i\Delta_{x_i} = 0 \quad , \quad \Delta_{\beta_i}(u_i - x'_i) - \beta'_i\Delta_{x_i} = 0 \tag{4.8}$$

Noticing that $x'_i + \theta\Delta_{x_i} < u_i$ for all $i \in L \cup N$ and for $\theta > 0$ small enough, from the complementarity conditions $(\beta'_i + \theta\Delta_{\beta_i})(u_i - x'_i - \theta\Delta_{x_i}) = 0$ it yields $\beta'_i + \theta\Delta_{\beta_i} = 0$; analogously, we also have $\alpha'_i + \theta\Delta_{\alpha_i} = 0$ for all $i \in U \cup N$. Since $\theta > 0$, for (4.3) and (4.4) these conditions imply:

$$\Delta_{\alpha_i} = 0 \ \forall i \in N \cup U \quad , \quad \Delta_{\beta_i} = 0 \ \forall i \in L \cup N$$

so that:

$$\Delta_{\alpha_i}(x'_i - l_i) = \Delta_{\beta_i}(u_i - x'_i) = 0 \quad \forall i = 1, \dots, n$$

and the result is proved. □

Note that from the first and the second equations of (4.6) and from the positive definiteness of Q we obtain again $\Delta_x \neq 0$ and:

$$\Delta_\lambda = \Delta_x^T Q \Delta_x > 0$$

while from (4.3), (4.4) and (4.6) we have:

$$\Delta_{\alpha_i} \geq 0 \ \forall i \in L \cup E \text{ such that } \alpha'_i = 0$$
$$\Delta_{\beta_i} \geq 0 \ \forall i \in U \cup E \text{ such that } \beta'_i = 0$$

From the two last conditions of (4.6) it yields:

$$\Delta_{x_i} = 0 \ \forall i \in L \ s.t.\ \alpha'_i > 0 \ , \ \forall i \in U \ s.t.\ \beta'_i > 0$$

As a conclusion, we have the following explicit solution, depending on Δ_x, of the multipliers in (4.6):

$$\begin{aligned}
\Delta_\lambda &= \Delta_x^T Q \Delta_x \\
\Delta_{\alpha_i} &= \begin{cases} 0 & \forall i \in N \cup U \\ Q_i\Delta_x - d_i\Delta_\lambda & \forall i \in L \\ \max\{0, Q_i\Delta_x - d_i\Delta_\lambda\} & \forall i \in E \end{cases} \\
\Delta_{\beta_i} &= \begin{cases} 0 & \forall i \in L \cup N \\ d_i\Delta_\lambda - Q_i\Delta_x & \forall i \in U \\ \max\{0, d_i\Delta_\lambda - Q_i\Delta_x\} & \forall i \in E \end{cases}
\end{aligned}$$

In order to determine vector Δ_x it is worth using the partitions $L = L_p \cup L_0$ and $U = U_p \cup U_0$ defined as follows:

$$L_p = \{i \in L : \ \alpha'_i > 0\} \quad , \quad L_0 = \{i \in L : \ \alpha'_i = 0\}$$
$$U_p = \{i \in U : \ \beta'_i > 0\} \quad , \quad U_0 = \{i \in U : \ \beta'_i = 0\}$$

Vector Δ_x is then the unique solution (recall that $\overline{P}_{\xi'+\theta}$ is a strictly convex problem) of the following system:

$$\begin{cases} Q_i\Delta_x = d_i\Delta_x^T Q\Delta_x \ \forall i \in N \\ d^T\Delta_x = 1 \\ \Delta_{x_i} = 0 \ \forall i \in L_p \ , \ \forall i \in U_p \ , \ \forall i \in E \\ (Q_i\Delta_x - d_i\Delta_x^T Q\Delta_x)\Delta_{x_i} = 0 \quad \forall i \in L_0 \cup U_0 \\ Q_i\Delta_x \geq d_i\Delta_x^T Q\Delta_x \ , \ \Delta_{x_i} \geq 0 \quad \forall i \in L_0 \\ d_i\Delta_x^T Q\Delta_x \geq Q_i\Delta_x \ , \ \Delta_{x_i} \leq 0 \quad \forall i \in U_0 \end{cases} \tag{4.9}$$

which is suitable for values of $\theta \geq 0$ which verify the following conditions:

$$\begin{array}{lcl} \textit{feasibility conditions} & : & l \leqq (x' + \theta\Delta_x) \leqq u \\ \textit{optimality conditions} & : & \alpha'_i + \theta\Delta_{\alpha_i} \geq 0 \ \forall i \in L, \ \beta'_i + \theta\Delta_{\beta_i} \geq 0 \ \forall i \in U \end{array}$$

