Università degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all'Economia

Report n. 280

ADJOINTS OF A MATRIX

Franco Fineschi and Riccardo Giannetti

Pisa, Gennaio 2006
- Stampato in Proprio –

Via Cosimo Ridolfi, 10 - 56124 PISA - Tel. Segr. Amm. 050 2216231 Segr. Stud. 050 2216317 Fax 050 2216375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

ADJOINTS OF A MATRIX

FRANCO FINESCHI* AND RICCARDO GIANNETTI†

Abstract. A set of adjoints for a matrix is defined. Basic concepts in linear algebra and matrix analysis, such as the most important inverses of a matrix and the differential of a determinant, are expressed by adjoints.

Key words. Matrix, Adjoints, Derivative.

AMS subject classifications. 15A15, 15A09.

1. Introduction. Basic definitions such as cofactors and determinants are extended to rectangular matrices. The Volume of an arbitrary matrix, introduced by A.Ben-Israel in [3], is extended by a general definition of determinant. Some new relations, such as "the square of product of the nonnull eigenvalues of a normal matrix A of rank r is its determinant", follow. The definition of adjoint matrices A^{ad}, for an arbitrary matrix A, is especially useful in order to obtain formal expressions of general and special (1,2)-inverses of A by entries of the same A. For an example, the Moore-Penrose inverse is well known by numerical methods (as reductions, full rank factorization, SVD, small perturbations and so on) or by the property "$r\times r$ minors of A^+ are proportional to the corresponding minors of A'", see e.g. Miao [11], Miao and Ben-Israel [12] and Bapat [1]. As for square A, the formula $|A|^{-1} Adj\, A$ proves useful in theoretical considerations, as well as the same form $|A|_r^{-1} A^{ad}$ (Theo. 5.11) for the MP-inverse is also useful. The examples for small matrices emphasize the possibility for writing inverses with symbolical entries. If A is square then also characteristic polynomials of inverses are obtained. Two new inverses of A are presented. A link is obtained between EP and Compound matrices. In the final section the differential of determinant for general matrix functions is produced. For the sake of brevity, other results in multilinear algebra are not considered here. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex field respectively, $R(A)$ the range of the matrix A, $N(A)$ the nullity of A.

2. Rectangular determinant and cofactors. The multindex I_r^n of length r is defined by

$$I_r^n = \{(i_1, \dots, i_r):\ 1 \le i_1 < \dots < i_r \le n\}$$

besides we define, for a fixed natural number k,

$$(I_r^n)_k = \{(i_1, \dots i_p \dots, i_r):\ 1 \le i_1 < \dots < i_p = k < \dots < i_r \le n, \\ \text{where } 1 \le k \le n\}$$

finally, $(I_r^n) - i$ is the set I_r^{n-1} on the indices $1, \dots, i-1, i+1, \dots, n$. If $A \in \mathfrak{F}^{m\times n}$, with $\mathfrak{F}$ a field with $char(\mathfrak{F}) \neq 2$, then A_β^α is the submatrix of A determined by rows

*Department of Business and Social Studies, Siena University, Piazza S.Francesco, 53100 Siena, Italia (fineschi@unisi.it).

†Department of Business and Social Studies, Siena University, Piazza S.Francesco, 53100 Siena, Italia (giannetti@unisi.it).

indexed by α and columns indexed by β, $\alpha \in I_r^m$, $\beta \in I_r^n$. The determinant of A is denoted by $|A|$, its rank by $r(A)$.
For the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, with $r(A) = 2$, consider

$$Vol^2 A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}^2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}^2 = |A_{12}^{12}|^2 + |A_{13}^{12}|^2 + |A_{23}^{12}|^2$$

the sum is defined as the square volume of a matrix A. A.Ben-Israel shows in [3] that $VolA$ generalizes to arbitrary matrices the absolute value of the determinant, also see [7]. More generally, given the 2×3 matrix A with $r(A) = 2$ we consider the linear form in the parameters $\lambda_\beta^\alpha \in \mathfrak{F}$, $\Delta_r A = |A_{12}^{12}|\lambda_{12}^{12} + |A_{13}^{12}|\lambda_{13}^{12} + |A_{23}^{12}|\lambda_{23}^{12}$. For an arbitrary matrix

DEFINITION 2.1. Let $\Delta_r : \mathfrak{F}^{m \times n} \to (\mathfrak{F}^{\binom{m}{r}\binom{n}{r}})^*$ be a map defined by

$$\Delta_r A = \sum_{\alpha,\beta} |A_\beta^\alpha| \lambda_\beta^\alpha, \quad \alpha \in I_r^m,\ \beta \in I_r^n.$$

The map Δ_r has the following properties

(i) If $\lambda_\beta^\alpha = |A_\beta^\alpha|$ then $\Delta_r A$ is a symmetric map with respect to the rows and columns of A

(ii) $\Delta_r A = 0$, for any λ_β^α iff $|A_\beta^\alpha| = 0$, for any $\alpha \in I_r^m$, $\beta \in I_r^n$.

(iii) If $m = r = r(A)$, then $|A_\beta^\alpha|$, in Δ_m, are the Plücker coordinates of the subspace spanned by the rows of A.

DEFINITION 2.2. Any $m \times n$ matrix A is said to be r-nonsingular iff it satisfies anyone of these two equivalent conditions

(i) $\Delta_r A \neq 0$

(ii) there exists $\alpha \in I_r^m$, $\beta \in I_r^n$ such that $|A_\beta^\alpha| \neq 0$, i.e. $r(A) = r$.

Generally $\Delta_r A = \Delta_r A'$ only if A is square.
If $\lambda_\beta^\alpha = |A_\beta^\alpha|$, or, in the complex field $\mathbb{C}$ $\lambda_\beta^\alpha = |\overline{A_\beta^\alpha}|$, for any $\alpha \in I_r^m$, $\beta \in I_r^n$, then we denote $\Delta_r A$ by $det_r A$ or $|A|_r$. This notation is justified by A.Ben-Israel's result

$$(det_r A)^{\frac{1}{2}} = VolA = \prod_{i=1}^{r} \sigma_i$$

where σ_i are the nonnull singular values of A, and by the next theorems.
The definition of cofactor α_{ij} can be extended to an arbitrary matrix.

EXAMPLE 2.3. *If A is a 2×3 matrix with $r(A) = 2$*

$$\alpha_{11} = \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \end{vmatrix}^{\underline{1}} = \begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} \lambda_{12}^{12} + \begin{vmatrix} 1 & 0 \\ a_{21} & a_{23} \end{vmatrix} \lambda_{13}^{12}$$

$$\alpha_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \end{vmatrix}^{\underline{2}} = \begin{vmatrix} a_{11} & a_{12} \\ 1 & 0 \end{vmatrix} \lambda_{12}^{12} + \begin{vmatrix} a_{11} & a_{13} \\ 1 & 0 \end{vmatrix} \lambda_{13}^{12}$$

and so on. More generally

DEFINITION 2.4. Let $A \in \mathfrak{F}^{m\times n}$ be a matrix with $r(A) = r$, the cofactor α^r_{ij} of the entry a_{ij} of A, for $m < n$, is defined by

$$\alpha^r_{ij} = \begin{vmatrix} a_{11} & \dots & \dots & \dots & a_{1n} \\ & \dots & \dots & \dots & \\ 0 & \dots & 1 & \dots & 0 \\ & \dots & \dots & \dots & \\ a_{m1} & \dots & \dots & \dots & a_{mn} \end{vmatrix}^{\underline{i}}$$

$= \Delta_r A:\ a_{ix} = 0$ for $x = 1, \dots, j-1, j+1, \dots, n$, $a_{ij} = 1$ and $\lambda^\alpha_\beta \neq 0$ only if $i \in \alpha$

$= \sum_{\substack{i_1 \dots i \dots i_r \\ j_1 \dots j \dots j_r}} \lambda^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r} |(A^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r})$ i-th row $= e'_j|$

where in the last determinant the i-th row is $e'_j = (0, \dots, 1, \dots, 0)$, i.e. the j-th unit vector. The sum is over all $i_1, \dots, i, \dots, i_r \in (I^m_r)_i$ and $j_1, \dots, j, \dots, j_r \in (I^n_r)_j$. Similarly, for $m > n$,

$$\alpha^r_{ij} = \begin{vmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & 1 & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & 0 & \dots & a_{mn} \end{vmatrix}_{\underline{j}}$$

The following relation is immediate between cofactors of square and rectangular matrices

$$\alpha^r_{ij} = \sum_{\substack{i_1 \dots i \dots i_r \\ j_1 \dots j \dots j_r}} \lambda^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r} \alpha(A^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r})_{ij} \tag{2.1}$$

where $\alpha(A^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r})_{ij}$ denotes the classic cofactor of a_{ij} for the matrix $(A^{i_1 \dots i \dots i_r}_{j_1 \dots j \dots j_r})$.

EXAMPLE 2.5. *For* $A \in \mathfrak{F}^{3\times 4}$ *and* $r(A) = 2$

$$\alpha^r_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}^{\underline{2}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & 0 \end{vmatrix}^{12} +$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}^{23} = \left(\begin{vmatrix} a_{11} & a_{13} \\ 0 & 1 \end{vmatrix} \lambda^{12}_{13} + \begin{vmatrix} a_{12} & a_{13} \\ 0 & 1 \end{vmatrix} \lambda^{12}_{23} + \begin{vmatrix} a_{13} & a_{14} \\ 1 & 0 \end{vmatrix} \lambda^{12}_{34} \right) + \left(\begin{vmatrix} 0 & 1 \\ a_{31} & a_{33} \end{vmatrix} \lambda^{23}_{13} + \begin{vmatrix} 0 & 1 \\ a_{32} & a_{33} \end{vmatrix} \lambda^{23}_{23} + \begin{vmatrix} 1 & 0 \\ a_{33} & a_{34} \end{vmatrix} \lambda^{23}_{34} \right)$$

As in the example, it will be useful to write the numbers of the rows (columns) involved in the expansion of the cofactor like apexes (indexes).
Let $C = (\alpha^r_{ij})$ the r-cofactor matrix of A. The transpose of C, i.e. the r-adjoints of A will be denoted by A^{ad}.

REMARK 2.6. The set A^{ad} exists for any matrix A .
THEOREM 2.7. *If* $A \in \mathfrak{F}^{n\times n}$ *and* $r(A) = n$, *then for* $\lambda_\beta^\alpha = |A_\beta^\alpha|$

$$A^{ad} = Adj\, A\, |A|$$
$$A^{ad} = Adj\, A\, |\overline{A}| \text{ if } \mathfrak{F} = \mathbb{C}$$

Proof. It is immediate by $\alpha_{ij}^r = \alpha_{ij}|A|$, $i, j = 1\ldots, n$, where α_{ij} is the classic cofactor of a_{ij} in A. □

For any scalar λ, the scalar λ^+ is defined by $\lambda^+ = \begin{cases} \lambda^{-1} & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \end{cases}$.

The r-cofactors have several properties
THEOREM 2.8. *Let* $A \in \mathfrak{F}^{m\times n}$, *where* $r(A) = r$, *then*

(i) $\alpha_{ij}^r = \frac{\partial \Delta_r A}{\partial a_{ij}}$
as a special case
$\alpha_{ij}^r = \frac{1}{2}\frac{\partial |A|_r}{\partial a_{ij}} \qquad \lambda_\beta^\alpha = |A_\beta^\alpha|$

(ii) $\sum_{j=1}^n a_{ij}\alpha_{ij}^r = \sum_{\substack{i_1\ldots i\ldots i_r \\ \beta}} \lambda_\beta^{i_1\ldots i\ldots i_r}|A_\beta^{i_1\ldots i\ldots i_r}| \qquad i_1\ldots i\ldots i_r \in (I_r^m)_i,\ \beta \in I_r^n,\ m \leq n$

(iii) $a_{h1}\alpha_{i1}^r + a_{h2}\alpha_{i2}^r + \ldots + a_{hn}\alpha_{in}^r = \Delta_r A^{ih} - \frac{m-r}{m}$
$\lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$, $m < n$ *(det by rows)*

(iv) $\alpha_{1k}^r a_{1j} + \alpha_{2k}^r a_{2j} + \ldots + \alpha_{mk}^r a_{mj} = \Delta_r A_{kj} - \frac{n-r}{n}$
$\lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$, $m < n$ *(det by rows)*

(v) $a_{i1}\alpha_{i1}^r + a_{i2}\alpha_{i2}^r + \ldots + a_{in}\alpha_{in}^r = \frac{r}{m} \qquad \lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$

(vi) $\alpha_{1j}^r a_{1j} + \alpha_{2j}^r a_{2j} + \ldots + \alpha_{mj}^r a_{mj} = \frac{r}{n} \qquad \lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$

the matrix A^{ih} *is* A *where the i-th row is substituted by the h-th row, the matrix* A_{kj} *is* A *where the k-th column is substituted by the j-th column. Similarly for* $m > n$.

