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The aim of this paper is to develop some ideas proposed by Antoci and Al. [2] about possible connections between social capital development and economic growth. With rispect to their works, we study a more articulated model in which social capit also concurs in private sector as a productive factor. According to our previous work on theme, we concentrate especially on the agents' time allocation between private and social activities and on the causes that create pressures on this process. The main novelty in the present contribution is to inset in this framework constant returns of scale, that is the possibility of sustined endogenous growth both in social and in private sector.

Keywords: Social Capital, Well-being, Time Allocation, Endogenous Growth, Maximum Principle

1. Introduction

In Sodini [16] we have studied a productive economy with capital accumulation, characterized by multiple no growth steady states, both in economic activities and in social capital. Essentially, the results of that model find their foundamentals on the diminishing returns of scale trough which we have described the two sectors. Instead, in the present paper, we explore the case in which costant returns of scale could be engine of endogenous growth, both in private and social accumulation.
Nonetheless, the main featuere is that the virtuous result of perpetual growth in social sector is not an automatic occurrence of the model and deeply depends on the initial conditions and on depreciation rate of social capital. As we'll explain below, the decreasement rate of social capital, creates a kind of threshold on development of social endowment.
According to other models on the theme, to undestand the social dynamics

in a community, we rather focalize on the role of the time allocation between social and private activity. That is, we assume that if agents devote too short time to social relationships these links tends to be depleted.
In contrast to much part of modern literature on economic growth that studies models in which economy experiences a balanced path[a] where all sectors grew, we show the possibility of 'unbalanced' development path where private sector could depress the social one.
Considering the well being of the agents, in such a case the whole result in instantaneous utility is ambiguous because of the positive weight that social capital exerts on private sector and on social side of utility function.
Analytically, to make tractable and no dispersive the analysis[b], we consider some strong assumptions:
1) To create the possibility of perpetual growth in private sector, we consider a modified AK production function and a positive externality due to social capital, social activity and average labour supply.
2) We assume that productivity factor A is not constant and it is described by a step-like function that could take two values, low and high, in relation with the development of private sector: if the economy surpasses the threshold, the private productivity rises[c]. Surely this assumption makes simpler the analysis, but *a*) it could be considered an approximation of a logistic development of productivity[d]; *b*) in many works, for example in Rostow [15], Azariadis [3], discontinuous stages in economic growth are considered.
3) To describe the dynamics of social capital, we assume that the creation of social links shows constant returns of scale both in time and social endowment, and that this process is not influenced by the private sector.
To make possible the study of the dynamics carachterized by a discontinuus state variable, we use a generalization of Maximum Principle. This approach allow us to find temporary possible steady states that vanish by the time

[a]See for example Lucas [11] or Jones et Al. [7] and more recently the works of Ladron and Al. [10], Ortigueira [12]. with endogenous labour supply
[b]Anyway, AK model is used in Jones and Al. [7] to study some phenomena related to endogenous growth and, about Social Capital, in Bartolini [4] (even if he centates on no growth steady state solutions).
[c]For example [8] uses this idea of different productivities to study the shift of resources to human capital sector in developed country.
[d]Probably, an explicit use of this form make no analitically tractable the problem of stability of steady states.

2. The model

We consider an economy inhabited by infinitely lived homogeneous agents whose utility depends on private consumption, on leisure and on social capital. Formally we have that instantaneous utility of each agent is

$$U(C(t), s(t), S(t), K_s) = \log C(t) + Hs(t)^{\alpha} S(t)^{\beta} + K_s^{\rho} \qquad (1)$$

where C denotes consumption, s and S are respectively the individual and the average time devoted to social activities[e], K_s is the stock of social capital and H is a positive parameter. According to Antoci and Al. [1],[2] we assume that the quality of no working time depends drastically on the others' choices, but differently by their framework and by Sodini [16], we consider an additive form of instantaneous utility function where K_s^{ρ} represents the social component of utility of the agent and it is considered given by each individual. Nonetheless, as a direct consequence of the specification of the model (see below), the social resources of a community could be consumed and produced, in long run, if and only if agents devote a part of their time to no working activities.
We impose the following simplifying assumption:

Condition 2.1. $\alpha+\beta = 1$: this implies that, at aggregate level, the utility is linear with respect to leisure[f].