Notice that only the components Δ_{x_i} such that $i \in L_0 \cup N \cup U_0$ are left to be determined in (4.9). Notice also that the assumption $\xi' < \xi_{max}$ implies $L \cup N \neq \emptyset$. We are finally able to determine the values of $\theta > 0$ which guarantee both the optimality and the feasibility of $x'(\theta)$. From the feasibility conditions we have:

$$\theta \leq \hat{F} = \min \left\{ \min_{i \in L_0 \cup N:\ \Delta_{x_i} > 0} \left\{ \frac{u_i - x'_i}{\Delta_{x_i}} \right\}, \min_{i \in N \cup U_0:\ \Delta_{x_i} < 0} \left\{ \frac{l_i - x'_i}{\Delta_{x_i}} \right\} \right\}$$

Let us recall that whenever $\xi' < \xi_{max}$ then $\Delta_x \neq 0$, $L \cup N \neq \emptyset$ and hence $\hat{F} > 0$. On the other hand, denoting $L_p^- = \{i \in L_p : \ \Delta_{\alpha_i} < 0\}$ and $U_p^- = \{i \in U_p : \ \Delta_{\beta_i} < 0\}$, from the optimality conditions we have:

$$\theta \leq \hat{O} = \begin{cases} \min \left\{ \min_{i \in L_p^-} \left\{ \frac{\alpha'_i}{-\Delta_{\alpha_i}} \right\}, \min_{i \in U_p^-} \left\{ \frac{\beta'_i}{-\Delta_{\beta_i}} \right\} \right\} & \text{if } L_p^- \cup U_p^- \neq \emptyset \\ +\infty & \text{if } L_p^- \cup U_p^- = \emptyset \end{cases}$$

so that $\hat{O} > 0$.

As a consequence, $x'(\theta)$ is an optimal level solution for all θ such that:

$$0 \leq \theta \leq \theta_m = \min \left\{ \hat{F}, \hat{O} \right\}$$

where $\theta_m > 0$ whenever $\xi' < \xi_{max}$.

4.3 Box constraints and diagonal matrix Q

In the case matrix Q is diagonal several further improvements can be done to the solution method. In particular, it is possible to explicitly determine all of the multipliers in the Karush-Kuhn-Tucker system. The results which are going to be given in this subsection have been already stated in [8].

First of all, notice that the parametric subproblem $\overline{P}_\xi$ becomes:

$$\overline{P}_\xi : \begin{cases} \min \frac{1}{2} x^T D x + q^T x + q_0 \\ x \in X_\xi^B = \{x \in \Re^n : l \leqq x \leqq u, d^T x + d_0 = \xi\} \end{cases}$$

where $D = \text{diag}(\delta_1, \ldots, \delta_n) \in \Re^{n \times n}$, $\delta_i > 0 \ \forall i = 1, \ldots, n$. As a preliminary result, it is worth pointing out that it is possible to determine explicitly the optimal value for all the variables x_i such that $d_i = 0$ (see [8]).

Theorem 4.1 *Consider the subproblems $\overline{P}_\xi$, with $\xi \in [\xi_{min}, \xi_{max}]$. Then, for all indices $i = 1, \ldots, n$ such that $d_i = 0$ the optimal level solution is reached at*

$$x_i^* = \begin{cases} l_i & \textit{if } -\frac{c_i}{\delta_i} \leq l_i \\ u_i & \textit{if } -\frac{c_i}{\delta_i} \geq u_i \\ -\frac{c_i}{\delta_i} & \textit{if } l_i < -\frac{c_i}{\delta_i} < u_i \end{cases}$$

As a consequence, the feasible region can be reduced *a priori*, without loosing the optimal solution, by means of the following commands:

- if $-\frac{c_i}{\delta_i} \leq l_i$ then set $u_i := l_i$,
- if $-\frac{c_i}{\delta_i} \geq u_i$ then set $l_i := u_i$,
- if $l_i < -\frac{c_i}{\delta_i} < u_i$ then set $l_i := -\frac{c_i}{\delta_i}$ and $u_i := -\frac{c_i}{\delta_i}$.

