Università degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all'Economia

Report n. 289

Multiobjective Problems with Set Constraints: from Necessary Optimality Conditions to Duality Results

Riccardo Cambini

Pisa, Dicembre 2006
- Stampato in Proprio –

Via Cosimo Ridolfi, 10 – 56124 PISA – Tel. Segr. Amm. 050 2216231 Segr. Stud. 050 2216317 Fax 050 2216375
Cod. Fisc. 80003670504 - P. IVA 00286820501 - Web http://statmat.ec.unipi.it/ - E-mail: dipstat@ec.unipi.it

Multiobjective Problems with Set Constraints: from Necessary Optimality Conditions to Duality Results

Riccardo Cambini[1]

Abstract

In this paper multiobjective problems having equality, inequality and set constraints are studied. Necessary optimality conditions are stated in both the image space and in the decision space. Then, mixed duality results are proved under suitable generalized concavity assumptions.

Keywords Vector Optimization, Optimality Conditions, Image Space, Maximum Principle Conditions.
AMS - 2000 Math. Subj. Class. 90C29, 90C46, 90C30
JEL - 1999 Class. Syst. C61, C62

1 Introduction

The aim of this paper is to study optimality conditions for vector valued problems having three kinds of constraints: inequality constraints, equality constraints, and a set constraint (which covers the constraints that cannot be expressed by means of either equalities or inequalities). The partial ordering in the image of the objective function is given by a closed convex pointed cone C with nonempty interior (that is a solid cone, not necessarily the Paretian one), while the inequality constraints are expressed by means of a partial ordering given by a closed convex pointed cone V with nonempty interior.

Problems of this kind have been widely studied in the literature obtaining necessary optimality conditions in the decision space, that is conditions involving derivatives and multipliers. These conditions have been called "maximum/minimum principle" conditions or "generalized Lagrange multiplier rules" and have been stated for differentiable scalar problems (see for example [44, 49, 6, 35]) and for multiobjective paretian ones (see for example [46]). Recall also that these problems are used also in infinite dimensional spaces, for instance in optimal control theory (see for all [29, 41, 50, 55]).

Recently, this problem has been studied by the author with the aim to generalize the results known in the literature to multiobjective nondifferentiable problems (see [20, 21, 22, 23, 24]). The study has been carried on by means of the so called image space approach, first suggested in [43] and already used in [12, 13, 14, 15, 16, 19, 17].

Very recently, such a kind of problems have been studied in [25] stating duality results which generalize the ones proposed in [2, 52, 54, 63]. Notice that in the literature several classes of generalized concave functions have been used to study duality results (just recall the concepts of invexity, generalized invexity [1, 10, 11, 37, 42, 47, 57], ρ-concavity [32, 45, 61], F-concavity [39] and, more recently, $(F,\rho)-$concavity [2, 8, 9, 52, 54, 56, 63]).

[1]Dept. of Statistics and Applied Mathematics, Faculty of Economics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, ITALY. E-mail: cambric@ec.unipi.it

In this paper some necessary optimality results are first provided in the image space and in the decision space. Then some mixed type duality results are given. With this aim and for the sake of simplicity, some classical generalized concavity concepts have been used (see [25] for results concerning more general concavity properties).

In Section 2 the primary problem is defined and the main notations are introduced. In Section 3 some general necessary optimality condition in the image space are proved, while in Section 4 the necessary optimality conditions are stated in the decision space. In particular, the maximum principle type conditions are characterized in the image space allowing us to realize the necessity of studying some regularity conditions. The results are stated assuming the functions to be Hadamard differentiable or Fréchet differentiable. In Section 5 it is shown that assuming suitable generalized concavity properties the maximum principle type necessary optimality conditions become sufficient. In Section 6 mixed type duality results (covering as particular cases both Mond-Weir dual and Wolfe dual) are stated. For the sake of completeness, an Appendix providing some classical results regarding to differentiability, conical approximations, Lyusternik theorem, and generalized concavity, is given.

2 Preliminary definitions

The aim of this paper is to study optimality results for vector optimization problems having both inequality and equality constraints as well as a set constraint (which covers the constraints which cannot be expressed by means of neither equalities nor inequalities). In particular, we will consider the following class of problems:

$$P : \begin{cases} C\_\max \ f(x) & \\ g(x) \in V & \textit{inequality constraints} \\ h(x) = 0 & \textit{equality constraints} \\ x \in X & \textit{set constraint} \end{cases}$$

where $A \subseteq \Re^n$ is an open set, $f : A \longrightarrow \Re^s$, $g : A \longrightarrow \Re^m$ and $h : A \longrightarrow \Re^p$ are Hadamard differentiable vector valued functions with $s \geq 1$ and $m, p \geq 0$, $C \subset \Re^s$ and $V \subset \Re^m$ are closed convex pointed cones with nonempty interior (that is to say convex pointed solid cones), the set $X \subseteq A$ is not required to be convex, closed or open. In other words, problem P has inequality constraints, equality ones and a set constraint which covers those constraints that cannot be expressed by means of either equalities or inequalities.

For the sake of convenience, P can be also expressed as

$$P : \begin{cases} C\_\max \ f(x) \\ g(x) \in V \\ x \in X \cap H \end{cases} \qquad \text{or} \qquad P : \begin{cases} C\_\max \ f(x) \\ x \in S_P \end{cases}$$

where $H = \{x \in A : h(x) = 0\}$ and $S_P = \{x \in A : g(x) \in V, h(x) = 0, x \in X\}$.

As usual, a feasible point $x_0 \in X$ is said to be a *local efficient point* if there exists a suitable neighbourhood I_{x_0} of x_0 such that:

$$\nexists y \in I_{x_0} \cap S_P \text{ such that } f(y) \in f(x_0) + C, \ \ f(y) \neq f(x_0).$$

For the definition and the properties of Hadamard differentiable functions [31, 3, 4, 51, 60] see Subsection 7.1 in the appendix.

A key tool in studying optimality conditions in the image space is the so called *Bouligand Tangent cone to X at $x_0 \in Cl(X)$*, denoted with $T(X, x_0)$, as well as the *cone of feasible directions to X at x_0*, denoted with $F(X, x_0)$, and the *cone of interior directions to X at x_0*, denoted with $I(X, x_0)$ (see for example [6, 33, 34]). The definitions and the properties of these conical approximations are summarized in Subsection 7.2 of the appendix.

Note finally that, for any set $B \subseteq \Re^n$, the following notations will be used:

- B^C is the set $\Re^n \setminus B$,
- $Cl(B)$ is the closure of B,
- $Int(B)$ is the interior of B,
- $Co(B)$ is the convex hull of B,
- B^+ is the positive polar cone of B,
- $cone(B, x_0)$ is the cone generated by $B - \{x_0\}$:

$$cone(B, x_0) = \{y \in \Re^n : y = \lambda(x - x_0), \lambda \geq 0, x \in B\}.$$

3 Necessary optimality condition in the image space

The equality constraints play an important role in the study of optimality conditions. For this reason, a key tool is the so called *linearizing cone of H at x_0*, denoted with $L(H, x_0)$. For its definition and properties see Subsection 7.3 in the appendix.