Proof. (i)

$$\frac{\partial \Delta_r A}{\partial a_{ij}} = \frac{\partial(\sum_{\alpha,\beta}\lambda_\beta^\alpha|A_\beta^\alpha|)}{\partial a_{ij}} = \sum_{\alpha,\beta}\frac{\lambda_\beta^\alpha \partial|A_\beta^\alpha|}{\partial a_{ij}} = \sum_{\substack{i_1,\ldots,i,\ldots,i_r \\ j_1,\ldots,j,\ldots,j_r}} \lambda_\beta^\alpha \frac{\partial}{\partial a_{ij}}|A_{j_1,\ldots,j,\ldots,j_r}^{i_1,\ldots,i,\ldots,i_r}|$$
$$= \sum_{\substack{i_1,\ldots,i,\ldots,i_r \\ j_1,\ldots,j,\ldots,j_r}} \lambda_\beta^\alpha|(A_{j_1,\ldots,j,\ldots,j_r}^{i_1,\ldots,i,\ldots,i_r})_{i-th\ row = e_j'}| = \alpha_{ij}^r$$

(ii)By the linearity of the r-cofactors. (iii)

$$a_{h1}\alpha_{i1}^r + a_{h2}\alpha_{i2}^r + \ldots + a_{hn}\alpha_{in}^r = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{h1} & \cdots & a_{hn} \\ \cdots & \cdots & \cdots \\ a_{h1} & \cdots & a_{hn} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}^{\underline{i}}$$
$$= (\binom{m}{r}\binom{n}{r})^{-1}(\sum_{\substack{i_1,\ldots,i_r \\ h_1,\ldots,h_r \\ j_1,\ldots,j_r}} |A_{j_1,\ldots,j_r}^{i_1,\ldots,i_r}||A_{j_1,\ldots,j_r}^{h_1,\ldots,h_r}|^{-1})$$

where $i_1,\dots,i_r \in (I_r^m)_h - i$, $h_1,\dots,h_r \in (I_r^m)_i - h$, $j_1,\dots,j_r \in I_r^n$. By $(I_r^m)_h - i$ we denote all ordered subsets $i_1,\dots,i_r$ of $1,2,\dots,i-1,i+1,\dots m$ which have h. Furthermore

$$\Delta_r \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{h1} & \cdots & a_{hn} \\ \cdots & \cdots & \cdots \\ a_{h1} & \cdots & a_{hn} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = (\binom{m}{r}\binom{n}{r})^{-1}(\sum_{\substack{i_1,\dots,i_r\\ h_1,\dots,h_r\\ j_1,\dots,j_r}} |A^{i_1,\dots,i_r}_{j_1,\dots,j_r}||A^{h_1,\dots,h_r}_{j_1,\dots,j_r}|^{-1} + \binom{n}{r}\binom{m-1}{r})$$

then (iii). (iv) Similar to (iii). (v) A special case of (iii) for $h = i$. (vi) A special case of (iv) for $k = j$. □

THEOREM 2.9. *If $A \in \mathfrak{F}^{m\times n}$, where $r(A) = r$, then*

$$\Delta_r A = \frac{1}{r}(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\alpha_{ij}^r) = \frac{1}{r} tr\, AA^{ad} = \frac{1}{r} tr\, A^{ad}A$$

Proof. Suppose $m > n$. Since the cofactor α_{ij}^r is linear with respect to j-th column, then

$$\begin{aligned}
&\sum_{i=1}^m \sum_{j=1}^n a_{ij}\alpha_{ij}^r \\
&= \{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{\underline{1}} + \cdots + \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{\underline{1}}\} + \cdots \\
&\cdots + \{\begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{21} & \cdots & a_{2\,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{m\,n-1} & 0 \end{vmatrix}_{\underline{n}} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & 0 \\ a_{21} & \cdots & a_{2\,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{m\,n-1} & a_{mn} \end{vmatrix}_{\underline{n}}\} \\
&= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{\underline{1}} + \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{\underline{n}} \\
&= \sum_{\alpha\beta_1} |A^{\alpha}_{\beta_1}|\lambda^{\alpha}_{\beta_1} + \cdots + \sum_{\alpha\beta_n} |A^{\alpha}_{\beta_n}|\lambda^{\alpha}_{\beta_n} = r\sum_{\alpha\beta} |A^{\alpha}_{\beta}|\lambda^{\alpha}_{\beta} = r\Delta_r A
\end{aligned}$$

for $\alpha \in I_r^n$, $\beta \in I_r^m$, $\beta_i \in (I_r^m)_i$. In the same way we may prove the property for $m < n$. □

REMARK 2.10. For $m = n = r$, theo. 2.9 is Laplace's theorem.

3. Products. In 1950 A.Horn proved the following, see [9]

Let $A \in \mathfrak{F}^{m\times p}$ and $B \in \mathfrak{F}^{p\times n}$ be given, let $q = min\{n,p,q\}$ and denote the ordered singular values of A, B, and AB by $\sigma_1(A) \geq \cdots \geq \sigma_{min\{m,p\}}(A) \geq 0$, $\sigma_1(B) \geq \cdots \sigma_{min\{p,n\}}(B) \geq 0$, and $\sigma_1(AB) \geq \cdots \geq \sigma_{min\{m,n\}}(AB) \geq 0$. Then

$$\prod_{i=1}^{k} \sigma_i(AB) \leq \prod_{i=1}^{k} \sigma_i(A)\sigma_i(B) \qquad k = 1, \dots, q$$

if $n = p = m$, then equality holds for $k = n$.
In 1992 A.Ben-Israel proved the following, see [3]
Let $A, B \in \mathfrak{F}^{n\times n}$, where $R(A) = R(B')$, $R(A') = R(B)$, then

$$\prod_{i=1}^{r} \sigma_i(AB) = \prod_{i=1}^{r} \sigma_i(A)\sigma_i(B)$$

We present a more general proposition below

THEOREM 3.1. *Let $A \in \mathfrak{F}^{m\times n}$ and $B \in \mathfrak{F}^{n\times p}$ be matrices and let $r(A) = r(B) = r(AB) = n$, $\Delta_n A = \sum |A^{\alpha}_{1,\dots,n}|\eta^{\alpha}_{1,\dots,n}$, $\Delta_n B = \sum |B^{1,\dots,n}_{\beta}|\theta^{1,\dots,n}_{\beta}$, $\Delta_n(AB) = \sum |(AB)^{\alpha}_{\beta}|\lambda^{\alpha}_{\beta}$, then*

(i) $\Delta_n(AB) = \Delta_n A \cdot \Delta_n B$ iff $\lambda^{\alpha}_{\beta} = (\eta^{\alpha}_{1,\dots,n})(\theta^{1,\dots,n}_{\beta})$
as a special case
(ii) $det_n(AB) = det_n(A) \cdot det_n(B)$ i.e. $\prod_{i=1}^{n} \sigma_i(AB) = \prod_{i=1}^{n} \sigma_i(A)\sigma_i(B)$
if $\eta^{\alpha}_{1,\dots,n} = |A^{\alpha}_{1,\dots,n}|$, $\theta^{1,\dots,n}_{\beta} = |B^{1,\dots,n}_{\beta}|$, $\lambda^{\alpha}_{\beta} = |(AB)^{\alpha}_{\beta}|$
for any $\alpha \in I^m_n$, $\beta \in I^p_n$.

Proof. (i) by $r(A) = r(B) = r(AB) = n$, $(AB)^{\alpha}_{\beta} = (A^{\alpha}_{1,\dots,n})(B^{1,\dots,n}_{\beta})$, then

$$\begin{aligned}\Delta_n(AB) &= \textstyle\sum_{\alpha,\beta} |(AB)^{\alpha}_{\beta}|\lambda^{\alpha}_{\beta} = (\sum_{\alpha,\beta} |A^{\alpha}_{1,\dots,n}||B^{1,\dots,n}_{\beta}|\lambda^{\alpha}_{\beta})\\ &= \textstyle(\sum_{\alpha} |A^{\alpha}_{1,\dots,n}|\eta^{\alpha}_{1,\dots,n})(\sum_{\beta} |B^{1,\dots,n}_{\beta}|\theta^{1,\dots,n}_{\beta}) = \Delta_n A \cdot \Delta_n B\end{aligned}$$

(ii) $det_n(AB) = \sum_{\alpha,\beta} |(AB)^{\alpha}_{\beta}|^2 = (\sum_{\alpha} |A^{\alpha}_{1,\dots,n}|^2)(\sum_{\beta} |B^{1,\dots,n}_{\beta}|^2) = det_n A \cdot det_n B$
where the summation $\sum_{\alpha,\beta}$ is over all subsets α of I^m_n and β of I^p_n. □

REMARK 3.2. For $m = p$, ii) is the Binet-Cauchy formula.

THEOREM 3.3. *Let $A \in \mathfrak{F}^{m\times n}$ be a matrix*
(i) if $r(A) = m$, then $(AA^{ad})_{ij} = \sum_{k=1}^{n} a_{ik}\alpha^r_{jk} = \delta_{ij}\Delta_m A$
(ii) if $r(A) = n$, then $(A^{ad}A)_{ij} = \sum_{k=1}^{m} \alpha^r_{ki}a_{kj} = \delta_{ij}\Delta_n A$

Proof. (i) $(AA^{ad})_{ij} = \sum_{k=1}^{n} a_{ik}\alpha^r_{jk}$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}^{j} + \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{i2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{vmatrix}^{j} + \cdots + \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{vmatrix}^{j}$$

$$= \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{vmatrix} = \delta_{ij}\Delta_m A \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(ii) is proved similarly. □

COROLLARY 3.4. *Let $A \in \mathfrak{F}^{m\times n}$ be a matrix*

(i) if $r(A) = m$, then $AA^{ad} = (\Delta_m A)I_m$

(ii) if $r(A) = n$, then $A^{ad}A = (\Delta_n A)I_n$

Proof. Immediate by Theo.3.3. □

By the Binet-Cauchy formula, the minors of the product of two rectangular matrices are expressed in terms of minors of the factors. If $A \in \mathfrak{F}^{m\times n}$ and $B \in \mathfrak{F}^{n\times p}$, then

$$|(AB)^{i_1,\dots,i_q}_{j_1,\dots,j_q}| = \sum_{k_1,\dots,k_q} |B^{k_1,\dots,k_q}_{j_1,\dots,j_q}||A^{i_1,\dots,i_q}_{k_1,\dots,k_q}|$$

where $i_1,\dots,i_q \in I^m_q$, $j_1,\dots,j_q \in I^p_q$ and the sum over all $k_1,\dots,k_q \in I^n_q$.
We can prove the following statement

THEOREM 3.5. *Let $A \in \mathfrak{F}^{m\times n}$ and $B \in \mathfrak{F}^{n\times p}$ be matrices and let $m \le p$. In the product AB consider the submatrix $((AB)^{i_1,\dots,i_n}_{j_1,\dots,j_n})_{i-th\ row=e'_j}$, where $i_1,\dots,i_n \in I^m_n$ and $j_1,\dots,j_n \in I^p_n$, and the unit vector $e'_j = 0,\dots,1,\dots,0$ substitutes for the i-th row, then*

$$\begin{aligned}&|((AB)^{i_1,\dots,i_n}_{j_1,\dots,j_n})_{i-th\ row=e'_j}|\\ &= \textstyle\sum_{1,\dots,k,\dots,n} |(B^{1,\dots,k,\dots,n}_{j_1,\dots,j,\dots,j_n})_{k-th\ row=e'_j}||(A^{i_1,\dots,i,\dots,i_n}_{1,\dots,k,\dots,n})_{k-th\ column=e_j}|\end{aligned}$$

where the sum is over all $1,\dots,k,\dots,n$ for $k = 1,\dots,n$.

The relation $(AB)^{ad} = B^{ad}A^{ad}$ is known only for square matrices, see e.g., Magnus and Neudecker [10]. The next theorem generalizes this relation to rectangular matrices.