The output y is produced with a modified AK technology with elastic labor supply and a positive role of social capital, i.e.:

$$y = A(t)K(1-s)^{a} + j(SK_s)(1-S)^{a} \qquad (2)$$

where $a \in (0,1)$, $\beta \in (0,1)$, A is a productivity parameter, j captures the weight of social capital externality, social activity and average labour supply on private production. The choice of this additive formulation is due essentially to have perpetual private accumulation and no-acceleration phenomena[g] in time allocation which make difficult to study the stability properties of the equilibria.

[e]We normalize the stock of time to 1 at every age, hence $s \in [0,1]$.
[f]This condition allow us to have an explicit solution of the maximization problem, and to exclude prepetual growth (or decline) of labour supply.
[g]Due to non-existence of reduced form with no growth steady for both the possibilities: rise and decline of Social Capital

Following the idea of Kejak[h], we consider that the productivity in private sector is influenced by the level of development of the economy. To make simpler the analysis and to point out the occurrence in the economy of different stages of development with different links between social capital and private sector, we introduce a step like function

$$A(t) = \begin{cases} A_h \text{ if } K \geq K^* \\ A_l \text{ if } K < K^* \end{cases} \tag{3}$$

with $A_h > A_l$ and K^* is the critical value of capital accumulation such that below this level the productivity of private sector is low (A_l), while above this level the productivity of private sector is high. The advantege of such description of productivity is that it is piece-wise constant and enables us to split the possible results of the model.
The capital dynamics are

$$\dot{K} = A(t)K(1-s)^a + j(SK_s)^\beta(1-S)^a - mk - c \tag{4}$$

while social capital dynamics are described by the following expression[i]

$$\dot{K_s} = K_s(S-p) \tag{5}$$

where p is the natural depreciation rate of social capital. Note that if $S > p$ the social capital tends to growth indefinitely at the rate $(S-p)$ and it tends to 0 if $S < p$[j].

3. Agent's problem

In maximizing his utility, the agents seek a lifetime consumption and time allocation between labor and social activities. Anyway, differently by the mechanisms studied in many endogenous growth model with educational sector (and even used by Glaeser and Al. [6] to study the evolution of social

[h]See the introduction.
[i]This formulation is analogous to those usually assumed to describe human capital dynamics. Nonetheless, we underline the presence of the depreciation rate that crates a kind of threshold for the perpetual growth of the social endowment.
[j]We think that the linearity of time devoted to social activities in the "social production" is not a strong assumption, considering that the time is a finite resource for the community: $S \in [0,1]$.

capital), but according to the modelling considered in Antoci [1] and in Sodini [16] we consider that each agent doesn't consider the role of own choices on social capital sector. So the problem solved by each agent is :

$$\max_{s,c} \int_0^{+\infty} e^{-rt}(\log c(t) + Hs(t)^\alpha S(t)^\beta + K_s^\rho)dt \tag{6}$$

$$s.t. \; \dot{K}_s = K_s(S-p) \tag{7}$$

$$\dot{K} = AK(1-s)^a + j(SK_s)^\beta(1-S)^a - mK - c \tag{8}$$

$$s \in [0,1], K_{s_0} > 0, K_0 > 0 \tag{9}$$

and the transversality condition

$$\lim_{t \to +\infty} e^{-rt}\frac{K}{C} = 0 \tag{10}$$

Where K_0 and K_{s_0} are, respectively, the initial endowment of private and social capital in the community. In what follows, we'll consider Symmetric Nash Equilibria in which each agents' time allocation is equal to average one, i.e.:

Condition 3.1. s=S

To analyze this problem, we have to consider more cases, according to the different time allocations of the agents, to the different configurations of paramenters and with respect to initial conditions. We concentrate on the more interesting cases.

3.1. *Case $K_0 > K^*$ and $p < S < 1$*

In this section we study the case in which the initial condition and the time allocation are favorable to create perpetual growth of physical and social capital.
The current value Hamiltonian for the agent's problem is

$$H = \log c + Hs^\alpha S^\beta + \lambda\left[AK(1-s)^a + j(SK_s)^\beta(1-S)^a - mK - c\right] + \eta\left[K_s(S-p)\right] \tag{11}$$

From the Pontryagin Maximum Principle (PMP) of optimal control we get that the necessary conditions are

$$\dot{K} = \frac{\partial H}{\partial \lambda} = A_i K(1-s)^a + j(SK_s)^\beta (1-S)^a - mK - c \quad (12)$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial K} = \lambda(r + m - A_i(1-s)^a) \quad (13)$$

$$\dot{K}_s = K_s(S-p) \quad (14)$$

We omit the dynamic of η, the shadow price of social capital, since equations (12), (13), (14) are independent of it[k].The first order conditions characterizing the interior solution are

$$\frac{\partial H}{\partial c} = \lambda - 1/c = 0 \quad (15)$$

$$\frac{\partial H}{\partial s} = H\alpha S^\beta s^{\alpha-1} - aA_i K\lambda(1-s)^{a-1} = 0 \quad (16)$$