The following further cones will be helpful in the rest of this section:

$$\begin{aligned} L(H, x_0)^C &= \Re^n \setminus L(H, x_0) = \left\{v \in \Re^n \setminus \{0\} : \frac{\partial h}{\partial v}(x_0) \neq 0\right\} \\ T(X, H, x_0) = \mathcal{T} &= T(X \cap H, x_0) \cup L(H, x_0)^C \end{aligned}$$

For any cone $U \subseteq \Re^n$ also the following sets are worth to be defined:

$$\begin{aligned} D(U, x_0) &= \left\{t \in \Re^{m+s+p} : t = \left(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0) + g(x_0), \frac{\partial h}{\partial v}(x_0)\right), v \neq 0, v \in U\right\} \\ K(U, x_0) &= D(U, x_0) - (C \times V \times 0) \end{aligned}$$

Obviously, it is $D(U, x_0) \subseteq K(U, x_0)$. Notice also that these sets are cones, since the Hadamard directional differentiability of f, g and h implies that $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are positively homogeneous (of the first degree) as functions of the direction v.

The following necessary optimality condition in the image space can now be proved.

Theorem 3.1 *Consider problem P. If the feasible point $x_0 \in S_P$ is a local efficient point then the following condition holds:*

$$K(\mathcal{T}, x_0) \cap (Int(C) \times Int(V) \times 0) = \emptyset \tag{3.1}$$

Proof As a preliminary result notice that since the cones C and V are solid, convex and pointed then (3.1) is equivalent to:

$$D(\mathcal{T}, x_0) \cap (Int(C) \times Int(V) \times 0) = \emptyset \tag{3.2}$$

so that just (3.2) has to be proved. With this aim, suppose by contradiction that $\exists v \in \mathcal{T}(X, H, x_0)$, $v \neq 0$, such that

$$t = \left(\frac{\partial f}{\partial v}(x_0), \frac{\partial g}{\partial v}(x_0) + g(x_0), \frac{\partial h}{\partial v}(x_0)\right) \in (Int(C) \times Int(V) \times 0).$$

Since $\frac{\partial h}{\partial v}(x_0) = 0$ then $v \notin L(H, x_0)^C$ so that $v \in T(X \cap H, x_0)$. By means of the definition of $T(X \cap H, x_0)$ it yields that $\exists \{x_k\} \subset (X \cap H)$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re$, $\lambda_k > 0$, $\lambda_k \to +\infty$, such that $v = \lim_{k \to +\infty} v_k$ where $v_k = \lambda_k(x_k - x_0)$. Since functions f and g are Hadamard directionally differentiable it results:

$$\lim_{k \to +\infty} \frac{f(x_k) - f(x_0)}{\frac{1}{\lambda_k}} = \lim_{k \to +\infty} \frac{f(x_0 + \frac{1}{\lambda_k} v_k) - f(x_0)}{\frac{1}{\lambda_k}} = \frac{\partial f}{\partial v}(x_0) \in Int(C)$$

and, in the same way:

$$\begin{aligned} \lim_{k \to +\infty} \frac{g(x_k) - g(x_0)}{\frac{1}{\lambda_k}} &= \lim_{k \to +\infty} \frac{g(x_0 + \frac{1}{\lambda_k} v_k) - g(x_0)}{\frac{1}{\lambda_k}} = \\ &= \frac{\partial g}{\partial v}(x_0) \in -g(x_0) + Int(V) \end{aligned}$$

By means of a well known limit theorem it then exists $\bar{k} > 0$ such that $\lambda_k > 1$, $\lambda_k(f(x_k) - f(x_0)) \in Int(C)$ and $\lambda_k(g(x_k) - g(x_0)) + g(x_0) \in Int(V)$ for any $k > \bar{k}$. Since $\lambda_k > 0$ it follows that $f(x_k) \in f(x_0) + Int(C)$ and $g(x_k) \in g(x_0)(1 - \frac{1}{\lambda_k}) + Int(V)$ $\forall k > \bar{k}$. Since $g(x_0) \in V$ and $(1 - \frac{1}{\lambda_k}) > 0$ it follows that $g(x_0)(1 - \frac{1}{\lambda_k}) \in V$; as a consequence $g(x_k) \in Int(V)$ since V is a convex pointed solid cone. The sequence $\{x_k\} \subset (X \cap H)$, $x_k \to x_0$, then results to be feasible for $k > \bar{k}$ with $f(x_k) \in f(x_0) + Int(C)$; this means that x_0 is not a local efficient point, which is a contradiction. □

The following example points out that the cone $K(\mathcal{T}, x_0)$ might be nonconvex.

Example 3.1 Consider the following problem:

$$P : \{\max f(x_1, x_2) = x_1, g(x_1, x_2) = x_2 \geq 0, x \in X\}$$

where $X = X_1 \cup X_2 \cup X_3$ with:

$$\begin{aligned} X_1 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, 2x_1 + x_2 \leq 0\}, \\ X_2 &= \{(x_1, x_2) \in \Re^2 : x_1 \leq 0, x_2 \leq 0\}, \\ X_3 &= \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0, x_1 + 2x_2 \leq 0\} \end{aligned}$$

and $x_0 = (0, 0)$; since the problem has no equality constraints it is $S = \Re^2$ and $\mathcal{T}(X, H, x_0) = \mathcal{T}(X, x_0) = X$. Notice also that $(Int(C) \times Int(V)) = \Re^2_{++}$ and that $X = D(\mathcal{T}, x_0)$ since $[J_f(x_0), J_g(x_0)]$ is equal to the identity matrix. It results:

$$K(\mathcal{T}, x_0) = \{(x_1, x_2) \in \Re^2 : 2x_1 + x_2 \leq 0 \text{ or } x_1 + 2x_2 \leq 0\}$$

so that $K(\mathcal{T}, x_0) \cap (Int(C) \times Int(V)) = K(\mathcal{T}, x_0) \cap \Re^2_{++} = \emptyset$. The point x_0 is the global efficient point of the problem and the necessary optimality condition in the image space is verified. Nevertheless, the sets X, $T(X, x_0)$, $T(X, H, x_0)$, $D(\mathcal{T}, x_0)$ and $K(\mathcal{T}, x_0)$ are not convex.

4 Necessary optimality condition in the decision space

Problem P has been already studied in the literature in the particular case of a scalar objective function and assuming the differentiability of functions f, g and h (see for all [6, 44, 49]). Under such assumptions some necessary optimality conditions (known as "maximum/minimum principle" conditions) have been stated in the decision space.

The aim of this section is to generalize those conditions for multiobjective problems with Hadamard directionally differentiable functions. It will be shown also that the condition previously stated in the image space (Theorem 3.1) is more general than the ones which will be stated in the decision space.

4.1 Characterization in the image space of maximum principle conditions

The study of this section is based on the following fundamental preliminary result, which provides a characterization in the image space of the generalized maximum principle conditions given in the decision space. With this aim, the following further notation is introduced:

$$Im_{\partial h}(U) = \{0\} \cup \{t \in \Re^p : t = \frac{\partial h}{\partial v}(x_0), v \neq 0, v \in U\}$$

Theorem 4.1 *Consider problem P and let $U \subseteq \Re^n$ be a cone. Then, at least one of the following conditions holds:*

i) $p \geq 1$ *and* $Co(Im_{\partial h}(U)) \neq \Re^p$,

ii) $Co(D(U, x_0)) \cap (Int(C) \times Int(V) \times 0) = \emptyset$,

if and only if $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad and \quad \alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \;\; \forall v \in Cl(U) \setminus \{0\}$$

Proof $\Rightarrow$*)* Suppose condition *i)* holds. Since $Co(Im_{\partial h}(U)) \neq \Re^p$ there exists a support hyperplane for the convex cone $Co(Im_{\partial h}(U))$, so that $\exists \alpha_h \in \Re^p$, $\alpha_h \neq 0$, such that $\alpha_h^T t \leq 0$ $\forall t \in Co(Im_{\partial h}(U))$; this implies that $\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0$ $\forall v \in U$, $v \neq 0$. Since the directional derivative $\frac{\partial h}{\partial v}(x_0)$ is continuous as a function of direction v and because of the Hadamard directional differentiability of h, it then follows that $\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0$ $\forall v \in Cl(U)$, $v \neq 0$. The whole result is then proved just assuming $\alpha_f = 0$ and $\alpha_g = 0$.