THEOREM 3.6. *Let $A \in \mathfrak{F}^{m\times n}$ and $B \in \mathfrak{F}^{n\times p}$ be matrices and let $r(A) = r(B) = r(AB) = n$, $\Delta_n A = \sum |A^{\alpha}_{1,\dots,n}|\eta^{\alpha}_{1,\dots,n}$, $\Delta_n B = \sum |B^{1,\dots,n}_{\beta}|\theta^{1,\dots,n}_{\beta}$, $\Delta_n(AB) = \sum |(AB)^{\alpha}_{\beta}|\lambda^{\alpha}_{\beta}$, then*

(i) $(AB)^{ad} = B^{ad}A^{ad}$ iff $\lambda^{\alpha}_{\beta} = (\eta^{\alpha}_{1,\dots,n})(\theta^{1,\dots,n}_{\beta})$
as special cases

(ii) $(AB)^{ad} = B^{ad}A^{ad}$ if $\eta^{\alpha}_{1,\dots,n} = |A^{\alpha}_{1,\dots,n}|$, $\theta^{1,\dots,n}_{\beta} = |B^{1,\dots,n}_{\beta}|$, $\lambda^{\alpha}_{\beta} = |(AB)^{\alpha}_{\beta}|$

(iii) $(AB)^{ad} = B^{ad}A^{ad}$ if $\eta^{\alpha}_{1,\dots,n} = |A^{\alpha}_{1,\dots,n}|^{-1}\binom{m}{n}^{-1}$, $\theta^{1,\dots,n}_{\beta} = |B^{1,\dots,n}_{\beta}|^{-1}\binom{p}{n}^{-1}$, $\lambda^{\alpha}_{\beta} = |(AB)^{\alpha}_{\beta}|^{-1}(\binom{p}{n}\binom{m}{n})^{-1}$

Proof. (i) Let $r(A) = r(B) = r(AB) = n$. Suppose $m \le p$. Let the cofactor $\gamma^n_{ij} = (AB)^{ad}_{ij}$ be the ji-entry of $(AB)^{ad}$, β^n_{rs} the sr-entry of B^{ad} and α^n_{tv} the vt-entry of A^{ad}. The relation

$$\gamma^n_{ij} = \beta^n_{1j}\alpha^n_{i1} + \dots + \beta^n_{nj}\alpha^n_{in}$$

must be proven. Using Theo. 3.5

$$\begin{aligned}
&\textstyle\sum_{r=1}^{n} \beta_{rj}^{n}\alpha_{ir}^{n} = \sum_{j_1,\dots,j,\dots,j_n}\{|(B_{j_1,\dots,j,\dots,j_n}^{k,2,\dots,n})_{k-th\ row=e'_j}|\theta_{j_1,\dots,j,\dots,j_n}^{1,\dots,n}\}\cdot\\
&\cdot\textstyle\sum_{i_1,\dots,i,\dots,i_n}\{|(A_{k,2,\dots,n}^{i_1,\dots,i,\dots,i_n})_{k-th\ column=e_j}|\eta_{1,\dots,n}^{i_1,\dots,i,\dots,in}\}+\cdots\\
&\cdots+\textstyle\sum_{j_1,\dots,j,\dots,j_n}\{|(B_{j_1,\dots,j,\dots,j_n}^{1,\dots,n-1,k})_{k-th\ row=e'_j}|\theta_{j_1,\dots,j,\dots,j_n}^{1,\dots,n}\}\cdot\\
&\cdot\textstyle\sum_{i_1,\dots,i,\dots,i_n}\{|(A_{1,\dots,n-1,k}^{i_1,\dots,i,\dots,i_n})_{k-th\ column=e_j}|\eta_{1,\dots,n}^{i_1,\dots,i,\dots,in}\}=\\
&=\textstyle\sum_{\substack{i_1,\dots,i,\dots,i_n\\ j_1,\dots,j,\dots,j_n}}\{\sum_{1,\dots,k,\dots,n}|(B_{j_1,\dots,j,\dots,j_n}^{1,\dots,k,\dots,n})_{k-th\ row=e'_j}|\cdot\\
&\cdot|(A_{1,\dots,k,\dots,n}^{i_1,\dots,i,\dots,i_n})_{k-th\ column=e_j}|\}\cdot\theta_{j_1,\dots,j,\dots,j_n}^{1,\dots,n}\cdot\eta_{1,\dots,n}^{i_1,\dots,i,\dots,in}=\\
&=\textstyle\sum_{\substack{i_1,\dots,i,\dots,i_n\\ j_1,\dots,j,\dots,j_n}}|(AB)_{j_1,\dots,j,\dots,j_n}^{i_1,\dots,i,\dots,i_n})_{i-th\ row=e'_j}|\lambda_{j_1,\dots,j,\dots,j_n}^{i_1,\dots,i,\dots,i_n}=\gamma_{ij}^{n}
\end{aligned}$$

(ii) and (iii), it is sufficient to notice that

$$|B_{j_1,\dots,j,\dots,j_n}^{1,\dots,n}||A_{1,\dots,n}^{i_1,\dots,i,\dots,i_n}| = |(AB)_{j_1,\dots,j,\dots,j_n}^{i_1,\dots,i,\dots,i_n}|$$

Similarly the theorem is proved for $m \geq p$. □

4. Isometries. THEOREM 4.1. *Let $V \in \mathbb{C}^{m\times r}$ be a matrix and let $V^*V = I_r$, then*

(i) $det_r V = |V^*V| = 1$

(ii) $V^{ad} = V^* \qquad \lambda_{1,\dots,r}^{\alpha} = |\overline{V_{1,\dots,r}^{\alpha}}|$

(iii) $V^{ad}V = I_r \qquad \lambda_{1,\dots,r}^{\alpha} = |\overline{V_{1,\dots,r}^{\alpha}}|^{+}\binom{m}{k}^{-1}$

where k, $1 \leq k \leq \binom{m}{r}$, is the number of nonnull $|\overline{V_{1,\dots,r}^{\alpha}}|$ for $\alpha \in I_r^m$.

Proof. (i) By the Binet-Cauchy formula

$$1 = |V^*V| = \sum_{\alpha} |(V^*)_{\alpha}^{1,\dots,r}| \cdot |V_{1,\dots,r}^{\alpha}| = det_r V$$

where the sum is over all $\alpha \in I_r^m$.

(ii) By r-cofactor definition and Binet-Cauchy formula

$$(V^{ad})_{ij} = \alpha_{ji}^{r} = \begin{vmatrix} v_{11} & \cdots & 0 & \cdots & v_{1r}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ v_{j1} & \cdots & 1 & \cdots & v_{jr}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ v_{m1} & \cdots & 0 & \cdots & v_{mr}\end{vmatrix}_{\underline{i}}$$

$$\begin{aligned}
&= \textstyle\sum_{j_1,\dots,j,\dots,j_r}|(V_{1,\dots,r}^{j_1,\dots,j,\dots,j_r})_{i-th\ column=e_j}|\cdot|\overline{V_{1,\dots,r}^{j_1,\dots,j,\dots,j_r}}|\\
&= \left|V^*\begin{bmatrix} v_{11} & \cdots & 0 & \cdots & v_{1r}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ v_{j1} & \cdots & 1 & \cdots & v_{jr}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ v_{m1} & \cdots & 0 & \cdots & v_{mr}\end{bmatrix}\right| = \begin{vmatrix} 1 & \cdots & \overline{v_{j1}} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \overline{v_{ji}} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \overline{v_{jr}} & \cdots & 1\end{vmatrix}\\
&= \overline{v_{ji}} = (V^*)_{ij}
\end{aligned}$$

(iii) By Cor. 3.4 $V^{ad}V = (\Delta_r V)I_r$. Since $\Delta_r V = 1$ for $\lambda_{1,\dots,r}^{\alpha} = |\overline{V_{1,\dots,r}^{\alpha}}|^{+}\binom{m}{k}^{-1}$, then $V^{ad}V = I_r$. □

THEOREM 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix of rank r. Suppose $\lambda_1, \dots, \lambda_{r_1}$ are the nonnull eigenvalues of A, $r_1 \le r$, then*

$$(\prod_{i=1}^{r_1} \lambda_i)^2 = det_r A$$

Proof A is unitarily diagonalizable, then a subunitary matrix U, $U^*U = I_r$, exists such that $A = U\Lambda U^*$, where $\Lambda = diag(\lambda_1, \dots, \lambda_{r_1})$. By Theorem 3.1 and 4.1, it follows that

$$det_r A = det_r(U\Lambda U^*) = det_r U\,|\Lambda|^2\, det_r U^* = |\Lambda|^2 = (\prod_{i=1}^{r_1} \lambda_i)^2 \qquad \square$$

5. Inverses. ◇ By the following propositions, for any matrix A, an X such that $AXA = A$ or $XAX = X$ or both, is an r-adjoint of A.

THEOREM 5.1. *Let $A \in \mathbb{C}^{m\times n}$ and $r(A) = r$. Then $(\Delta_r A)^{-1}A^{ad}$ is the set of $(2)-$inverses of A iff $r(A^{ad}) \le r$.* *Proof* Suppose $r(A^{ad}) \le r$. By Theo.3.1 , 3.6 and Cor.3.4

$$\begin{aligned}&(\Delta_r A)^{-1}A^{ad}A(\Delta_r A)^{-1}A^{ad} = (\Delta_r(V\Sigma W^*))^{-2}W^{*ad}\Sigma^{ad}V^{ad}V\Sigma W^*W^{*ad}\Sigma^{ad}V^{ad}\\ &= (\Delta_r V\, \Delta_r\Sigma\, \Delta_r W^*)^{-2}W^{*ad}\Sigma^{ad}(\Delta_r V)\,\Sigma(\Delta_r W^*)\,\Sigma^{ad}V^{ad}\\ &= (\Delta_r V\, \Delta_r\Sigma\, \Delta_r W^*)^{-1}(\Delta_r\Sigma)^{-1}\, W^{ad}(\Delta_r\Sigma)\,\Sigma^{ad}V^{ad} = (\Delta_r A)^{-1}A^{ad}\end{aligned}$$

Conversely, if $(\Delta A)^{-1}A^{ad}$ is a (2)-inverse of A, then

$$r((\Delta A)^{-1}A^{ad}) = r(A^{ad}) = r(A^{ad}\cdot A\cdot A^{ad}) \le r(A) = r \qquad \square$$

By a similar proof we obtain the dual proposition

THEOREM 5.2. *Let $A \in \mathbb{C}^{m\times n}$ and $r(A) = r$. Then $(\Delta_r A)^{-1}A^{ad}$ is the set of $(1)-$inverses of A iff $r(A^{ad}) \ge r$.* Then, the characterization of the (1,2)-inverse set of A is immediate.

COROLLARY 5.3. *Let $A \in \mathbb{C}^{m\times n}$ and $r(A) = r(A^{ad}) = r$, then $(\Delta_r A)^{-1}A^{ad}$ are the (1,2)-inverses of A.* Explicitly, by Theorem 2.8 the (1,2)-inverse set of a matrix A is given by

$$(\Delta_r A)^{-1}\begin{bmatrix}\frac{\partial\Delta_r A}{\partial a_{11}} & \dots & \frac{\partial\Delta_r A}{\partial a_{m1}}\\ \dots & \dots & \dots\\ \frac{\partial\Delta_r A}{\partial a_{1n}} & \dots & \frac{\partial\Delta_r A}{\partial a_{mn}}\end{bmatrix} \qquad \lambda^\alpha_\beta \in \mathbb{C}$$

where the matrix has rank r.

For $A \in \mathbb{C}^{m\times n}$, it is known that , if $A^- \in \mathbb{C}^{n\times m}$ is a (1)-inverse of A, then A^- has the property that A^-b is a solution of $Ax = b$ for every $b \in \mathbb{C}^m$ for which $Ax = b$ is consistent. Then, the next statement follows

COROLLARY 5.4. *Any consistent linear system $Ax = b$, $b \ne 0$, where $A \in \mathbb{C}^{m\times n}$, $r(A) = r$, has as set of solutions*

$$x_j = \frac{\alpha^r_{1j}b_1 + \dots + \alpha^r_{mj}b_m}{\Delta_r A} \qquad j = 1, \dots, n$$

for any $\lambda_\beta^\alpha \in \mathbb{C}$ *such that* $r(A^{ad}) \geq r$.

Proof. $(\Delta_r A)^{-1} A^{ad}$ are the (1)-inverses of A, then the solutions of $Ax = b$ are $x = (\Delta_r A)^{-1} A^{ad} b$. □

REMARK 5.5. The Cor. 5.4 is an evident Cramer's rule generalization. ◇ It is known, see Wei [14], that the most common generalized inverses of a matrix A are all generalized inverse $A^{(2)}_{TS}$ with prescribed range T and null space S. By Theorem 5.1, $A^{(2)}_{TS}$ is the matrix $(\Delta_r A)^{-1} A^{ad}$ such that

(i) $(\Delta_r A)^{-1} A^{ad} At = t \qquad t \in T$

(ii) $(\Delta_r A)^{-1} A^{ad} As = 0 \qquad s \in S$

We give a direct and numerically efficient expression for $A^{(2)}_{TS}$ by A and arbitrary bases of T and S.

THEOREM 5.6. *Let* $A \in \mathbb{C}^{m \times n}$ *and* $r(A) = r$. *Let* $t_1, \dots, t_{r_1}$ *and* $s_1, \dots, s_{m-r_1}$ *be arbitrary bases of* T *and* S *respectively,* $r_1 \leq r$. *Denote by* $t = \alpha_1 t_1 + \cdots + \alpha_{r_1} t_{r_1}$ *and* $s = \beta_1 s_1 + \cdots + \beta_{m-r_1} s_{m-r_1}$ *arbitrary vectors in* T *and* S. *Then*

$$A^{(2)}_{TS} = (y^* y)^+ t y^* (I - (s^* s)^+ s s^*)$$

where $y = (I - (s^* s)^+ s s^*) At$.