Imposing Condition 2.1 we have that the individual labour supply is

$$(1-S) = (\frac{K\lambda a A_i}{\alpha H})^{\frac{1}{1-a}} = \left(\frac{K}{C}\frac{aA_i}{\alpha H}\right)^{\frac{1}{1-a}} \quad (17)$$

where $S \in (0,1)$.
Among other things, notice that labour supply is inceasing function of a and decreasing of α
In this section we assume that $K_0 > K^*$
The equilibrium dynamics are described by the equations (12), (13), (14) in which we subtitute optimizing conditions. Using the change of variables $\chi = \frac{C}{K}, \pi = \frac{F(K,K_s,S)}{K}$ we can study a transformation of the previous dynamical system in which the model presents steady states. Note the first variable is a like-control one[l] and the second is predetermined. So we have that right side of equations (12), (13), (14) could be expressed though the use the new variables and we pass to study the following two dimensional dynamical

[k]As in Antoci [2], it is due to the fact that agents consider both S and K_s (and therefore $\dot{K}_s$) as exogenous.
[l]Because C is a control variable for the agent, individual could detrmine its value.

system[m]

$$\dot{\chi} = \chi(\frac{\dot{C}}{C} - \frac{\dot{K}}{K}) \quad (18)$$

$$\dot{\pi} = (s^*\pi - A)(\beta\frac{\dot{K}}{K} - \frac{\dot{K}_s}{K_s}) \quad (19)$$

A steady state of the system is such that C and K grows at the same rate, s^* is constant, and K_s grows at a constant rate that is a fraction of the rate of the private sector ($\frac{\dot{K}_s}{K_s} = \beta\frac{\dot{K}}{K}$)
It follows the next proposition, for which we omit the proof.

Proposition 3.1. *There exists an unique (economic relevant)steady state equilibrium and it is a saddle stable.*

Remark 3.1. In order to verify the transversality condition we have to impose that long run growth rate is less then r.

3.2. *Case $S < p$ and $K_0 > K^*$*

If we consider the case $S < p$ and $K > K^*$ social capital could not persist in the long run even if the private sector has high performances. So the growth is irremediably accompanied by the depletion of the social resources.
In such a case the agent's problem formulation is the same respect to the previous paragraph, but it conduce to a decupled sistem in which private sector grews as in the standard AK model, meanwhile social capital tends to be depleted by the time allocation of the agents. To understand the result we consider the extreme case in which labour supply is equal to one.

$$s.t.\ \dot{K}_s = K_s(-p) \quad (20)$$

$$\dot{K} = AK(1-s)^a - mK - c \quad (21)$$

$$\dot{\lambda} = \lambda(r + m - A_i(1-s)^a) \quad (22)$$

The system ha the attratting point $K = +\infty$, $\lambda = 0$, $K_s = 0$.

[m]In which we find constant labour supply and s^* is the optimal time allocation.

4. Mixed cases

It is now interesting to study the case in which economy starts at $K_0 < K^*$. In such a case if the economy could endogenously grows, we can't use PMP because of the non regularity of function (i.d. for the like-step definition of $A(t)$)involved in the maximization problem. Anyway we can reformulate the problem since (6) to (9) as a two-stage problem. During the first stage (form 0 to T), the agent maximizes his utility given the scrap function V (see below) and on the interval $(T, +\infty)$ he solve his problem with $A = A_h$ where T is the time for which the economy reach the threshold K_0. So the problem becomes

$$\max_{s,C} \int_0^T e^{-rt}(\log C(t) + Hs(t)^\alpha S(t)^\beta + K_s^\rho)dt + V(K_t, K_s)e^{-rt} \qquad (23)$$

$$s.t.\ \dot{K}_s = K_s(S-p) \qquad (24)$$

$$\dot{K} = A_l K(1-s)^a + j(SK_s)^\beta (1-S)^a - mK - C \qquad (25)$$

$$s \in [0,1], (K_s)_0 > 0, K_0 > 0 \qquad (26)$$

plus transversality conditions for free time problem and

$$K_T = K^* \qquad (27)$$

where
$V(K_t, K_s) =$

$$\max_{s,C} \int_T^{+\infty} e^{-rt}(\log C(t) + Hs(t)^\alpha S(t)^\beta + K_s^\rho)dt \qquad (28)$$