Suppose now condition *ii)* holds. By means of a well known separation theorem between convex sets, $\exists(\alpha_f, \alpha_g, \alpha_h) \in (Int(C) \times Int(V) \times 0)^+$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that $(\alpha_f, \alpha_g, \alpha_h)^T t \leq 0$ $\forall t \in Co(D(U, x_0)) \supseteq D(U, x_0)$. A known result on polar cones ([2])

[2] Let $C_1, \ldots, C_n$ be cones, then $(C_1 \times \ldots \times C_n)^+ = (C_1^+ \times \ldots \times C_n^+)$.

implies that $(Int(C) \times Int(V) \times 0)^+ = (Int(C)^+ \times Int(V)^+ \times \Re^p)$ and hence, being C and V convex cones ([3]), $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, such that:

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \left(\frac{\partial g}{\partial v}(x_0) + g(x_0)\right) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in U, \ v \neq 0. \tag{4.1}$$

The directional derivatives $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are positively homogeneous (of the first degree) as functions of direction v, hence denoting with $\overline{v} = \frac{v}{||v||}$ we have:

$$||v|| \left(\alpha_f^T \frac{\partial f}{\partial \overline{v}}(x_0) + \alpha_g^T \frac{\partial g}{\partial \overline{v}}(x_0) + \alpha_h^T \frac{\partial h}{\partial \overline{v}}(x_0)\right) + \alpha_g^T g(x_0) \leq 0 \ \forall v \in U, v \neq 0.$$

This condition implies that $\alpha_g^T g(x_0) \leq 0$; to prove this inequality suppose by contradiction that $\alpha_g^T g(x_0) > 0$, then for a vector $v \in U$, $v \neq 0$, with $||v||$ small enough, we have

$$||v|| \left(\alpha_f^T \frac{\partial f}{\partial \overline{v}}(x_0) + \alpha_g^T \frac{\partial g}{\partial \overline{v}}(x_0) + \alpha_h^T \frac{\partial h}{\partial \overline{v}}(x_0)\right) + \alpha_g^T g(x_0) > 0$$

which is a contradiction.

Since $\alpha_g \in V^+$ and $g(x_0) \in V$ it is $\alpha_g^T g(x_0) \geq 0$ and hence, since $\alpha_g^T g(x_0) \leq 0$, it follows that $\alpha_g^T g(x_0) = 0$.

Condition (4.1) becomes:

$$\alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in U, v \neq 0$$

and the result then follows since the directional derivatives $\frac{\partial f}{\partial v}(x_0)$, $\frac{\partial g}{\partial v}(x_0)$ and $\frac{\partial h}{\partial v}(x_0)$ are continuous as functions of direction, because of the Hadamard directional differentiability at x_0 of f, g and h.

$\Leftarrow$*)* Let us first prove, as a preliminary result, that if condition *i)* does not hold then $(\alpha_f, \alpha_g) \neq 0$. If $p = 0$ this result is trivial since there is no multiplier vector α_h; if $p \geq 1$ and $Co(Im_{\partial h}(U)) = \Re^p$ just assume by contradiction that both $\alpha_f = 0$ and $\alpha_g = 0$, so that $\alpha_h \neq 0$ and

$$\alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in Cl(U) \setminus \{0\},$$

which yields $\alpha_h^T t \leq 0 \, \forall t \in Im_{\partial h}(U)$. As a consequence, it is $\alpha_h^T t \leq 0 \, \forall t \in Co(Im_{\partial h}(U)) = \Re^p$ and this implies $\alpha_h = 0$, which is a contradiction.

Suppose now by contradiction that both conditions *i)* and *ii)* are not verified, so that $(\alpha_f, \alpha_g) \neq 0$ and

$$\exists(t_f, t_g, t_h) \in Co(D(U, x_0)) \cap (Int(C) \times Int(V) \times 0) \neq \emptyset.$$

Since $\alpha_f \in C^+$, $\alpha_g \in V^+$, $(\alpha_f, \alpha_g) \neq 0$, $t_f \in Int(C)$, $t_g \in Int(V)$ and $t_h = 0$ it is:

$$\alpha_f^T t_f + \alpha_g^T t_g + \alpha_h^T t_h > 0 \tag{4.2}$$

[3]Let C be a cone; it is known (see for all [59]) that $C^+ = Cl(C)^+$ so that $Int(C)^+ = Cl(Int(C))^+$ too. If C is a convex cone we also have (see for instance [7]) that $Cl(Int(C)) = Cl(C)$ so that $Int(C)^+ = C^+$.

Since $(t_f, t_g, t_h) \in Co(D(U, x_0))$ $\exists q \in \mathbf{N}$, $q > 0$, $\exists u_1, \ldots, u_q \in U$, such that

$$(t_f, t_g, t_h) = \sum_{i=1}^{q} \left(\frac{\partial f}{\partial u_i}(x_0), \frac{\partial g}{\partial u_i}(x_0) + g(x_0), \frac{\partial h}{\partial u_i}(x_0) \right)$$

hence

$$\begin{aligned} \alpha_f^T t_f + \alpha_g^T t_g + \alpha_h^T t_h &= \sum_{i=1}^{q} \left(\alpha_f^T \frac{\partial f}{\partial v_i}(x_0) + \alpha_g^T \frac{\partial g}{\partial v_i}(x_0) + \alpha_g^T g(x_0) + \alpha_h^T \frac{\partial h}{\partial v_i}(x_0) \right) \\ &= \sum_{i=1}^{q} \left(\alpha_f^T \frac{\partial f}{\partial v_i}(x_0) + \alpha_g^T \frac{\partial g}{\partial v_i}(x_0) + \alpha_h^T \frac{\partial h}{\partial v_i}(x_0) \right) \leq 0 \end{aligned}$$

and this contradicts (4.2). □

4.2 Hadamard differentiability assumptions

The previous fundamental result allow us to state the following necessary optimality conditions in the decision space.

Theorem 4.2 *Consider problem P and let $U \subseteq \Re^n$ be a cone which verifies at least one of the following regularity conditions N1)-N2):*

N1) $p \geq 1$ *and* $Co(Im_{\partial h}(U)) \neq \Re^p$,

N2) $Co(D(U, x_0)) \subseteq K(\mathcal{T}, x_0)$.

If the feasible point $x_0 \in S_P$ is a local efficient point then the following maximum principle condition holds:

(C_N) $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad \alpha_f^T \frac{\partial f}{\partial v}(x_0) + \alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \;\; \forall v \in Cl(U) \setminus \{0\}$$

Moreover, the following further results hold:

i) if $p = 0$ or $Co(Im_{\partial h}(U)) = \Re^p$ then condition (C_N) is verified with $(\alpha_f, \alpha_g) \neq 0$,

ii) if the constraint qualification $Co(Im_{\partial g, \partial h}(U)) = \Re^{m+p}$ holds, where

$$Im_{\partial g, \partial h}(U) = \{0\} \cup \left\{ t \in \Re^{m+p} : t = \left(\frac{\partial g}{\partial v}(x_0), \frac{\partial h}{\partial v}(x_0) \right), v \neq 0, v \in U \right\}$$

then condition (C_N) is verified with $\alpha_f \neq 0$,

iii) assuming $p = 0$ or $Co(Im_{\partial h}(U)) = \Re^p$, if the constraint qualification

$$\left\{ d \in \Re^n \setminus \{0\} : \frac{\partial g}{\partial d}(x_0) \in Int(V) \right\} \cap L(H, x_0) \cap Co(U) \neq \emptyset$$

holds, then condition (C_N) is verified with $\alpha_f \neq 0$.