Proof. It is known, see e.g., [13] or [10] , that the matrix equation $XB = C$ has a solution iff $CB^+ B = C$, in which case the general solution is $X = CB^+ + Q(I - BB^+)$, where Q is an arbitrary matrix of appropriate order. $N(A^{(2)}_{TS}) = S$ implies $A^{(2)}_{TS} s = 0$, that is

$$A^{(2)}_{TS} = Q(I - ss^+) = Q(I - (s^* s)^+ s s^*) \qquad Q \in \mathbb{C}^{n \times m}$$

By $R(A^{(2)}_{TS}) = T$ we have $A^{(2)}_{TS} At = t$ and substituting $Q(I - (s^* s)^+ s s^*) At = t$. Denote $y = (I - (s^* s)^+ s s^*) At$, then $Qy = t$, so

$$Q = ty^+ + Q_1 (I - yy^+) \qquad Q_1 \in \mathbb{C}^{n \times m}$$

assuming $Q_1 = 0$, it follows $Q = (y^* y)^+ t y^*$, whence $A^{(2)}_{TS} = (y^* y)^+ t y^* (I - (s^* s)^+ s s^*)$. □

EXAMPLE 5.7. *For the matrix* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, *let* $t = (t_1, t_2, t_3)$ *and* $s = (s_1, s_2)$ *be arbitrary vectors in* T *and* S *respectively, then*

$$A^{(2)}_{TS} = (s_2(a_{11}t_1 + a_{12}t_2 + a_{13}t_3) - s_1(a_{21}t_1 + a_{22}t_2 + a_{23}t_3))^{-1} \begin{bmatrix} s_2 t_1 & -s_1 t_1 \\ s_2 t_2 & -s_1 t_2 \\ s_2 t_3 & -s_1 t_3 \end{bmatrix}$$

◇ Campbell and Meyer write in [5] "one of the major shortcomings of the Moore Penrose inverse is that the 'reverse order law', does not always hold, that is $(AB)^+$ is not always $B^+ A^+$". A known result is the next proposition, see [5]
Suppose that $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$, and $r(A) = r(B) = n$. Then $(AB)^+ = B^+ A^+$.
We want to define a different kind of inverse of A such that the reverse order law holds.

DEFINITION 5.8. Let $A \in \mathbb{C}^{m\times n}$ be a matrix of full rank. The (1,2)-inverse of A defined by

$$A^r = (\Delta_n A)^{-1} A^{ad} \qquad \text{for } \lambda^\alpha_\beta = 1$$

is a reverse order inverse of A.

THEOREM 5.9. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ such that $r(A) = r(B) = r(AB) = n$, then*

$$\begin{aligned} &AA^rA = A \\ &A^rAA^r = A^r \\ &(AB)^r = B^rA^r \end{aligned}$$

Proof. By Cor. 5.3 the (1,2)-inverses of A and B are $(\Delta_n A)^{-1}A^{ad}$ and $(\Delta_n B)^{-1}B^{ad}$ respectively. Then the following relation must be proven

$$(\Delta_n AB)^{-1}(AB)^{ad} = \Delta_n A\, \Delta_n B\, B^{ad}A^{ad} \tag{5.1}$$

By Theorem 3.1 and 3.6 it follows that

$$(AB)^{ad} = B^{ad}A^{ad} \quad \text{and} \quad \Delta_n(AB) = \Delta_n A\, \Delta_n B \quad \text{iff} \quad \lambda^\alpha_\beta = (\eta^\alpha_{1,\dots,n})(\theta^{1,\dots,n}_\beta)$$

and as a special case, if $(\eta^\alpha_{1,\dots,n}) = |\overline{A^\alpha_{1,\dots,n}}|$, $(\theta^{1,\dots,n}_\beta) = |\overline{B^{1,\dots,n}_\beta}|$, $\lambda^\alpha_\beta = |\overline{(AB)^\alpha_\beta}|$. Then the relation 5.1 is immediate. The condition $\lambda^\alpha_\beta = (\eta^\alpha_{1,\dots,n})(\theta^{1,\dots,n}_\beta)$ is satisfied by $\lambda^\alpha_\beta = (\eta^\alpha_{1,\dots,n}) = (\theta^{1,\dots,n}_\beta) = 1$, that is, if the chosen (1,2)-inverses of A, B, AB are reverse order inverses. For $(\eta^\alpha_{1,\dots,n}) = |\overline{A^\alpha_{1,\dots,n}}|$, $(\theta^{1,\dots,n}_\beta) = |\overline{B^{1,\dots,n}_\beta}|$, $\lambda^\alpha_\beta = |\overline{(AB)^\alpha_\beta}|$ the reverse order law holds. In this case, by the next Theorem 5.11, the (1,2)-inverses of A, B, AB are $A^+, B^+, (AB)^+$ respectively. □

EXAMPLE 5.10. *For the 2×3 matrix $A = (a_{ij})$, $r(A) = 2$*

$$A^r = (|A^{12}_{12}| + |A^{12}_{13}| + |A^{12}_{23}|)^{-1} \cdot \begin{bmatrix} a_{22} + a_{23} & -a_{12} - a_{13} \\ -a_{21} + a_{23} & a_{11} - a_{13} \\ -a_{21} - a_{22} & a_{11} + a_{12} \end{bmatrix}$$

It is immediate that AA^r and A^rA are projectors. By $(A^{ad})' = (A')^{ad}$ it follows that $(A^r)' = (A')^r$.

◇ The next theorem gives an elementary and intuitive form for the MP inverse, moreover it generalizes to any matrix the well known

$$A^{-1} = \frac{1}{detA} Adj\, A$$

THEOREM 5.11. *Let $A \in \mathbb{C}^{m\times n}$, $r(A) = r$ and let $\lambda^\alpha_\beta = |\overline{A^\alpha_\beta}|$, then*

$$A^+ = \frac{1}{det_r A} A^{ad} = \frac{1}{det_r A} \begin{bmatrix} \alpha^r_{11} & \dots & \alpha^r_{m1} \\ \dots & \dots & \dots \\ \alpha^r_{1n} & \dots & \alpha^r_{mn} \end{bmatrix}$$

Proof. Let $A = V\Sigma W^*$ be a singular value decomposition, where $V^*V = W^*W = I_r$ and Σ is a $r \times r$ positive definite diagonal matrix. By Theo. 3.6

$$A^{ad} = (V\Sigma W^*)^{ad} = W^{*ad}\Sigma^{ad}V^{ad}$$

By Theorem 3.1

$$det_r A = det_r(V\Sigma W^*) = det_r V|\Sigma|^2 det_r W^*$$

By Theorem 4.1

$$\begin{array}{l} (\frac{1}{det_r V}V^{ad})V = I_r, \quad W^*(\frac{1}{det_r W^*}W^{*ad}) = I_r \\ VV^{ad} = VV^*, \quad W^{*ad}W^* = WW^* \end{array}$$

Then $A^+ = \frac{1}{det_r A}A^{ad}$ satisfies the MP axioms

(1)

$$\begin{array}{l} A(det_r A)^{-1}A^{ad}A = (det_r(V\Sigma W^*))^{-1}V\Sigma W^*W^{*ad}\Sigma^{ad}V^{ad}V\Sigma W^* \\ = (det_r V\,|\Sigma|^2\, det_r W^*)^{-1}V\Sigma det_r W^*\Sigma^{ad} det_r V\,\Sigma W^* \\ = |\Sigma|^{-1}V\Sigma\, Adj\Sigma\,\Sigma W^* = V\Sigma W^* = A \end{array}$$

(2) Similar to the previous one.

(3)

$$\begin{array}{l} A(det_r A)^{-1}A^{ad} = (det_r(V\Sigma W^*))^{-1}V\Sigma W^*W^{*ad}\Sigma^{ad}V^{ad} \\ = VV^{ad} = VV^* \text{ is Hermitian} \end{array}$$

(4) Similar to the previous one.

□

EXAMPLE 5.12. *For the* 3×3 *matrix* $A = (a_{ij})$, $r(A) = 2$

$$A^+ = (|A^{12}_{12}| + |A^{13}_{12}| + |A^{23}_{12}| + |A^{12}_{13}| + |A^{13}_{13}| + |A^{23}_{13}| + |A^{12}_{23}| + |A^{13}_{23}| + |A^{23}_{23}|)^{-1}\cdot$$

$$\begin{bmatrix} a_{22}|A^{12}_{12}| + a_{23}|A^{12}_{13}| + a_{32}|A^{13}_{12}| + a_{33}|A^{13}_{13}| & -a_{12}|A^{12}_{12}| - a_{13}|A^{12}_{13} + a_{32}|A^{23}_{12}| + a_{33}|A^{23}_{13}| & -a_{12}|A^{13}_{12}| - a_{13}|A^{13}_{13}| - a_{22}|A^{23}_{12}| - a_{23}|A^{23}_{13}| \\ -a_{21}|A^{12}_{12}| + a_{23}|A^{12}_{23}| - a_{31}|A^{13}_{12}| + a_{33}|A^{13}_{23}| & a_{11}|A^{12}_{12}| - a_{13}|A^{12}_{23}| - a_{31}|A^{23}_{12}| + a_{33}|A^{23}_{23}| & a_{11}|A^{13}_{12}| - a_{13}|A^{13}_{23}| + a_{21}|A^{23}_{12}| - a_{23}|A^{23}_{23}| \\ -a_{21}|A^{12}_{13}| - a_{22}|A^{12}_{23}| - a_{31}|A^{13}_{13}| - a_{32}|A^{13}_{23}| & a_{11}|A^{12}_{13}| + a_{12}|A^{12}_{23}| - a_{31}|A^{23}_{13}| - a_{32}|A^{23}_{23}| & a_{11}|A^{13}_{13}| + a_{12}|A^{13}_{23}| + a_{21}|A^{23}_{13}| + a_{22}|A^{23}_{23}| \end{bmatrix}$$

REMARK 5.13. Theorem 5.11 generalizes the known relation, see e.g., [5]

$$A^+ = \frac{1}{tr\, A^*A}A^*$$

where $r(A) = 1$.

COROLLARY 5.14. *Let* $A \in \mathbb{C}^{n\times n}$, $r(A) = r$. *The characteristic polynomial of* A^+ *is*

$$p_{A^+}(t) = t^n - (2|A|_r)^{-1}\sum_{i=1}^n \frac{\partial |A|_r}{\partial a_{ii}}t^{n-1} + (2|A|_r)^{-2}\sum_{i_1,i_2}\begin{vmatrix} \frac{\partial |A|_r}{\partial a_{i_1 i_1}} & \frac{\partial |A|_r}{\partial a_{i_1 i_2}} \\ \frac{\partial |A|_r}{\partial a_{i_2 i_1}} & \frac{\partial |A|_r}{\partial a_{i_2 i_2}} \end{vmatrix} t^{n-2} -$$

$$\cdots \pm (2|A|_r)^{-r}\sum_{i_1,\dots,i_r}\begin{vmatrix} \frac{\partial |A|_r}{\partial a_{i_1 i_1}} & \dots & \frac{\partial |A|_r}{\partial a_{i_1 i_r}} \\ \dots & \dots & \dots \\ \frac{\partial |A|_r}{\partial a_{i_r i_1}} & \dots & \frac{\partial |A|_r}{\partial a_{i_r i_r}} \end{vmatrix} t^{n-r}$$

where $i_1, i_2 \in I_2^n$, $i_1, i_2, i_3 \in I_3^n$, $\ldots, i_1, \ldots, i_r \in I_r^n$.

Proof. By (i) of Theorem 2.8 $A^+ = |A|_r^{-1}(a_{ji}^r) = |A|_r^{-1}(\frac{1}{2}\frac{\partial |A|_r}{\partial a_{ji}})$, then the expression for $p_{A^+}(t)$ follows. □

COROLLARY 5.15. *Denote by* $B_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}}$ *the* $n \times m$ *matrix defined by*

$$B_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}} = \begin{cases} B_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} = Adj\,(A_{j_1,\ldots,j_r}^{i_1,\ldots,i_r})|A_{j_1,\ldots,j_r}^{i_1,\ldots,i_r}| \\ \text{remaining entries of } B_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}} \text{ are null} \end{cases}$$

then

$$A^+ = \frac{1}{det_r A} \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}} B_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}}$$

Proof By Theorem 5.11 and 2.1

$$A_{ji}^+ = \frac{1}{det_r A}\alpha_{ij}^r = \frac{1}{det_r A} \sum_{\substack{i_1,\ldots,i,\ldots,i_r\\ j_1,\ldots,j,\ldots,j_r}} |A_{j_1,\ldots,j,\ldots,j_r}^{i_1,\ldots,i,\ldots,i_r}|\alpha(A_{j_1,\ldots,j,\ldots,j_r}^{i_1,\ldots,i,\ldots,i_r})_{ij}$$

so

$$A^+ = \frac{1}{det_r A} \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}} B_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_r}} \qquad \square$$

COROLLARY 5.16. *If* $A \in \mathbb{C}^{m\times n}$, $r(A) = r$, *then*

$$|A|_r \cdot |A^+|_r = |AA^+|_r = |A^+A|_r = 1$$

Proof. By the Theorem 3.1 we have $|A|_r \cdot |A^+|_r = |AA^+|_r = |A^+A|_r$. Since $AA^+ = V\Sigma W^*(det_r V|\Sigma|^2 det_r W^*)^{-1}W^{*ad}\Sigma^{ad}V^{ad} = V\Sigma\frac{Adj\,\Sigma}{|\Sigma|}V^{ad} = VV^{ad}$, then $|AA^+|_r = |VV^{ad}|_r = |V|_r|V^{ad}|_r = 1$. □

This is a partially known result, see Ben-Israel [3], obtained by Plücker coordinates and compound matrices.