$$s.t.\ \dot{K}_s = K_s(S-p) \qquad (29)$$

$$\dot{K} = A_h K(1-s)^a + j(SK_s)^\beta (1-S) - mK - C \qquad (30)$$

$$K_s^{T^-} = K_s^{T^+}, K^{T^-} = K^{T^+} = K^*, C^{T^-} = C^{T^+} \qquad (31)$$

and (31) are "connecting" conditions between the two stages[n].

It is worth noting that at moment of the jump of A_t, we have even a negative jump in s^o.

The dynamics are so described by the following two system: the first before the jump, and the second after.

$$\dot{K} = A_l(1 - s_1^*)^a + j(K_s s^*)^\beta (1 - s^*)^a - m - C \quad (32)$$

$$\frac{\dot{C}}{C} = (A_l(1 - s_1^*)^a - m - r) \quad (33)$$

$$\dot{K}_s = K_s(s_1^* - p) \quad (34)$$

and

$$\dot{K} = A_h K(1 - s_2^*)^a + j(s^* K_s)^\beta (1 - s^*)^a - m - C \quad (35)$$

$$\frac{\dot{C}}{C} = (A_h(1 - s_2^*)^\alpha - m - r) \quad (36)$$

$$\dot{K}_s = K_s(s_2^* - p) \quad (37)$$

with $s_1^* < s_2^*$.

In this case the first system defines a temporary attracting point for the economy (that is a path with increasing social capital), but when the system approaches to the threshold K^*, it loses its stability and the dynamics are driven by the second one.

We can assist at two different cases: s could stay over the threshold defined by p and so the social capital still grows, even if at a lower rate[p]; or s becomes too low (eventually 0) and social capital declines to 0. In both the cases the overall effect on private sector and on utility of the jump is uncertain[q]. We omit the technical analysis of the dynamical systems anyway some numerical simulations shows interesting reversed U-shaped or sigma-shaped evolution of social capital: it is interesting to note that this cases

[n] We have no (quite unrealistic) jump in C.

[o] $(1 - s^1) \equiv (\frac{K\lambda a A_l}{\alpha H})^{\frac{1}{1-a}} > \left(\frac{K}{C}\frac{aA_h}{\alpha H}\right)^{\frac{1}{1-a}} \equiv (1 - s^2)$

[p] In such a case the second dynamical system could be tranformaed in the variables $\chi, and\ \pi$

[q] Being the dynamic of K_s no internalized, contrary to Kejak, the jump could be no positive for the well-being of the agents.

could present a kind of "surprise" in the development of social capital. In fact because of the saddle-stability of the equilibrium, it we have found that economy could approach the threshold at increasing rate in social capital growth and rapidly, after T, we assist to a complete depletion of social structure[r].

5. Conclusions

We have explored a no static model about possible interactions between Social Capital and Private Production. Following many theoretical and empirical studies (Coleman (1988); Putnam (1993) and (2000)), we have considered models with constant infinitely lived homogeneous population in which the time spent in social activities plays the determinant role to explain the social development of a community and each agent doesn't consider the effect of his own time allocation on the evolution of social capital.

Respect to the literature on the theme, our analysis has introduced a certain number of innovative features. Considering, at the same time, the effects of social capital on private production and on utility of the agents, our work suggests that, the weight and the direction of the contribute of social capital on economic growth is based on a complex mix of interplay between social structure, individual preferences and productive sector.

The model is centered on the study of a model that guarantees long run private accumulation. Considering a modified AK production function we have studied a two stages model that formalizes a typical result underlined by many theoretical and empirical studies (for example Costa and Al. [5])), that is the economic growth have imposed a modification in life style and social links of a community.

The reason why rational individuals could create this change is that the perpetual growth of private sector generates, by the time, a jump in the productivity that pushes agents from a more balanced time allocation (that preserves and stimulates the social sector) to a productive oriented one that generates a falling down of social resources of the community. It is important to note that this process could be no optimal because driven by the different level of knowledge of the returns of the private and social sector. Moreover, the no internalized effects of the Social Capital could have a negative impact even in transitional dynamics of economic growth.

[r]Note the analogy with the results by Putnam [14] in his empirical study of the American society.

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