From now on we can then assume that:

$$i \in E \text{ for all } i = 1, \ldots, n \text{ such that } d_i = 0. \tag{4.10}$$

where $L \cup U \cup N \cup E = \{1, \ldots, n\}$ with:

$$\begin{array}{lcl} L = \{i: \ l_i = x'_i < u_i\} & , & N = \{i: \ l_i < x'_i < u_i\} \\ U = \{i: \ l_i < x'_i = u_i\} & , & E = \{i: \ l_i = x'_i = u_i\} \end{array}$$

Notice that assumption (4.10) implies also $X^B_{\xi_{min}} = \{l\}$ and $X^B_{\xi_{max}} = \{u\}$, so that there is no need to solve the starting quadratic problem $\overline{P}_{\xi_{start}}$ in *Step 1)* since we can simply choose $x' := l$.

By means of assumption (4.10) and the results stated in the previous subsections, the following explicit solutions of the Karush-Kuhn-Tucker systems can be determined (see also [8]):

$$\begin{array}{rcl} \lambda' & = & \left\{ \begin{array}{cl} \frac{\delta_i x'_i + q_i}{d_i}, \text{ for any } i \in N & \text{if } N \neq \emptyset \\ \min_{i \in L} \left\{ \frac{\delta_i l_i + q_i}{d_i} \right\} & \text{if } N = \emptyset \text{ and } L \neq \emptyset \\ \max_{i \in U} \left\{ \frac{\delta_i u_i + q_i}{d_i} \right\} & \text{if } N = \emptyset \text{ and } L = \emptyset \end{array} \right. \\ \alpha'_i & = & \left\{ \begin{array}{cl} 0 & \forall i \in N \cup U \\ \delta_i l_i + q_i - \lambda' d_i & \forall i \in L \\ \max\{0, \delta_i l_i + q_i - \lambda' d_i\} & \forall i \in E \end{array} \right. \\ \beta'_i & = & \left\{ \begin{array}{cl} 0 & \forall i \in L \cup N \\ \lambda' d_i - \delta_i u_i - q_i & \forall i \in U \\ \max\{0, \lambda' d_i - \delta_i u_i - q_i\} & \forall i \in E \end{array} \right. \end{array}$$

By defining the following further partition of indices $L = L^+ \cup L^0$:

$$L^+ = \{i \in L: \ \alpha'_i > 0\} \quad , \quad L^0 = \{i \in L: \ \alpha'_i = 0\}$$

we also have that (see [8]):

$$\begin{array}{rcl} \Delta_\lambda & = & \dfrac{1}{\sum_{i \in L^0 \cup N} \frac{1}{\delta_i} d_i^2} > 0 \\ \Delta_{x_i} & = & \left\{ \begin{array}{cl} 0 & \text{if } i \in L^+ \cup U \cup E \\ \Delta_\lambda \frac{d_i}{\delta_i} > 0 & \text{if } i \in L^0 \cup N \end{array} \right. \end{array}$$

$$\Delta_{\alpha_i} = \begin{cases} 0 & \text{if } i \notin L^+ \\ -\Delta_\lambda d_i < 0 & \text{if } i \in L^+ \end{cases}$$

$$\Delta_{\beta_i} = \begin{cases} 0 & \text{if } i \in L \cup N \\ \Delta_\lambda d_i \geq 0 & \text{if } i \in U \cup E \end{cases}$$

Finally, notice that it is:

$$\hat{F} = \begin{cases} \min_{i \in L^0 \cup N} \left\{ \frac{u_i - x'_i}{\Delta_{x_i}} \right\} & \text{if } L^0 \cup N \neq \emptyset \\ 0 & \text{if } L^0 \cup N = \emptyset \end{cases}$$

$$\hat{O} = \begin{cases} \min_{i \in L^+} \left\{ \frac{-\alpha'_i}{\Delta_{\alpha_i}} \right\} & \text{if } L^+ \neq \emptyset \\ +\infty & \text{if } L^+ = \emptyset \end{cases}$$

where $\theta_m > 0$ if and only if $x' \neq u$.

As a conclusion, let us point out that:

- in *Step 2a)* all of the parameters of the solution algorithm can be computed explicitly without the need of solving any further system;
- since $x'(\theta)$ and $\alpha'(\theta)$ are, respectively, increasing and decreasing with respect to θ (this follows from the nonnegativity of Δ_x and the nonpositivity of Δ_α) then, it can be proved that the algorithm stops after no more than $2n - 1$ iterations (see [8]).

In [8] the solution method has been studied and specified for the particular case of $f(y) = y$, $g_1(y) = 1$ and $g_2(y) = \frac{1}{2}ky^2$; the obtained algorithm has been fully implemented in a symbolic calculus environment and the results related to a deep computational test have been presented.

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