COROLLARY 5.17. *Let* P *be a symmetric and idempotent matrix, i.e. an orthogonal projector, where* $r(P) = r$, *then*

$$|P|_r = 1$$

Proof. Since P is symmetric and idempotent, then $P = P^+$ and $|P|_r \cdot |P^+|_r = |P|_r^2 = 1$. □

COROLLARY 5.18. *For any* $A \in \mathbb{C}^{m\times n}$, $r(A) = r$

$$|A^{ad}|_r = |A|_r^{2r-1}$$

Proof. Since $r(A) = r(A^+)$, $|A^+|_r = ||A|_r^{-1}A^{ad}|_r = |A|_r^{-2r} \cdot |A^{ad}|_r$ and by Cor. 5.16, $|A^+|_r = |A|_r^{-1}$. □

The basic identity $AA^{ad} = A^{ad}A = |A|I_n$ for square matrices is generalized by

COROLLARY 5.19. *Let $A \in \mathbb{C}^{m\times n}$ have a singular value decomposition $A = V\Sigma W^*$, then*

(i) $AA^{ad} = VV^*|A|_r$

(ii) $A^{ad}A = WW^*|A|_r$

Proof.

(i) $AA^{ad} = AA^+|A|_r = V\Sigma W^*W\Sigma^{-1}V^*|A|_r = VV^*|A|_r = VV^{ad}|A|_r$

(ii) $A^{ad}A = A^+A|A|_r = W\Sigma^{-1}V^*V\Sigma W^*|A|_r = WW^*|A|_r$

□

COROLLARY 5.20. *Let $A \in \mathbb{C}^{m\times n}$ and $r(A) = n$. Suppose $m \geq n$ and set $\alpha = 1,\dots,m$, $\beta = 1,\dots,n$. Then*

$$(det_n A)^n = \sum_{\alpha_i} |A_\beta^{\alpha_i}||(A^{ad})_{\alpha_i}^\beta| \tag{5.2}$$

where the sum is over all the subsets α_i of α that have n elements.

Proof. By the Binet-Cauchy formula

$$|A^+A| = \frac{1}{(det_n A)^n}\sum_{\alpha_i} |A_\beta^\alpha||(A^{ad})_{\alpha_i}^\beta|$$

using Cor. 5.16, we have $|A^+A| = 1$ and 5.2. □

◇ For any $A \in \mathbb{C}^{m\times n}$, $r(A) = r$, denote by A^Δ the matrix A^{ad} for $\lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$. The next proposition characterizes the unique $(R(A^\Delta), N(A^\Delta))$-generalized inverse for A (or inverse of prescribed range $R(A^\Delta)$, null space $N(A^\Delta)$).

THEOREM 5.21. *For $A \in \mathbb{C}^{m\times n}$, $r(A) = r$,*

$$A^\Delta = A^{ad} \qquad \lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$$

is the $(R(A^\Delta), N(A^\Delta))$-generalized inverse for A.

Proof. Let $A = V\Sigma W^*$ be a singular value decomposition, where $V^*V = W^*W = I_r$ and Σ a $r\times r$ positive definite diagonal matrix. By (iii) of Theorem 4.1 and (iii) of Theorem 3.6 it follows that $A^\Delta = W^{*ad}\Sigma^{-1}V^{ad}$ for $\lambda_\beta^\alpha = |A_\beta^\alpha|^+(\binom{m}{r}\binom{n}{r})^{-1}$. Then

$$AA^\Delta A = V\Sigma W^*W^{*ad}\Sigma^{-1}V^{ad}V\Sigma W^* = V\Sigma W^* = A$$

and similarly $A^\Delta AA^\Delta = A^\Delta$. □

It is straightforward to prove the following properties

THEOREM 5.22. *Let $A \in \mathbb{C}^{m\times n}$, $r(A) = r$. Then*

(i) $\Delta_r A = \Delta_r A^\Delta = 1, \quad |A^\Delta|_r = \frac{1}{|A|_r} = |A^+|_r$

(ii) $(A')^\Delta = (A^\Delta)'$

(iii) $(A^\Delta)^\Delta = A$

(iv) $(\lambda A)^{\Delta} = \lambda^{+} A^{\Delta} \qquad \lambda \in \mathbb{C}$
(v) $(AA^{\Delta})^{\Delta} = AA^{\Delta}, \qquad (A^{\Delta}A)^{\Delta} = A^{\Delta}A$
(vi) $A^{+}A = A^{*}(AA^{*})^{\Delta}A, \qquad AA^{+} = A(A^{*}A)^{\Delta}A^{*}$
(vii) $AA^{\Delta} = AA^{*}(AA^{*})^{\Delta}, \qquad A^{\Delta}A = A^{*}A(A^{*}A)^{\Delta}$

THEOREM 5.23. *Let $A \in \mathbb{C}^{m\times n}$, $r(A) = r$. Suppose $m < n$, then the range of A^{Δ} is characterized by*

$$R(A^{\Delta}) = \{(x_1, \dots, x_n) : ((\textstyle\sum_{i=2}^{n} \Delta_r A_{i1}) - n + r)x_1 + \\ +((\textstyle\sum_{\substack{i=1\\ i\neq 2}}^{n} \Delta_r A_{i2}) - n + r)x_2 + \dots + ((\textstyle\sum_{i=1}^{n-1} \Delta_r A_{in}) - n + r)x_n = 0\}$$

the matrix A_{kj} is A where the k-th column is substituted by the j-th column. Similarly for $m > n$.

Proof. For any $x \in R(A^{\Delta})$ is $A^{\Delta}(Ax) = x$, then

$$A^{\Delta}(Ax) = \begin{cases} (\sum_{i=1}^{m} a_{i1}\alpha_{i1}^{r})x_1 + \dots + (\sum_{i=1}^{m} a_{in}\alpha_{i1}^{r})x_n & = x_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (\sum_{i=1}^{m} a_{i1}\alpha_{in}^{r})x_1 + \dots + (\sum_{i=1}^{m} a_{in}\alpha_{in}^{r})x_n & = x_n \end{cases}$$

by (iv) and (vi) of Theorem 2.8

$$\begin{cases} (\frac{r}{n} - 1)x_1 + (\Delta_r A_{12} - \frac{n-r}{n})x_2 + \dots + (\Delta_r A_{1n} - \frac{n-r}{n})x_n & = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (\Delta_r A_{n1} - \frac{n-r}{n})x_1 + (\Delta_r A_{n2} - \frac{n-r}{n})x_2 + \dots + (\frac{r}{n} - 1)x_n & = 0 \end{cases}$$

$$\left(\begin{bmatrix} 0 & \Delta_r A_{12} & \cdots & \Delta_r A_{1n} \\ \dots & \dots & \dots & \dots \\ \Delta_r A_{n1} & \Delta_r A_{n2} & \cdots & 0 \end{bmatrix} + \frac{r-n}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \cdots & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

multiplying by the row vector $(1, 1, \dots, 1)$

$$\left(\left(\sum_{i=2}^{n} \Delta_r A_{i1}, \sum_{\substack{i=1\\ i\neq 2}}^{n} \Delta_r A_{i2}, \dots, \sum_{i=1}^{n-1} \Delta_r A_{in}\right) + (r - n)(1, 1, \dots, 1)\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

Then the expression for the range of A^{Δ} follows. □

THEOREM 5.24. *Let $A \in \mathbb{C}^{n\times n}$, $r(A) = r$. The characteristic polynomial of A^{Δ} is*

$$p_{A^{\Delta}}(t) = t^n - \sum_{i=1}^{n} \frac{\partial \Delta_r A}{\partial a_{ii}} t^{n-1} + \sum_{i_1,i_2} \begin{vmatrix} \frac{\partial \Delta_r A}{\partial a_{i_1 i_1}} & \frac{\partial \Delta_r A}{\partial a_{i_1 i_2}} \\ \frac{\partial \Delta_r A}{\partial a_{i_2 i_1}} & \frac{\partial \Delta_r A}{\partial a_{i_2 i_2}} \end{vmatrix} t^{n-2} - \dots$$
$$\dots \pm \sum_{i_1,\dots,i_r} \begin{vmatrix} \frac{\partial \Delta_r A}{\partial a_{i_1 i_1}} & \cdots & \frac{\partial \Delta_r A}{\partial a_{i_1 i_r}} \\ \cdots & \cdots & \cdots \\ \frac{\partial \Delta_r A}{\partial a_{i_r i_1}} & \cdots & \frac{\partial \Delta_r A}{\partial a_{i_r i_r}} \end{vmatrix} t^{n-r}$$

for $\lambda_{\beta}^{\alpha} = |A_{\beta}^{\alpha}|^{+}(\binom{m}{r}\binom{n}{r})^{-1}$, where $i_1, i_2 \in I_2^n$, $i_1, i_2, i_3 \in I_3^n$, $\dots$, $i_1, \dots, i_r \in I_r^n$.

Proof. By (i) of Theorem 2.8 $A^{\Delta} = (a_{ji}^{r}) = (\frac{\partial \Delta_r A}{\partial a_{ji}})$ for $\lambda_{\beta}^{\alpha} = |A_{\beta}^{\alpha}|^{+}(\binom{m}{r}\binom{n}{r})^{-1}$, then the expression for $p_{A^{\Delta}}(t)$ follows. □

6. Drazin and Group inverses. The Drazin inverse provides solutions for systems of linear differential equations and linear difference equations. We recall the Canonical Form Representation for A and A^D.
Let $A \in \mathfrak{F}^{n\times n}$ and $Ind\,A = k$, then there exist nonsingular matrices P, C and a nilpotent matrix N of index k such that

$$A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}, \quad A^D = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

By the next proposition, the Drazin inverse of A is expressed by the set A^{ad}.

THEOREM 6.1. *Let $A \in \mathfrak{F}^{n\times n}$ and $Ind\,A = k$, then for each integer $l \geq k$*

$$A^D = (\Delta_r A^{2l+1})^{-1} A^l (A^{2l+1})^{ad} A^l$$

where $r(A^{2l+1}) = r$. *Proof* By the Canonical Form Representation

$$A^{2l+1} = P\begin{bmatrix} C^{2l+1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

Since $(\Delta_r A^{2l+1})^{-1}(A^{2l+1})^{ad}$ is a 1-inverse of A^{2l+1}, then

$$(\Delta_r A^{2l+1})^{-1}(A^{2l+1})^{ad} = P\begin{bmatrix} C^{-2l+1} & X_1 \\ X_2 & X_3 \end{bmatrix} P^{-1}$$

and by multiplying the block matrices

$$A^l(\Delta_r A^{2l+1})^{-1}(A^{2l+1})^{ad} A^l = A^D \qquad \square$$

If $Ind\,A \leq 1$, a special and important case of the Drazin inverse is the Group inverse $A^\#$ of A.

THEOREM 6.2. *Let $A \in \mathfrak{F}^{n\times n}$ be a matrix with $r(A) = r$ and $Ind\,A \leq 1$, then*

$$A^\# = A^{ad} \quad \textit{for } \lambda^\alpha_\beta = |A^\beta_\alpha|$$

for any $\alpha, \beta \in I^n_r$.