Proof If *N1)* or *N2)* holds then (C_N) follows directly from Theorems 3.1 and 4.1.

i) This result has been already proved within the proof of Theorem 4.1.

ii) Suppose by contradiction that $\alpha_f = 0$, so that $(\alpha_g, \alpha_h) \neq 0$. Then

$$\alpha_g^T \frac{\partial g}{\partial v}(x_0) + \alpha_h^T \frac{\partial h}{\partial v}(x_0) \leq 0 \ \forall v \in Cl(U) \setminus \{0\},$$

and this yields $\alpha_g^T t_g + \alpha_h^T t_h \leq 0 \ \forall (t_g, t_h) \in Im_{\partial g, \partial h}(U)$. As a consequence, we have $\alpha_g^T t_g + \alpha_h^T t_h \leq 0 \ \forall (t_g, t_h) \in Co(Im_{\partial g, \partial h}(U)) = \Re^{m+p}$ and this implies $(\alpha_g, \alpha_h) = 0$, which is a contradiction.

iii) For *i)* we have that (C_N) is verified with $(\alpha_f, \alpha_g) \neq 0$. Suppose now by contradiction that $\alpha_f = 0$ and $\alpha_g \neq 0$. Taken $d \in Co(U)$ such that $\frac{\partial g}{\partial d}(x_0) \in Int(V)$ and $\frac{\partial h}{\partial d}(x_0) = 0$ we get

$$\alpha_f^T \frac{\partial f}{\partial d}(x_0) + \alpha_g^T \frac{\partial g}{\partial d}(x_0) + \alpha_h^T \frac{\partial h}{\partial d}(x_0) > 0$$

which is a contradiction. □

It is worth noticing that Theorems 4.1 and 4.2 point out that in order to obtain a necessary optimality condition in the decision space some additional hypotheses must be assumed. Specifically speaking, Example 3.1 points out that the necessary optimality condition in the image space

$$K(\mathcal{T}, x_0) \cap (Int(C) \times Int(V) \times 0) = \emptyset$$

holds even if the cone $K(\mathcal{T}, x_0)$ is not convex. On the other hand, in order to obtain a necessary optimality condition in the decision space we have to use a cone U such that:

$$Co(D(U, x_0)) \cap (Int(C) \times Int(V) \times 0) = \emptyset$$

This means that in order to obtain a necessary optimality condition in the decision space a suitable cone U must be chosen (see Example 4.1), that is to say that some additional assumptions are needed. These additional assumptions, such as conditions *N1)-N2)* in Theorem 4.2, are nothing but regularity conditions and have been called U-regularity conditions in [21, 22, 23].

Example 4.1 Consider now Example 3.1 again. If the cone $U = X_1 \cup X_2 \cup X_3 = \mathcal{T}(X, H, x_0)$ is chosen then it is not possible to obtain the corresponding necessary optimality condition in the decision space since $Co(D(U, x_0)) = \Re^2$. If we choose $U = X_1 \cup X_3$ we get $Co(D(U, x_0)) = \{(x_1, x_2) \in \Re^2 : x_1 + x_2 \geq 0\}$ and hence the corresponding optimality condition in the decision space does not hold. In order to obtain condition (C_N), we can choose for example $U = X_1 \cup X_2$, $U = X_2 \cup X_3$, $U = X_i$, $i = 1, 2, 3$, which are cones verifying the regularity condition *N2)*.

4.3 Fréchet differentiability assumptions

The aim of this subsection is to deep on the study of necessary optimality conditions in the decision space in the case the directional derivatives of functions f, g and h are linear as functions of the direction. This property, together with the Hadamard differentiability assumption, implies the use of Fréchet differentiable functions (see Subsection 7.1 in the appendix). In this light, note that if f, g and h are Fréchet differentiable at $x_0 \in A$ then

- $L(H, x_0) = Ker(J_h(x_0))$,
- $Co(D(U, x_0)) = D(Co(U), x_0)$,
- $Co(Im_{\partial h}(U)) = J_h(x_0)[Co(U)]$.

where, given the Jacobian matrix $J_h(x_0)$ of h at x_0 and given a set $B \subseteq \Re^n$, the following notation is used:

$$J_h(x_0)[B] = \{t \in \Re^p : t = J_h(x_0)v, v \in B\}$$

Taking into account the Fréchet differentiability of the functions f, g and h, Theorem 4.2 can be rewritten as follows.

Corollary 4.1 *Consider problem P and assume functions f, g and h to be Fréchet differentiable at $x_0 \in S_P$. Moreover, let $U \subseteq \Re^n$ be a cone which verifies one of the following regularity conditions F1)-F2):*

F1) $p \geq 1$ *and* $J_h(x_0)[U] \neq \Re^p$,

F2) U *is convex and* $D(U, x_0) \subseteq K(\mathcal{T}, x_0)$;

If the feasible point $x_0 \in S_P$ is a local efficient point then the following maximum principle condition holds:

(C_F) $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)]v \leq 0 \quad \forall v \in Cl(U)$$

In particular, if $D(\Re^n, x_0) \subseteq K(\mathcal{T}, x_0)$ then (C_F) reduces to the following:

(C_{FJ}) $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)] = 0$$

Moreover, the following further results hold:

i) if $p = 0$ or $J_h(x_0)[U] = \Re^p$ then condition (C_F) is verified with $(\alpha_f, \alpha_g) \neq 0$,

ii) if the constraint qualification

$$J_{g,h}(x_0)[U] = \Re^{m+p} \quad \text{where} \quad J_{g,h}(x_0) = [J_g(x_0), J_h(x_0)]$$

holds, then condition (C_F) is verified with $\alpha_f \neq 0$,

iii) assuming $p = 0$ or $J_h(x_0)[U] = \Re^p$, if the constraint qualification

$$\{d \in \Re^n : J_g(x_0)d \in Int(V)\} \cap Ker(J_h(x_0)) \cap U \neq \emptyset$$

holds, then condition (C_F) is verified with $\alpha_f \neq 0$.

Notice that condition (C_{FJ}) is nothing but the Fritz John criterion, which gets to the Karush-Kuhn-Tucker conditions if a constraint qualification condition is verified.

Remark 4.1 It is also worth pointing out that the convex cone U in *F2)* does not have to be a subset of $\mathcal{T}(X, H, x_0)$. For instance, consider again Example 4.1 and let $U = Co(X_1 \cup X_2)$ or $U = Co(X_2 \cup X_3)$; these cones verify the regularity condition *F2)* even if they are not subsets of $\mathcal{T}(X, H, x_0)$.

4.4 Regularity conditions in the decision space

The linearity of the directional derivatives of f, g and h, allow us to state the following further regularity condition *F3)* which yields in the decision space, and not in the image space like the previous ones.

Corollary 4.2 *Consider problem P and assume functions f, g and h to be Fréchet differentiable at $x_0 \in S_P$. Moreover, let $U \subseteq \Re^n$ be a cone which verifies the following regularity condition:*

F3) U is convex and $(U \cap L(H, x_0)) \subseteq T(X \cap H, x_0)$;

If the feasible point $x_0 \in S_P$ is a local efficient point then condition (C_F) holds. In particular, if $x_0 \in Int(X)$ and $L(H, x_0) = T(H, x_0)$ then (C_{FJ}) holds.