Proof. By Cor. 5.3 $(\Delta_r A)^{-1}A^{ad}$, for $r(A^{ad}) = r$, is the set of (1,2)-inverses of A. In order to obtain $A^\#$ impose the $AA^{ad} = A^{ad}A$. Let $(AA^{ad})_{ij} = \sum_{k=1}^n a_{ik}\alpha^r_{jk}$ and $(A^{ad}A)_{ij} = \sum_{k=1}^n \alpha^r_{kj}a_{kj}$ be the (i,j) entries of AA^{ad} and $A^{ad}A$ respectively, then

$(AA^{ad})_{ij} = (A^{ad}A)_{ij}$ must be proven.

$$(AA^{ad})_{ij} = a_{i1}\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^j + \cdots + a_{in}\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^j$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^j = \sum_{s_1\dots,i,\dots s_r}\begin{vmatrix} a_{s_1 1} & \cdots & a_{s_1 n} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{in} \\ \cdots & \cdots & \cdots \\ a_{s_r 1} & \cdots & a_{s_r n} \end{vmatrix}^j$$

$$= \sum_{s_1\dots,i,\dots s_r}\sum_{t_1,\dots,t_r} |A^{s_1\dots,i,\dots s_r}_{t_1,\dots,t_r}|\lambda^{s_1\dots,j,\dots s_r}_{t_1,\dots,t_r}$$

Developing by columns

$$(A^{\#}A)_{ij}$$
$$= a_{1j}\begin{vmatrix} a_{11} & \cdots & 1 & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix}_i + \cdots + a_{nj}\begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & 1 & \cdots & a_{nn} \end{vmatrix}_i$$
$$= \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}_i = \sum_{s_1,\dots,j,\dots,s_r}\begin{vmatrix} a_{1s_1} & \cdots & a_{1j} & \cdots & a_{1s_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{ns_1} & \cdots & a_{nj} & \cdots & a_{ns_r} \end{vmatrix}_i$$
$$= \sum_{s_1,\dots,j,\dots,s_r}\sum_{t_1,\dots,t_r} |A^{t_1,\dots,t_r}_{s_1,\dots,j,\dots,s_r}|\lambda^{t_1,\dots,t_r}_{s_1,\dots,i,\dots,s_r}$$

for any $(s_1,\dots,i,\dots,s_r), (s_1,\dots,j,\dots,s_r)$ and $(t_1,\dots,t_r)$ in I^n_r. Then

$$\sum_{s_1,\dots,i,\dots,s_r}\sum_{t_1,\dots,t_r} |A^{s_1,\dots,i,\dots,s_r}_{t_1,\dots,t_r}|\lambda^{s_1,\dots,j,\dots,s_r}_{t_1,\dots,t_r}$$
$$= \sum_{s_1,\dots,j,\dots,s_r}\sum_{t_1,\dots,t_r} |A^{t_1,\dots,t_r}_{s_1,\dots,j,\dots,s_r}|\lambda^{t_1,\dots,t_r}_{s_1,\dots,i,\dots,s_r}$$

holds for $\lambda^\alpha_\beta = |A^\beta_\alpha|$, where $\alpha, \beta \in I^n_r$. □

7. EP and Compound matrices. Let $A \in \mathbb{C}^{n\times n}$ be a matrix where $r(A) = r$. If $A^+A = AA^+$, or equivalently $R(A) = R(A^*)$, then A is called an EP matrix. The next theorem gives a necessary and sufficient condition for a matrix to be an EP matrix.

THEOREM 7.1. *Let $A \in \mathfrak{F}^{n\times n}$ be a matrix with $r(A) = r$ and $\Delta_r A = \sum |A^\alpha_\beta|\lambda^\alpha_\beta$, then A is an EP matrix iff $|A^\alpha_\beta| = |A^\beta_\alpha|$ for any $\alpha, \beta \in I^n_r$.*

Proof. If A is EP then A^+ satisfies $AA^+A = A$, $A^+AA^+ = A^+$, $AA^+ = A^+A$, it follows that $A^+ = A^D = A^{\#}$. By Theorem 5.11 and 6.2 $\lambda^\alpha_\beta = |A^\alpha_\beta| = |A^\beta_\alpha|$. Conversely, if $|A^\alpha_\beta| = |A^\beta_\alpha|$ for any $\alpha, \beta \in I^n_r$ then $A^+ = A^{\#}$ and $AA^+ = AA^{\#} = A^{\#}A = A^+A$ so that A must be EP. □

If A is an $m \times n$ matrix with $r(A) = r$, then the k-th compound of A is defined as an $\binom{m}{k} \times \binom{n}{k}$ matrix $A^{(k)}$ whose elements are $|A^\alpha_\beta|$ for $\alpha \in I^m_k$, $\beta \in I^n_k$, where α, β

are ordered lexicographically.
We recall, without proofs, elementary properties of the compound matrices, see e.g., Fiedler [6].

(i) Let A be a diagonal matrix with $a_{11}, \dots, a_{nn}$ the nonzero entries. Then $A^{(k)}$ is diagonal and its entries are $a_{i_1 i_1}, \dots, a_{i_k i_k}$ for any $\alpha = i_1, \dots, i_k \in I_k^n$.

(ii) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. For any positive integer k, $k \leq min(m, n, p)$, then

$$(AB)^{(k)} = A^{(k)} B^{(k)}$$

(iii) If A is an unitary matrix of order n and $1 \leq k \leq n$, then $A^{(k)}$ is unitary as well.

The last property actually holds for semiunitary matrices
If S is a semiunitary matrix $m \times r$ with $S^*S = I_r$ and if $1 \leq k \leq r$, then $S^{(k)}$ is semiunitary as well.

A link between EP and compound matrices is given by the next proposition

THEOREM 7.2. *Let $A \in \mathbb{C}^{n \times n}$ be a matrix with $r(A) = r$, then A is EP iff $A^{(r)}$ is symmetric.*

Proof. Immediate by Theorem 7.1. □

Not long ago Hartwig and Katz gave a necessary and sufficient geometric condition for the product of two EP matrices to be EP, see [8]. The next theorem shows a condition for the product AB to be EP.

THEOREM 7.3. *If $A, B \in \mathbb{C}^{n \times n}$ with $r(A) = r(B) = r(AB) = r$, $\Delta_r A = \sum_{\alpha,\beta} |A_\beta^\alpha| \lambda_\beta^\alpha$ and $\Delta_r B = \sum_{\alpha,\beta} |B_\beta^\alpha| \mu_\beta^\alpha$, then*

$$AB \text{ is EP iff } \sum_\beta |A_\beta^\alpha||B_\gamma^\beta| = \sum_\beta |A_\beta^\gamma||B_\alpha^\beta|$$

for any $\alpha, \beta, \gamma \in I_r^n$.

Proof. By Theorem 7.2 AB is EP iff $(AB)^{(r)}$ is symmetric. Since $(AB)^{(r)} = A^{(r)} B^{(r)}$ and $(A^{(r)} B^{(r)})_{ij} = \sum_\beta |A_\beta^i||B_j^\beta|$ for $i, j, \beta \in I_r^n$, then the symmetry implies $\sum_\beta |A_\beta^i||B_j^\beta| = \sum_\beta |A_\beta^j||B_i^\beta|$. □

Now we are going to extend to compound matrices properties partially known only for square matrices.

THEOREM 7.4. *Let $A \in \mathbb{C}^{m \times n}$ and $r(A) = r$. If $\sigma_1, \dots, \sigma_r$ are the singular values of A and k is a positive integer, $k \leq r$, then the singular values of $A^{(k)}$ are all the possible products $\sigma_{i_1}, \dots, \sigma_{i_k}$, where $\sigma = i_1, \dots, i_k \in I_k^r$.*

Proof. By the singular value decomposition of A there exist semiunitary matrices S and T such that $S^*S = T^*T = I_r$ and a $r \times r$ positive diagonal matrix Λ such that $A = S\Lambda^{\frac{1}{2}}T^*$. Using (ii) we have $A^{(k)} = (S\Lambda^{\frac{1}{2}}T^*)^{(k)} = S^{(k)}(\Lambda^{\frac{1}{2}})^{(k)}(T^*)^{(k)}$, where $S^{(k)}$ and $(T^*)^{(k)}$ are still semiunitary and $(\Lambda^{\frac{1}{2}})^{(k)}$ is a diagonal positive matrix of order $\binom{r}{k}$. Hence the singular values of $A^{(k)}$ are the diagonal entries of $(\Lambda^{\frac{1}{2}})^{(k)}$. By (i) we know the diagonal entries of $(\Lambda^{\frac{1}{2}})^{(k)}$ are all the numbers $\sigma_{i_1}, \dots, \sigma_{i_k}$ for any $\sigma = i_1, \dots, i_k \in I_k^r$. □

THEOREM 7.5. *Let $A \in \mathbb{C}^{m\times n}$ and $r(A) = r$, $r(A^{(k)}) = r_k$. Let k be a positive integer such that $k \leq r$, then*

$$det_{r_k} A^{(k)} = (det_r A)^{\binom{r-1}{k-1}}$$

Proof By $det_r A = \prod_{i=1}^{r} \sigma_i$ it follows that

$$det_{r_k} A^{(k)} = \prod_{\sigma} \sigma_{i_1}, \ldots, \sigma_{i_k} \quad \text{for } \sigma = i_1, \ldots, i_k,\ \sigma \in I_k^r$$

and some of the singular values may be identical. Each of the numbers σ_i occurs in $\binom{r}{k} - \binom{r-1}{k} = \binom{r-1}{k-1}$ terms on the right side. Hence

$$(\prod_{i=1}^{r} \sigma_i)^{\binom{r-1}{k-1}} = (det_r A)^{\binom{r-1}{k-1}} \qquad \square$$

THEOREM 7.6. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. If $r(A^{(k)}) = r(B^{(k)}) = r(AB)^{(k)} = \binom{n}{k}$ for positive integer k, then*

$$det_{\binom{n}{k}}(AB)^{(k)} = det_{\binom{n}{k}}(A)^{(k)} \cdot det_{\binom{n}{k}}(B)^{(k)}$$

Proof. By (i), $(AB)^{(k)} = A^{(k)}B^{(k)}$ and by Theorem 3.1. $\square$

8. Differentials. If a matrix function $F : S \to \mathbb{R}^{m\times m}$ is considered, where S is an open subset of $\mathbb{R}^{n\times q}$ and F is continuously differentiable on S, then the real valued function $|F| : S \to \mathbb{R}$, defined by $|F|(X) = |F(X)|$, is k times continuously differentiable and

$$d\,|F| = tr\ Adj(F(X))dF$$

see Magnus and Neudecker [10]. The next theorem generalizes the above known result to a matrix function $F : S \to \mathbb{R}^{m\times n}$.

THEOREM 8.1. *Let the matrix function $F : S \to \mathbb{R}^{m\times n}$ be k times continuously differentiable on open subset S of $\mathbb{R}^{p\times q}$, then likewise it is the real valued function $|F|_r : S \to \mathbb{R}$ defined by $|F|_r(X) = det_r(F(X))$ at points X where $r(F(X)) = r$. Moreover*

$$d\,|F|_r = 2\,tr\,F^{ad} d\,F$$

proof Let $\phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be the real valued function defined by $\phi(A) = det_r A$. The function ϕ is ∞ times differentiable for any $A \in \mathbb{R}^{m\times n}$. Since $det_r A = |A|_r = \sum_{\alpha,\beta} |A^{\alpha}_{\beta}|^2$, where the sum is over all $\alpha \in I_r^m$ and $\beta \in I_r^n$, then by the r-cofactor definition 2.4

$$\frac{\partial |A|_r}{\partial a_{ij}} = \frac{\partial(\sum_{\alpha,\beta} |A^{\alpha}_{\beta}|^2)}{\partial a_{ij}} = \sum_{\alpha,\beta} \frac{\partial |A^{\alpha}_{\beta}|^2}{\partial a_{ij}} = \sum_{\substack{i_1,\ldots,i,\ldots,i_r \\ j_1,\ldots,j,\ldots,j_r}} \frac{\partial}{\partial a_{ij}} |A^{i_1,\ldots,i,\ldots,i_r}_{j_1,\ldots,j,\ldots,j_r}|^2$$
$$= \sum_{\substack{i_1,\ldots,i,\ldots,i_r \\ j_1,\ldots,j,\ldots,j_r}} 2|A^{i_1,\ldots,i,\ldots,i_r}_{j_1,\ldots,j,\ldots,j_r}| \frac{\partial}{\partial a_{ij}} |A^{i_1,\ldots,i,\ldots,i_r}_{j_1,\ldots,j,\ldots,j_r}|$$
$$= 2\sum_{\substack{i_1,\ldots,i,\ldots,i_r \\ j_1,\ldots,j,\ldots,j_r}} |A^{i_1,\ldots,i,\ldots,i_r}_{j_1,\ldots,j,\ldots,j_r}||(A^{i_1,\ldots,i,\ldots,i_r}_{j_1,\ldots,j,\ldots,j_r})_{i-th\ row=e'_j}| = 2\alpha^r_{ij}$$

so

$$d\phi(A) = D\phi(A) \cdot vec\, d\,A = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial |A|_r}{\partial a_{ij}} d\,a_{ij} = 2\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}^{r} d\,a_{ij} = 2\,tr\,A^{ad} d\,A$$

Where $D\phi(A)$ denotes the Jacobian. By the $|F|_r = \phi \circ F$ and the Cauchy's rule of invariance

$$d\,|F|_r = 2\,tr\,F^{ad} d\,F \qquad \square$$

THEOREM 8.2. *Let $F : \xi \to A(\xi)$ be a matrix valued function of a scalar. If $F(\xi)$ is a differentiable $m \times n$ matrix at some point ξ, for $m < n$ and $r(F(\xi)) = r$, then*

$$D_\xi |F(\xi)|_r = 2\sum_{i=1}^{m} |(F(\xi))_i|_r^{\frac{i}{r}} = 2tr\,\frac{\partial F(\xi)}{\partial \xi} A^{ad}$$

where $(F(\xi))_i$ is the matrix that coincides with the matrix $F(\xi)$ except that every entry in the i-th row is differentiated with respect to ξ.