Proof First notice that

$$U \subseteq \mathcal{T}(X, H, x_0) \iff (U \cap L(H, x_0)) \subseteq T(X \cap H, x_0)$$

and that $U \subseteq \mathcal{T}$ implies $D(U, x_0) \subseteq D(\mathcal{T}, x_0) \subseteq K(\mathcal{T}, x_0)$. As a consequence, it holds *F3)*$\Rightarrow$*F2)* and hence the results follows directly from Corollary 4.1.

In the case $x_0 \in Int(X)$ and $L(H, x_0) = T(H, x_0)$ it is $T(X \cap H, x_0) = T(H, x_0) = L(H, x_0)$ so that the results follows by choosing $U = \Re^n$ in *F3)*. □

It is worth to point out that the regularity condition *F3)* (which requires the convex cone U to be a subset of $\mathcal{T}(X, H, x_0)$) is much stronger than *F2)*, as it has been discussed in Remark 4.1.

The following further regularity conditions in the decision space *F4)*, *F5)* and *F6)* can now be derived from *F3)* by means of the behaviour of conical approximations given in Property 7.5 and the generalizations of the Lyusternik theorem (see Subsection 7.3 in the appendix) given in Theorem 7.2 and in Corollary 7.1.

Corollary 4.3 *Consider problem P and assume functions f, g and h to be Fréchet differentiable at $x_0 \in S_P$. Moreover, let h be continuous on a neighborhood of x_0 and let $U \subseteq \Re^n$ be a cone which verifies one of the following regularity conditions:*

F4) U is convex and $(U \cap T(H, x_0)) \subseteq T(X \cap H, x_0)$,

F5) $U = I(X, x_0)$ is convex;

F6) $U = T(X, x_0)$ and X is convex;

If the feasible point $x_0 \in S_P$ is a local efficient point then condition (C_F) holds. In particular, if $x_0 \in Int(X)$ then (C_{FJ}) holds.

Proof Let us first prove *F4)*. If $J_h(x_0)$ is not surjective then $J_h(x_0)[U] \neq \Re^p$ and hence the result holds for *F1)* of Corollary 4.1. If $J_h(x_0)$ is surjective then for Corollary 7.1 it is $T(H, x_0) = L(H, x_0)$, so that the results follows from *F3)* of Corollary 4.2.

As regards to *F5)*, it follows directly from *F4)* and Property 7.5.

Let us now prove *F6)*. First of all, notice that since X is convex then $T(X, x_0)$ is convex too. If $J_h(x_0)[T(X, x_0)] \neq \Re^p$ then the result holds for *F1)* of Corollary 4.1. If

$J_h(x_0)[T(X,x_0)] = \Re^p$ then for Theorem 7.2 it is $T(X \cap H, x_0) = L(H,x_0) \cap T(X,x_0)$, hence the results yields from *F3)* of Corollary 4.2.

Finally, in the case $x_0 \in Int(X)$ it is $T(X \cap H, x_0) = T(H,x_0)$ so that the results follows by choosing $U = \Re^n$ in *F4)*. □

Notice that *F5)* represents a generalization to multiobjective problems of the result stated in [6] and related to a scalar objective function f. Notice also that *F6)* represents a generalization of the results in [44, 49].

Finally, it is worth pointing out the following further particular case which will be useful in the study of duality results.

Property 4.1 *Consider problem P, assume functions f, g and h to be Fréchet differentiable at $x_0 \in S_P$ and let h be continuous on a neighborhood of x_0. If X is convex and the feasible point $x_0 \in S_P$ is a local efficient point then the following maximum principle condition holds:*

- $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad [\alpha_f^T J_f(x_0) + \alpha_g^T J_g(x_0) + \alpha_h^T J_h(x_0)](y - x_0) \leq 0 \quad \forall y \in X$$

Proof Follows from *F6)* of Corollary 4.3 by choosing $U = F(X,x_0) \subseteq T(X,x_0)$. □

5 Sufficient optimality conditions

The aim of this section is to show how the necessary optimality conditions previously stated become sufficient assuming suitable generalized concavity properties (see Subsection 7.4 in the appendix).

With this aim, it is useful to introduce the following Lagrangean type function:

$$\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = \alpha_f^T f(x) + \alpha_g^T g(x) + \alpha_h^T h(x)$$

and its gradient with respect to variable x:

$$\nabla \mathcal{L}_x(x, \alpha_f, \alpha_g, \alpha_h)^T = \left[\alpha_f^T J_f(x) + \alpha_g^T J_g(x) + \alpha_h^T J_h(x)\right]$$

In this light, notice that the maximum principle condition given in Property 4.1 can be rewritten as:

$$\nabla \mathcal{L}_x(x_0, \alpha_f, \alpha_g, \alpha_h)^T (y - x_0) \leq 0 \quad \forall y \in X$$

The following result holds.

Theorem 5.1 *Consider problem P and assume functions f, g and h to be Fréchet differentiable. Let also $x_0 \in S_P$ be a feasible point verifying the following condition:*

- $\exists(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $(\alpha_f, \alpha_g, \alpha_h) \neq 0$, *such that:*

$$\alpha_g^T g(x_0) = 0 \quad \text{and} \quad \nabla \mathcal{L}_x(x_0, \alpha_f, \alpha_g, \alpha_h)^T (y - x_0) \leq 0 \quad \forall y \in X$$

The following properties hold:

i) if function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is strictly pseudoconcave with respect to variable x then $\nexists y \in S_P$ such that $f(y) \in f(x_0) + C$, $f(y) \neq f(x_0)$;

ii) if $\alpha_f \neq 0$ and function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is pseudoconcave with respect to variable x then $\nexists y \in S_P$ such that $f(y) \in f(x_0) + Int(C)$.

Proof Let us first prove property *i)*. Assume by contradiction that:

$$\exists y \in X \text{ such that } g(y) \in V,\ h(y) = 0,\ f(y) \in f(x_0) + C,\ f(y) \neq f(x_0)$$

Since $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$ it yields:

$$\alpha_g^T g(y) \geq 0,\ \alpha_h^T h(y) = 0,\ \alpha_f^T f(y) \geq \alpha_f^T f(x_0)$$

By means of the hypothesis it is also $h(x_0) = 0$, so that $\alpha_h^T h(x_0) = 0$, and $\alpha_g^T g(x_0) = 0$. As a consequence it results:

$$\mathcal{L}(y, \alpha_f, \alpha_g, \alpha_h) \geq \mathcal{L}(x_0, \alpha_f, \alpha_g, \alpha_h)$$

From the strict pseudoconcavity of $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ with respect to x it then follows:

$$\nabla \mathcal{L}_x(x_0, \alpha_f, \alpha_g, \alpha_h)^T (y - x_0) > 0$$

which is a contradiction.

Property *ii)* can be proved analogously. With this aim, just recall that since C a closed convex pointed cone with nonempty interior then conditions $\alpha_f \neq 0$ and $f(y) \in f(x_0) + Int(C)$ imply that $\alpha_f^T f(y) > \alpha_f^T f(x_0)$. □

6 Duality results

The aim of this section is to study duality results for problem P, which from now on will be referred to as the primal problem:

$$P : \begin{cases} C\_\max\ f(x) \\ x \in S_P \end{cases}$$

where $S_P = \{x \in A : g(x) \in V, h(x) = 0, x \in X\}$. For the sake of convenience, the open set $A \in \Re^n$ is assumed to be convex while functions f, g and h are assumed to be Fréchet differentiable.

6.1 Preliminary definitions

By following two classical approaches of the literature, the following dual problems can be defined by using the previously stated maximum principle conditions (notice that in the Wolfe type dual it is $c \in Int(C)$):

Mond-Weir type Dual

$$\begin{cases} C\_\min \ f(x) \\ \alpha_g^T g(x) \leq 0 \ , \ \alpha_h^T h(x) = 0 \\ \nabla \mathcal{L}_x(x, \alpha_f, \alpha_g, \alpha_h)^T (y - x) \leq 0 \ \forall y \in X, \\ x \in A, \ \alpha_f \in C^+ \setminus \{0\}, \ \alpha_g \in V^+, \ \alpha_h \in \Re^p \end{cases}$$

Wolfe type Dual

$$\begin{cases} C\_\min \ f(x) + \frac{c}{\alpha_f^T c}[\alpha_g^T g(x) + \alpha_h^T h(x)] \\ \nabla \mathcal{L}_x(x, \alpha_f, \alpha_g, \alpha_h)^T (y - x) \leq 0 \ \forall y \in X, \\ x \in A, \ \alpha_f \in C^+ \setminus \{0\}, \ \alpha_g \in V^+, \ \alpha_h \in \Re^p \end{cases}$$

A comparison of these two dual problems points out that the easiest is the objective function, the more complex are the constraints, and *vice versa*.

In the literature another approach has been proposed to manage in an unifying framework both the Mond-Weir and the Wolfe duals. With this aim the following notations are needed:

- $\delta \in \{0, 1\}$ is a $0 - 1$ parameter;
- $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ is a partition of $\mathcal{P} = \{1, \ldots, p\}$;
- $h(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]$ and $\alpha_h = (\alpha_{h_1}, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4})$ are partitioned accordingly to $\mathcal{J}$.

The following mixed type dual problem D will be referred to as the dual of P ([4]):

Mixed type Dual

$$D : \begin{cases} C\_\min \ \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) \\ (1 - \delta)\alpha_g^T g(x) + \alpha_{h_2}^T h_2(x) \leq 0 \\ \alpha_{h_3}^T h_3(x) = 0, \ \alpha_{h_4}^T h_4(x) \leq 0 \\ \nabla \mathcal{L}_x(x, \alpha_f, \alpha_g, \alpha_h)^T (y - x) \leq 0 \ \forall y \in X, \\ x \in A, \ \alpha_f \in C^+ \setminus \{0\}, \ \alpha_g \in V^+, \ \alpha_h \in \Re^p \end{cases}$$

where $\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) = f(x) + \frac{c}{\alpha_f^T c}[\delta \alpha_g^T g(x) + \alpha_{h_1}^T h_1(x)]$. From now on the feasible region of D will be denoted as S_D.

It is worth noticing that the Mond-Weir type dual can be obtained, as a particular case, with $\delta = 0$, $J_3 = \{1, \ldots, p\}$ and $J_1 = J_2 = J_4 = \emptyset$, while the Wolfe type dual can be obtained with $\delta = 1$, $J_1 = \{1, \ldots, p\}$ and $J_2 = J_3 = J_4 = \emptyset$.

[4] Notice that mixed type dual problems have been firstly introduced by Xu in [63] in the case of paretian cones C and V and a primal feasible region defined by just inequality constraints. Similar results for a primal feasible region defined by both equality and inequality constraints been proposed in [1, 2, 52]. In [25] mixed type duality have been studied for generic convex cones C and V and a primal feasible region defined by equality, inequality and set constraints.

It can be easily seen that when the dual problem has a more "complex" objective function then there is a smaller number of constraints and hence a bigger feasible region, while when the dual problem has a "simpler" objective function then there is a bigger number of constraints and hence a smaller feasible region.

For the sake of convenience, the following further function can be introduced:

$$\begin{aligned}\mathcal{D}(x, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4}) &= (1-\delta)\alpha_g^T g(x) + \alpha_{h_2}^T h_2(x) + \alpha_{h_3}^T h_3(x) + \alpha_{h_4}^T h_4(x) \\ &= \mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) - \alpha_f^T \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)\end{aligned}$$

Finally, notice that the following helpful properties follow directly from the definitions:

C_A) $\mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h) \in f(x_1) + C \quad \forall(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, $\forall x_1 \in S_P$

C_B) $\mathcal{D}(x_2, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4}) \leq 0 \leq \mathcal{D}(x_1, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4})$ $\forall(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$ and $\forall x_1 \in S_P$

6.2 Weak duality

The following weak duality result can now be proved assuming some suitable generalized concavity properties (see Subsection 7.4 in the appendix).

Theorem 6.1 *Let us consider the primal problem P and the dual problem D, and let C^* be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. Assume also that at least one of the following conditions (C_1), (C_2), (C_3) and (C_4) is verified for all multipliers $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$:*

(C_1) *$C^* = Int(C)$ and function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is pseudoconcave with respect to x;*

(C_2) *$C^* = C$ and function $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ is strictly pseudoconcave with respect to x;*

(C_3) *function $\mathcal{F}(x, \ldots)$ is C^*-pseudoconcave and function $\mathcal{D}(x, \ldots)$ is quasiconcave with respect to x;*

(C_4) *function $\mathcal{F}(x, \ldots)$ is C^*-quasiconcave and function $\mathcal{D}(x, \ldots)$ is strictly pseudoconcave with respect to x.*

Then, $\forall x_1 \in S_P$ and $\forall(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$ it is:

$$f(x_1) \notin \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

where in the case $C^* = C$ it is also assumed that $x_1 \neq x_2$.

Proof Suppose by contradiction that $\exists x_1 \in S_P$ and $\exists(x_2, \alpha_f, \alpha_g, \alpha_h) \in S_D$ such that

$$f(x_1) \in \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

For condition C_A) it is $\mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h) \in f(x_1) + C$ so that, being C a closed convex pointed cone with nonempty interior, it results

$$\mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) + C^*$$

It can be easily seen that $x_1 \neq x_2$, in fact if $C^* = C$ this is guaranteed by the hypothesis, while if $C^* = Int(C)$ this is implied by the previous condition.

[Case (C_1), (C_2)*]* Since $\alpha_f \in C^+ \setminus \{0\}$, it results:

$$\alpha_f^T \mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h) \geq \alpha_f^T \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) \quad [> 0 \text{ if } C^* = Int(C)]$$

For condition $C_B)$ it is $\mathcal{D}(x_1, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4}) \geq \mathcal{D}(x_2, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4})$ so that, being $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h) = \mathcal{D}(x, \alpha_g, \alpha_{h_2}, \alpha_{h_3}, \alpha_{h_4}) + \alpha_f^T \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)$, it yields:

$$\mathcal{L}(x_1, \alpha_f, \alpha_g, \alpha_h) \geq \mathcal{L}(x_2, \alpha_f, \alpha_g, \alpha_h) \quad [> 0 \text{ if } C^* = Int(C)]$$

As a consequence, being $\mathcal{L}(x, \alpha_f, \alpha_g, \alpha_h)$ strictly pseudoconcave with respect to x [just pseudoconcave if $C^* = Int(C)$], it follows:

$$\nabla \mathcal{L}_x(x_2, \alpha_f, \alpha_g, \alpha_h)^T (x_1 - x_2) > 0$$

This implies that $(x_2, \alpha_f, \alpha_g, \alpha_h) \notin S_D$, which is a contradiction.