Proof. $D_\xi |F(\xi)|_r = D_\xi(\sum_{\alpha,\beta} |F(\xi)_\beta^\alpha|^2) = 2\sum_{\alpha,\beta} |F(\xi)_\beta^\alpha| D_\xi |F(\xi)_\beta^\alpha|$, since $D_\xi |F(\xi)_\beta^\alpha| = \sum_{i=1}^{\alpha} |(F(\xi)_\beta^\alpha)_i|$, see Horn and Johnson [9] p.491, we have

$$\begin{aligned} D_\xi |F(\xi)|_r &= 2\textstyle\sum_{\alpha,\beta}\sum_{i=1}^{\alpha} |F(\xi)_\beta^\alpha||(F(\xi)_\beta^\alpha)_i| = 2\sum_{i=1}^{n} |(F(\xi))_i|_r^{\frac{i}{r}} \\ &= 2\{\begin{vmatrix} a'_{11} & \cdots & a'_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}^{\underline{1}} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a'_{m1} & \cdots & a'_{mn} \end{vmatrix}^{\underline{m}} \} \\ &= 2\textstyle\sum_{i,j} a'_{ij}\alpha_{ij}^{r} = 2tr\,\frac{\partial F(\xi)}{\partial \xi} A^{ad} \end{aligned}$$

similarly the proof for $m > n$. $\square$

COROLLARY 8.3. *Let A be a constant $n \times n$ matrix with $r(A) = r$, then*

$$D_\xi |\xi I - A| = 2tr\,(\xi I - A)^{ad}$$

Proof. Denote by β_{ij}^{r} the cofactors of $(\xi I - A)$,

$$\begin{aligned} D_\xi |\xi I - A| &= 2\{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & \xi - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & \xi - a_{nn} \end{vmatrix}^{\underline{1}} + \cdots \\ &\cdots + \begin{vmatrix} \xi - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \xi - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}^{\underline{n}} \} = 2\textstyle\sum_{i=1}^{n} \beta_{ii}^{r} = 2tr\,(\xi I - A)^{ad} \end{aligned}$$

$\square$

REFERENCES

[1] R.B. Bapat. *Linear Algebra and Linear Models.* Springer Verlag, 2000.
[2] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications.* J.Wiley & Sons, 1974.
[3] A. Ben-Israel. A Volume Associated With $m \times n$ Matrices. *Linear Algebra Appl.*, 167:87–111, 1992.
[4] L. Berg. Three results in connection with inverse matrices. *Linear Algebra Appl.*, 84:63–77, 1986.
[5] S.L. Campbell and C.D. Meyer,Jr. *Generalized Inverses of Linear Transformations.* Pitman, London, 1979.
[6] M. Fiedler. *Special Matrices and Their Applications in Numerical Mathematics.* Martinus Nijhoff Publishers, Dordrecht, 1986.
[7] F. Fineschi,R. Giannetti and K.B. Marathe. Generalized Products and Associated Structures on Euclidean Spaces. *Int.J.Math.Educ.Sci.Technol.*, 4:493–505, 1996.
[8] R.E. Hartwig and I.J. Katz. On Product of EP Matrices. *Linear Algebra Appl.*, 252:339–345, 1997.
[9] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, 1994.
[10] J.R. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics.* J.Wiley & Sons, Chichester, 1994.
[11] J. Miao. Reflexive Generalized Inverses and Their Minors. *Linear and Multilinear Algebra,* 35:153–163, 1993.
[12] J. Miao and A. Ben-Israel. Minors of the Moore-Penrose Inverse. *Linear Algebra Appl.*, 195:191–207, 1993.
[13] C.R. Rao and S.K. Mitra. *Generalized Inverse of Matrices and its Applications.* J.Wiley & Sons, 1971.
[14] Y. Wei. A characterization and representation of the generalized inverse $A^{(2)}_{T,S}$ and its applications. *Linear Algebra Appl.*, 280:87–96, 1998.

Elenco dei report pubblicati

Anno: 1987

- ☐ n. 1 Alberto Cambini - Laura Martein, Some Optimality Conditions in Vector Optimization
- ☐ n. 2 Alberto Cambini - Laura Martein - S.Schaibel, On Maximizing a Sum of Ratios
- ☐ n. 3 Giuliano Gasparotto, On the Charnes-Cooper Transformation in linear Fractional Programming.
- ☐ n. 4 Alberto Cambini, Non-linear separation Theorems, Duality and Optimality
- ☐ n. 5 Giovanni Boletto, Indicizzazione parziale: aspetti metodologici e riflessi economici
- ☐ n. 6 Alberto Cambini - Claudio Sodini, On Parametric Linear Fractional Programming
- ☐ n. 7 Alberto Bonaguidi, Alcuni aspetti meno noti delle migrazioni in Italia
- ☐ n. 8 Laura Martein - S. Schaible, On Solving a Linear Program with one Quadratic Constraint

Anno: 1988

- ☐ n. 9 Ester Lari, Alcune osservazioni sull'equazione funzionale Ø(x,y,z)=Ø(Ø(x,y,t),t,z)
- ☐ n. 10 F. Bartiaux, Une étude par ménage des migrations des personnes âgées: comparaison des résulttats pour l'Italie et les Etats-Unis
- ☐ n. 11 Giovanni Boletto, Metodi di scomposizione del tasso di inflazione
- ☐ n. 12 Claudio Sodini, A New Algorithm for the Strictly Convex Quadratic Programming Problem
- ☐ n. 13 Laura Martein, On Generating the Set of all Efficient Points of a Bicriteria Fractional Problem
- ☐ n. 14 Laura Martein, Applicazioni della programmazione frazionaria nel campo economico-finanziario
- ☐ n. 15 Laura Martein, On the Bicriteria Maximization Problem
- ☐ n. 16 Paolo Manca, Un prototipo di sistema esperto per la consulenza finanziaria rivolta ai piccoli risparmiatori
- ☐ n. 17 Paolo Manca, Operazioni Finanziarie di Soper e Operazioni di puro Investimento secondo Teichroew-Robichek-Montalbano
- ☐ n. 18 Paolo Carraresi - Claudio Sodini, A k - Shortest Path Approach to the Minimum Cost Matching Problem.
- ☐ n. 19 Odo Barsotti - Marco Bottai, Sistemi gravitazionali e fasi di transazione della crescita Demografica
- ☐ n. 20 Giovanni Boletto, Metodi di scomposizione dell'inflazione aggregata : recenti sviluppi.
- ☐ n. 21 Marc Termote - Alberto Bonaguidi, Multiregional Stable Populatioin as a Tool for Short-term Demographic Analysis .
- ☐ n. 22 Marco Bottai, Storie familiari e storie migratorie: un'indagine in Italia
- ☐ n. 23 Maria Francesca Romano - Marco Marchi, Problemi connessi con la disomogeneità dei gruppi sottoposti a sorveglianza statistico-epidemiologica.
- ☐ n. 24 Franca Orsi, Un approcio logico ai problemi di scelta finanziaria.

Anno: 1989

- ☐ n. 25 Vincenzo Bruno, Attrazione ed entropia.
- ☐ n. 26 Giorgio Giorgi - S. Mititelu, Invexity in nonsmooth Programming.
- ☐ n. 28 Alberto Cambini - Laura Martein, Equivalence in linear fractional programming.

Anno: 1990

- ☐ n. 27 Vincenzo Bruno, Lineamenti econometrici dell'evoluzione del reddito nazionale in relazione ad altri fenomeni economici
- ☐ n. 29 Odo Barsotti - Marco Bottai - Marco Costa, Centralità e potenziale demografico per l'analisi dei comportamenti demografici: il caso della Toscana
- ☐ n. 30 Anna Marchi, A sequential method for a bicriteria problem arising in portfolio selectiontheory.
- ☐ n. 31 Marco Bottai, Mobilità locale e pianificazione territoriale.
- ☐ n. 32 Anna Marchi, Solving a quadratic fractional program by means of a complementarity approach
- ☐ n. 33 Anna Marchi, Sulla relazione tra un problema bicriteria e un problema frazionario.

Anno: 1991

- ☐ n. 34 Enrico Gori, Variabili latenti e "self-selection" nella valutazione dei processi formativi.
- ☐ n. 35 Piero Manfredi - E. Salinelli, About an interactive model for sexual Populations.
- ☐ n. 36 Giorgio Giorgi, Alcuni aspetti matematici del modello di sraffa a produzione semplice
- ☐ n. 37 ALberto Cambini - S.Schaibl - Claudio Sodini, Parametric linear fractional programming for an unbounded feasible Region.
- ☐ n. 38 I.Emke - Poulopoulos - V.Gozàlves Pèrez - Odo Barsotti - Laura Lecchini, International migration to northern Mediterranean countries the cases of Greece, Spain and Italy.
- ☐ n. 39 Giuliano Gasparotto, A LP code implementation
- ☐ n. 40 Riccardo Cambini, Un problema di programmazione quadratica nella costituzione di capitale.
- ☐ n. 41 Gilberto Ghilardi, Stime ed errori campionari nell'indagine ISTAT sulle forze di lavoro.
- ☐ n. 42 Vincenzo Bruno, Alcuni valori medi, variabilità paretiana ed entropia.
- ☐ n. 43 Giovanni Boletto, Gli effetti del trascinamento dei prezzi sulle misure dell'inflazione: aspetti metodologici
- ☐ n. 44 P. Paolicchi, Gli abbandoni nell'università: modelli interpretativi.
- ☐ n. 45 Maria Francesca Romano, Da un archivio amministrativo a un archivio statistico: una proposta metodologica per i dati degli studenti universitari.
- ☐ n. 46 Maria Francesca Romano, Criteri di scelta delle variabili nei modelli MDS: un'applicazione sulla popolazione studentesca di Pisa.
- ☐ n. 47 Odo Barsotti - Laura Lecchini, Les parcours migratoires en fonction de la nationalité. Le cas de l'Italie.
- ☐ n. 48 Vincenzo Bruno, Indicatori statistici ed evoluzione demografica, economica e sociale delle province toscane.
- ☐ n. 49 Alberto Cambini - Laura Martein, Tangent cones in optimization.
- ☐ n. 50 Alberto Cambini - Laura Martein, Optimality conditions in vector and scalar optimization: a unified approach.

Anno: 1992

- ☐ n. 51 Gilberto Ghilardi, Elementi di uno schema di campionamento areale per alcune rilevazioni ufficiali in Italia.
- ☐ n. 52 Paolo Manca, Investimenti e finanziamenti generalizzati.
- ☐ n. 53 Laura Lecchini - Odo Barsotti, Le rôle des immigrés extra- communautaires dans le marché du travail

☐ **n. 112** Riccardo Cambini, Generalized concavity for bicriteria functions.

☐ **n. 113** Vincenzo Bruno, Variazioni dinamiche (1971-1981-1991)dei fenomeni demografici dei comuni (urbani e rurali) della Lombardia, in relazione ad alcune caratteristiche di mobilità territoriale.

Anno: **1997**

☐ **n. 114** Piero Manfredi - Fabio Tarini - J.R. Williams - A. Carducci - B. Casini, Infectious diseases: epidemiology, mathematical models, and immunization policies.

☐ **n. 115** Eugene Maurice Cleur - Piero Manfredi, One dimensional SDE models, low order numerical methods and simulation based estimation: a comparison of alternative estimators.

☐ **n. 116** Luciano.Fanti - Piero Manfredi, Point stability versus orbital stability (or instability): remarks on policy implications in classical growth cycle model.

☐ **n. 117** Piero Manfredi - Francesco Billari, transition into adulthood, marriage, and timing of life in a stable popolation framework.

☐ **n. 118** Laura Carosi, Una nota sul concetto di estremo superiore di insiemi ordinati da coni convessi.

☐ **n. 119** Laura Lecchini - Donatella Marsiglia, Reti sociali degli anziani: selezione e qualità delle relazioni.

☐ **n. 120** Piero Manfredi - Luciano Fanti, Gestation lags and efficiency wage mechanisms in a goodwin type growth model.

☐ **n. 121** G.Rivellini, La metodologia statistica multilevel come possibile strumento per lo studio delle interazioni tra il comportamento procreativo individuale e il contesto

☐ **n. 122** Laura Carosi, Una nota sugli insiemi C-limitati e L-limitati.

☐ **n. 123** Laura Carosi, Sull'estremo superiore di una funzione lineare fratta ristretta ad un insieme chiuso e illimitato.

☐ **n. 124** Piero Manfredi, A demographic framework for the evaluation of the impact of imported infectious diseases.

☐ **n. 125** Alessandro Valentini, Calo della fecondità ed immigrazione: scenari e considerazioni sul caso italiano.