[Case (C_3)*]* Let $J_{\mathcal{F}_x}(x, \alpha_f, \alpha_g, \alpha_h)$ and $\nabla \mathcal{D}_x(x, \ldots)$ be, respectively, the Jacobian matrix of $\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)$ and the gradient vector of $\mathcal{D}(x, \ldots)$ with respect to variable x. Since $\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)$ is C^*-pseudoconcave with respect to x it yields

$$J_{\mathcal{F}_x}(x_2, \alpha_f, \alpha_g, \alpha_h)(x_1 - x_2) \in Int(C)$$

hence, since $\alpha_f \in C^+ \setminus \{0\}$, it results

$$\alpha_f^T J_{\mathcal{F}_x}(x_2, \alpha_f, \alpha_g, \alpha_h)(x_1 - x_2) > 0$$

For condition $C_B)$ it is $\mathcal{D}(x_1, \ldots) \geq \mathcal{D}(x_2, \ldots)$ so that, being $\mathcal{D}(x, \ldots)$ quasiconcave with respect to the variable x, it results

$$\nabla \mathcal{D}_x(x_2, \ldots)^T (x_1 - x_2) \geq 0$$

As a consequence, it yields

$$\nabla \mathcal{L}_x(x_2, \ldots)^T (x_1 - x_2) = \nabla \mathcal{D}_x(x_2, \ldots)^T (x_1 - x_2) + \alpha_f^T J_{\mathcal{F}_x}(x_2, \ldots)(x_1 - x_2) > 0$$

This implies that $(x_2, \alpha_f, \alpha_g, \alpha_h) \notin S_D$, which is a contradiction.

[Case (C_4)*]* The proof is analogous to the one of the previous case. □

Note that, in the previous theorem, the bigger is the cone C^* the stronger is the proved necessary condition. Notice also that in [25] the authors proposed some other generalized concavity properties guaranteeing the weak duality result.

Corollary 6.1 *Let us consider the primal problem P and the dual problem D, and let C^* be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. Assume also that at least one of conditions (C_1), (C_2), (C_3) and (C_4) is verified for all multipliers $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$. If $(x, \alpha_f, \alpha_g, \alpha_h) \in S_D$ with $\delta \alpha_g^T g(x) = 0$ and $x \in S_P$ then*

$$x \in C^*\_\arg\max(P) \text{ and } (x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D).$$

Proof As a preliminary result, note that $x \in S_P$ implies $h_1(x) = 0$ so that from assumption $\delta\alpha_g g(x) = 0$ it yields $\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) = f(x)$.
Suppose by contradiction that $x \notin C^*\_\arg\max(P)$, that is to say that there exists $y \in S_p$ such that $f(y) \in f(x) + C^*$; hence $f(y) \in \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) + C^*$ and this contradicts the weak duality result.
Now suppose by contradiction that $(x, \alpha_f, \alpha_g, \alpha_h) \notin C^*\_\arg\min(D)$, that is to say that there exists $(\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) \in S_D$ such that $\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) \in \mathcal{F}(\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) + C^*$; hence $f(x) \in \mathcal{F}(\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) + C^*$ and this contradicts the weak duality result too. □

6.3 Strong duality

It is now possible to prove the following strong duality result. Notice that it requires the set X to be convex in order to verify the assumptions of Property 4.1.

Theorem 6.2 *Let us consider the primal problem P and the dual problem D, and let C^* be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. Assume also that at least one of conditions (C_1), (C_2), (C_3) and (C_4) is verified for all multipliers $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, that X is convex and that a constraint qualification condition holds for problem P. Then, $\forall x \in C^0\_\arg\max(P)$ $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$ such that:*

$$(x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D) \quad \textit{and} \quad f(x) = \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)$$

Proof Let $x \in C^0\_\arg\max(P)$; since a constraint qualification condition holds and X is convex, for Property 4.1 $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$ such that $\alpha_g^T g(x) = 0$ and

$$\nabla \mathcal{L}_x(x, \alpha_f, \alpha_g, \alpha_h)^T (y - x) \leq 0 \ \forall y \in X.$$

Since $h(x) = 0$ and $\alpha_g^T g(x) = 0$ it results $f(x) = \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h)$ and $(x, \alpha_f, \alpha_g, \alpha_h) \in S_D$. For the weak duality theorem $\nexists (\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) \in S_D$ such that

$$\mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) = f(x) \in \mathcal{F}(\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) + C^*$$

In other words, $\nexists (\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) \in S_D$ such that

$$\mathcal{F}(\hat{x}, \hat{\alpha}_f, \hat{\alpha}_g, \hat{\alpha}_h) \in \mathcal{F}(x, \alpha_f, \alpha_g, \alpha_h) - C^*$$

and hence $(x, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$. □

The following result follows directly from Theorem 6.2.

Corollary 6.2 *Let us consider the primal problem P and the dual problem D, and let C^* be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. Assume also that at least one of conditions (C_1), (C_2), (C_3) and (C_4) is verified for all multipliers $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, that X is convex and that a constraint qualification condition holds for problem P. If $C^*\_\arg\min(D) = \emptyset$. Then, $C^0\_\arg\max(P) = \emptyset$.*

The following further duality result follows from the weak and the strong duality theorems.

Corollary 6.3 *Let us consider the primal problem P and the dual problem D, and let C^* be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. Assume also that at least one of conditions (C_1), (C_2), (C_3) and (C_4) is verified for all multipliers $(\alpha_f, \alpha_g, \alpha_h) \in (C^+ \times V^+ \times \Re^p)$, $\alpha_f \neq 0$, that X is convex and that a constraint qualification condition holds for problem P. If $C^*\_\arg\min(D) = \emptyset$. Then,*

$$f(x_1) - \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin (C^* \cup -C^*)$$

$\forall x_1 \in C^0\_\arg\max(P)$ *and* $\forall (x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$.

Proof Let $x_1 \in C^0\_\arg\max(P)$ and $(x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$; for the weak duality theorem it is

$$f(x_1) - \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin C^*$$

For the strong duality theorem $\exists \alpha_f \in C^+ \setminus \{0\}$, $\exists \alpha_g \in V^+$, $\exists \alpha_h \in \Re^p$ such that $(x_1, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$ and $f(x_1) = \mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h)$. As a consequence, condition $(x_2, \alpha_f, \alpha_g, \alpha_h) \in C^*\_\arg\min(D)$ implies

$$\mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h) \notin \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) - C^*$$

and hence for the equality $f(x_1) = \mathcal{F}(x_1, \alpha_f, \alpha_g, \alpha_h)$ we have

$$f(x_1) - \mathcal{F}(x_2, \alpha_f, \alpha_g, \alpha_h) \notin -C^*$$

which proves the result. □

7 Appendix

The aim of this appendix is to summarize some general concepts and properties helpful in the study of optimality conditions.

7.1 Directional derivatives and differentiability

In this subsection the main concepts of directional derivability and differentiability are summarized. See for example [31] for a complete study of the subject.

Definition 7.1 Let $f : A \to \Re$, with $A \subseteq \Re^n$ open set, and let $x_0 \in A$. The following limit, if it exists:

$$f'_D(x_0, d) = \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda}$$

is called the *Dini derivative of function f at the point x_0 in a direction d.*

The following limit, if it exists:

$$f'_H(x_0, d) = \lim_{\lambda \to 0^+, v \to d} \frac{f(x_0 + \lambda v) - f(x_0)}{\lambda}$$

is called the *Hadamard derivative of function f at the point x_0 in a direction d.*

It is clear that the existence of the Hadamard derivative implies the existence of the Dini derivative and the two derivatives coincide. In this light, for the sake of simplicity, the same symbol $\frac{\partial f}{\partial d}(x_0)$ can be used to denote both the two derivatives. Notice that the directional derivative $f'_D(x_0, d)$ is positively homogeneous (of the first degree) as a function of the direction d, that is to say that

$$f'_D(x_0, \mu d) = \mu f'_D(x_0, d) \quad \forall \mu > 0$$

Definition 7.2 Let $f : A \to \Re$, with $A \subseteq \Re^n$ open set, and let $x_0 \in A$. Function f is said to be:

- *Dini differentiable at the point* x_0 if the limit $f'_D(x_0, d)$ exists and is finite for all directions $d \in \Re^n$;
- *Hadamard differentiable at the point* x_0 if the limit $f'_H(x_0, d)$ exists and is finite for all directions $d \in \Re^n$;
- *Dini uniformly differentiable at the point* x_0 if it is Dini differentiable at x_0 and:

$$f(x_0 + d) = f(x_0) + f'_D(x_0, d) + o(\|d\|)$$

The following fundamental result holds (see for example [31]).