☐ **n. 126** Alberto Cambini - Laura Martein, Second order optimality conditions.

Anno: **1998**

☐ **n. 127** Piero Manfredi and Alessandro Valentini, Populations with below replacementfertility: theoretical considerations and scenarioes from the italian laboratory.

☐ **n. 128** Alberto Cambini - Laura Martein - E. Moretti, Programmazione frazionaria e problemi bicriteria.

☐ **n. 129** Emilio Barucci - Fausto Gozzi - Andrej Swiech, Incentive compatibility constraints and dynamic programming in continuous time.

Anno: **1999**

☐ **n. 130** Alessandro Valentini, Impatto delle immigrazioni sulla popolazione italiana: confronto tra scenari alternativi.

☐ **n. 131** K. Iglicka - Odo Barsotti - Laura Lecchini, Recent developement of migrations from Poland to Europe with a special emphsis on Italy K.Iglika - Le Migrazioni est-ovest: le unioni miste in Italia

☐ **n. 132** Alessandro Valentini, Proiezioni demografiche multiregionali a due sessi, con immigrazioni internazionali e vincoli di consistenza.

☐ **n. 133** Fabio Antonelli - Emilio Barucci - Maria Elvira Mancino, Backward-forward stochastic differential utility: existence, consumption and equilibrium analysis.

☐ **n. 134** Emilio Barucci - Maria Elvira Mancino, Asset pricing with endogenous aspirations.

☐ **n. 135** Eugene Maurice Cleur, Estimating a class of diffusion models: an evaluation of the effects of sampled discrete observations.

☐ **n. 136** Luciano Fanti - Piero Manfredi, Labour supply, time delays, and demoeconomic oscillations in a solow-typegrowth model.

☐ **n. 137** Emilio Barucci - Sergio Polidoro - Vincenzo Vespri, Some results on partial differential equations and Asian options.

☐ **n. 138** Emilio Barucci - Maria Elvira Mancino, Hedging european contingent claims in a Markovian incomplete market.

☐ **n. 139** Alessandro Valentini, L'applicazione del modello multiregionale-multistato alla popolazione in Italia mediante l'utilizzo del Lipro: procedura di adattamento dei dati e particolarità tecniche del programma.

☐ **n. 140** I.M.Stancu-Minasian, optimality conditions and duality in fractional programming-involving semilocally preinvex and related functions.

☐ **n. 141** Alessandro Valentini, Proiezioni demografiche con algoritmi di consistenza per la popolazione in Italia nel periodo 1997-2142: presentazione dei risultati e confronto con metodologie di stima alternative.

☐ **n. 142** Laura Carosi, Competitive equilibria with money and restricted partecipation.

☐ **n. 143** Laura Carosi, Monetary policy and Pareto improvability in a financial economy with restricted partecipation

☐ **n. 144** Bruno Cheli, Misurare il benessere e lo sviluppo dai paradossi del Pil a misure di benessere economico sostenibile, con uno sguardo allo sviluppo umano

☐ **n. 145** Bruno Cheli - Laura Lecchini - Lucio Masserini, The old people's perception of well-being: the role of material and non material resources

☐ **n. 146** Eugene Maurice Cleur, Maximum likelihood estimation of one-dimensional stochastic differential equation models from discrete data: some computational resultas

☐ **n. 147** Alessandro Valentini - Francesco Billari - Piero Manfredi, Utilizzi empirici di modelli multistato continui con durate multiple

☐ **n. 148** Francesco Billari - Piero Manfredi - Alberto Bonaguidi - Alessandro Valentini, Transition into adulthoold: its macro-demographic consequences in a multistatew stable population framework

☐ **n. 149** Francesco Billari - Piero Manfredi - Alessandro Valentini, Becoming Adult and its Macro-Demographic Impact: Multistate Stable Population Theory and an Application to Italy

☐ **n. 150** Alessandro Valentini, Le previsioni demografiche in presenza di immigrazioni: confronto tra modelli alternativi e loro utilizzo empirico ai fini della valutazione dell'equilibrio nel sistema pensionistico

☐ **n. 151** Emilio Barucci - Roberto Monte, Diffusion processes for asset prices under bounded rationality

☐ **n. 152** Emilio Barucci - P. Cianchi - L. Landi - A. Lombardi, Reti neurali e analisi delle serie storiche: un modello per la previsione del BTP future

☐ **n. 153** Alberto Cambini - Laura Carosi - Laura Martein, On the supremum in fractional programming

☐ **n. 154** Riccardo Cambini - Laura Martein, First and second order characterizations of a class of pseudoconcave vector functions

☐ **n. 155** Piero Manfredi and Luciano Fanti, Embedding population dynamics in macro-economic models. The case of the goodwin's growth cycle

☐ **n. 156** Laura Lecchini e Odo Barsotti, Migrazioni dei preti dalla Polonia in Italia

☐ **n. 157** Vincenzo Bruno, Analisi dei prezzi, in Italia dal 1975 in poi

☐ **n. 158** Vincenzo Bruno, Analisi del commercio al minuto in Italia

☐ **n. 159** Vincenzo Bruno, Aspetti ciclici della liquidità bancaria, dal 1971 in poi

☐ **n. 160** Anna Marchi, A separation theorem in alternative theorems and vector optimization

- ☐ **n. 216** Luciano Fanti - Piero Manfredi, Population, unemployment and economic growth cycles: a further explanatory perspective
- ☐ **n. 217** J.R.Williams,P.Manfredi,S.Salmaso,M.Ciofi, Heterogeneity in regional notification patterns and its impact on aggregate national case notification data: the example of measles in Italy.
- ☐ **n. 218** Anna Marchi, On the connectedness of the efficient frontier: sets without local efficient maxima
- ☐ **n. 219** Laura Lecchini - Odo Barsotti, Les disparités territoriales au Maroc au travers d'une optique de genre.

Anno: 2002

- ☐ **n. 220** Gilberto Ghilardi - Nicola Orsini, Sull'uso dei modelli statistici lineari nella valutazione dei sistemi formativi.
- ☐ **n. 221** Andrea Mercatanti, Un'analisi descrittiva dei laureati dell'Università di Pisa
- ☐ **n. 222** E. Barucci - C. Impenna - R. Renò, The Italian Overnight Market: microstructure effects, the martingale hypothesis and the payment system.
- ☐ **n. 223** E. Barucci, P.Malliavin, M.E.Mancino, R.Renò, A.Thalmaier, The Price-volatility feedback rate: an implementable mathematical indicator of market stability.
- ☐ **n. 224** Andrea Mercatanti, Missing at random in randomized experiments with imperfect compliance
- ☐ **n. 225** Andrea Mercatanti, Effetto dell'uso di carte Bancomat e carte di Credito sulla liquidità familiare: una valutazione empirica
- ☐ **n. 226** Piero Manfredi - John R. Williams, Population decline and population waves: their impact upon epidemic patterns and morbidity rates for childhood infectious diseases. Measles in Italy as an example.
- ☐ **n. 227** Piero Manfredi - Marta Ciofi degli Atti, La geografia pre-vaccinale del morbillo in Italia. I. Comportamenti di contatto e sforzi necessari all'eliminazione: predizioni dal modello base delle malattie prevenibili da vaccino.
- ☐ **n. 228** I.M.Stancu-Minasian, Optimality Conditions and Duality in Fractional Programming Involving Semilocally Preinvex and Related
- ☐ **n. 229** Nicola Salvati, Un software applicativo per un'analisi di dati sui marchi genetici (Genetic Markers)
- ☐ **n. 230** Piero Manfredi, J. R. Williams, E. M. Cleur, S. Salmaso, M. Ciofi, The pre-vaccination regional landscape of measles in Italy: contact patterns and related amount of needed eradication efforts (and the "EURO" conjecture)
- ☐ **n. 231** Andrea Mercatanti, I tempi di laurea presso l'Università di Pisa: un'applicazione dei modelli di durata in tempo discreto
- ☐ **n. 232** Andrea Mercatanti, The weak version of the exclusion restriction in causal effects estimation: a simulation study
- ☐ **n. 233** Riccardo Cambini and Laura Carosi, Duality in multiobjective optimization problems with set constraints
- ☐ **n. 234** Riccardo Cambini and Claudio Sodini, Decomposition methods for nonconvex quadratic programs
- ☐ **n. 235** R.Cambini and L. Carosi and S.Schaible, Duality in fractional optimization problems with set constraints
- ☐ **n. 236** Anna Marchi, On the mix-efficient points

Anno: 2003

- ☐ **n. 237** Emanuele Vannucci, The valuation of unit linked policies with minimal return guarantees under symmetric and asymmetric information hypoteses
- ☐ **n. 238** John R Williams - Piero Manfredi, Ageing populations and childhood infections: the potential impact on epidemic patterns and morbidity
- ☐ **n. 239** Bruno Cheli, Errata Corrige del Manuale delle Impronte Ecologiche (2002) ed alcuni utili chiarimenti
- ☐ **n. 240** Alessandra Petrucci-Nicola Salvati-Monica Pratesi, Stimatore Combinato r Correlazione Spaziale nella Stima per Piccole Aree
- ☐ **n. 241** Riccardo Cambini - Laura Carosi, Mixed Type Duality for Multiobjective Optimization Problems with set constraints
- ☐ **n. 242** O.Barsotti, L.Lecchini, F.Benassi, Foreigners from central and eastern European countries in Italy: current and future perspectives of eu enlargement
- ☐ **n. 243** A. Cambini - L. Martein - S. Schaible, Pseudoconvexity under the Charnes-Cooper transformation
- ☐ **n. 244** Eugene M. Cleur, Piero Manfredi, and John R. William, The pre-and post-Vaccination regional dynamics of measles in Italy: Insights from time series analysis

Anno: 2004

- ☐ **n. 245** Emilio Barucci - Jury Falini, Determinants of Corporate Governance in Italy: Path dependence or convergence?
- ☐ **n. 246** R. Cambini - A. Marchi, A note on the connectedness of the efficient frontier
- ☐ **n. 247** Laura Carosi - Laura Martein, On the pseudoconvexity and pseudolinearity of some classes of fractional functions
- ☐ **n. 248** E. Barucci - R. Monte - B. Trivellato, Bayesian nash equilibrium for insider trading in continuous time
- ☐ **n. 249** Eugene M. Cleur, A Time Series Analysis of the Inter-Epidemic Period for Measles in Italy
- ☐ **n. 250** Andrea Mercatanti, Causal inference methods without exclusion restrictions: an economic application.
- ☐ **n. 251** Eugene M. Cleur, Non-Linearities in Monthly Measles data for Italy
- ☐ **n. 252** Eugene M. Cleur, A Treshold Model for Prevaccination Measles Data: Some Empirical Results for England and Italy
- ☐ **n. 253** Andrea Mercatanti, La gestione dei dati mancanti nei modelli di inferenza causale: il caso degli esperimenti naturali.
- ☐ **n. 254** Andrea Mercatanti, Rilevanza delle analisi di misture di distribuzioni nelle valutazioni di efficacia
- ☐ **n. 255** Andrea Mercatanti, Local estimation of mixtures in instrumental variables models
- ☐ **n. 256** Monica Pratesi - Nicola Salvati, Spatial EBLUP in agricultural surveys: an application based on italian census data.
- ☐ **n. 257** Emanuele Vannucci, A model analyzing the effects of information asymmetries of the traders
- ☐ **n. 258** Monica Pratesi-Emilia Rocco, Two-Step centre sampling for estimating elusive population size
- ☐ **n. 259** A.Lemmi, N.Pannuzi, P.Valentini, B.Cheli, G.Berti, Estimating Multidimensional Poverty: A Comparison of Three Diffused Methods°

Anno: 2005

- ☐ **n. 260** Nicola Salvati, Small Area estimation: the EBLUP estimator using the CAR model
- ☐ **n. 261** Monica Pratesi-Nicola Salvati, Small Area Estimation: the EBLUP estimator with autoregressive random area effects
- ☐ **n. 262** Riccardo Cambini-Claudio Sodini, A solution algorithm for a class of box constrained quadratic programming problems
- ☐ **n. 263** Andrea Mercatanti, A constrained likelihood maximization for relaxing the exclusion restriction in causal inference.
- ☐ **n. 264** Marco Bottai - Annalisa Lazzini - Nicola Salvati, Le proiezioni demografiche. Pisa 2003/2032
- ☐ **n. 265** Andrea Mercatanti, An exercise in estimating causal effects for non-compliers: the return to schooling in Germany and Austria
- ☐ **n. 266** Nicola Salvati, M-quantile Geographically Weighted Regression for Nonparametric Small Area Estimation
- ☐ **n. 267** Ester Rizzi, Alessandro Rosina, L'influsso della Luna sul comportamento sessuale
- ☐ **n. 268** Silvia Venturi, Linda Porciani, Moreno Toigo, Federico Benassi, Il migrate nello spazio sociale transnazionale: tra integrazione nel Paese di