Property 7.1 *A function f is Hadamard differentiable if and only if it is Dini uniformly differentiable and its directional derivative $\frac{\partial f}{\partial d}(x_0)$ is continuous as a function of direction.*

The following example points out that the Dini uniform differentiability does not guarantee the continuity of the directional derivative as a function of direction.

Example 7.1 Let $\{x_k\}$ be a sequence of different points on the unit circle of the space $\Re^2$ and define the function:

$$f(x) = \begin{cases} k\lambda & \text{if } x = \lambda x_k \text{ for some } k \text{ and } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

It can be verified that f is Dini uniformly differentiable at the point $x_0 = (0,0)$ even if the directional derivative $f'_D(x_0, d)$ is discontinuous as a function of direction.

Another important property is the following (see for example [31]).

Property 7.2 *If a function f is Hadamard differentiable at some point x_0, then it is continuous at this point.*

Notice that a Dini differentiable function (not Dini uniformly differentiable) with directional derivatives continuous as a function of direction may be discontinuous, as it is pointed out in the next example.

Example 7.2 Let $X = X_1 \cup X_2 \subset \Re^2$ where $X_1 = \{(x_1, x_2) \in \Re^2 : x_2 \geq x_1^2\}$ and $X_2 = \{(x_1, x_2) \in \Re^2 : x_2 \leq 0\}$, let $x_0 = (0,0)$ and define the function:

$$f(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$$

It can be verified that f is Dini differentiable at x_0 (but not Dini uniformly differentiable) with $f'_D(x_0, d) = 0$ for all directions d. Function f is clearly discontinuous at x_0, even if the directional derivative $f'_D(x_0, d)$ is continuous as a function of direction.

Definition 7.3 Let $f : A \to \Re$, with $A \subseteq \Re^n$ open set, and let $x_0 \in A$. Function f is said to be:

- *Gâteaux differentiable at the point* x_0 if it is Dini differentiable at x_0 with directional derivatives which are linear as functions of direction, that is to say that for all d:

$$f'_D(x_0, d) = \nabla f(x_0)^T d$$

- *Fréchet differentiable at the point* x_0 if it is Dini uniformly differentiable at x_0 with directional derivatives which are linear as functions of direction, that is to say that for all d:

$$f(x_0 + d) = f(x_0) + \nabla f(x_0)^T d + o(\|d\|)$$

The following further result follows directly from Property 7.1.

Property 7.3 *If a function f is Fréchet differentiable at some point x_0, then it is Hadamard differentiable at that point.*

The relationships among the various differentiability properties are summarized in the following scheme. It is worth pointing out that Example 7.1 provides a function which is Dini uniformly differentiable but not Hadamard differentiable, while Example 7.2 provides a function which is Gâteaux differentiable but neither Fréchet differentiable nor Dini uniformly differentiable. There is no need to recall that the absolute value single variable function is Hadamard differentiable but not Gâteaux differentiable since the directional derivatives are not linear as functions of direction.

Hadamard	$\Rightarrow$	Dini Uniformly	$\Rightarrow$	Dini
$\Uparrow$				$\Uparrow$
Fréchet		$\Longrightarrow$		Gâteaux

As a conclusion, it is worth recalling that a vector valued function $F : A \to \Re^s$ is [Dini, Hadamard, Gâteaux, Fréchet] differentiable at x_0 if all its components are [Dini, Hadamard, Gâteaux, Fréchet] differentiable at x_0.

7.2 Conical approximations

In this subsection the main conical approximations, useful in optimization, are summarized. See for example [6, 33, 34] for a complete study of the subject. Given a point $x_0 \in Cl(X) \subseteq \Re^n$ these cones are aimed to provide a local approximation of the set $X - \{x_0\}$, which is helpful for the investigation of optimality conditions.

Definition 7.4 Let $X \subseteq \Re^n$ be a nonempty set and let $x_0 \in Cl(X)$. The *Bouligand Tangent cone to* X *at* x_0, denoted with $T(X, x_0)$, the *cone of feasible directions to* X *at* x_0, denoted with $F(X, x_0)$, and the *cone of interior directions to* X *at* x_0, denoted with $I(X, x_0)$, are defined as follows:

$$\begin{aligned} T(X,x_0) &= \{d \in \Re^n : \exists \{x_k\} \subset X, x_k \to x_0, \exists \{\lambda_k\} \subset \Re^{++}, \lambda_k \to +\infty, \\ &\quad d = \lim_{k \to +\infty} \lambda_k (x_k - x_0)\}; \\ F(X,x_0) &= \{d \in \Re^n : \exists \epsilon > 0 \text{ such that } x_0 + \lambda d \in X \ \forall \lambda \in (0,\epsilon)\}; \\ I(X,x_0) &= \{d \in \Re^n : \exists \epsilon > 0, \exists \delta > 0 \text{ such that } \|y - d\| < \delta \text{ imply} \\ &\quad x_0 + \lambda y \in X \ \forall \lambda \in (0,\epsilon)\}. \end{aligned}$$

It is worth noticing that the Bouligand Tangent cone is closed, that $\{0\} \in T(X, x_0)$ and that $T(X, x_0)$ can be rewritten as follows:

$$\begin{aligned} T(X,x_0) &= \{d \in \Re^n : d = \mu v, \mu \geq 0, \exists \{x_k\} \subset X, x_k \to x_0, x_k \neq x_0, \\ &\quad v = \lim_{k \to +\infty} \frac{x_k - x_0}{\|x_k - x_0\|}\bigg\}. \end{aligned}$$

On the other hand, the cone of interior directions is open. Notice also that none of these cones is necessarily convex. The following main properties hold (see for example [34]):

- if $x_0 \in Int(X)$ then $I(X, x_0) = F(X, x_0) = T(X, x_0) = \Re^n$;
- if $X_1 \subseteq X_2$ then $I(X_1, x_0) \subseteq I(X_2, x_0)$, $F(X_1, x_0) \subseteq F(X_2, x_0)$, $T(X_1, x_0) \subseteq T(X_2, x_0)$;
- $T(X, x_0) = T(Cl(X), x_0)$;
- $I(X, x_0) = I(Int(X), x_0)$;
- $0 \in I(X, x_0)$ if and only if $x_0 \in Int(X)$;
- $0 \in F(X, x_0)$ if and only if $x_0 \in X$;
- $I(X, x_0) = [T(X^C, x_0)]^C$ and $T(X, x_0) = [I(X^C, x_0)]^C$.

The following inclusion relationships hold.

Property 7.4 *Let* $X \subseteq \Re^n$ *be a nonempty set and let* $x_0 \in Cl(X)$. *Then,*

$$\begin{aligned} I(X,x_0) \subseteq Int(F(X,x_0)) \subseteq \ F(X,x_0) \ \subseteq Cl(F(X,x_0)) \subseteq T(X,x_0) \\ I(X,x_0) \subseteq F(Int(X),x_0) \subseteq \ F(X,x_0) \ \subseteq F(Cl(X),x_0) \subseteq \ T(X,x_0) \end{aligned}$$

Proof Directly from the definitions we have $I(X, x_0) \subseteq F(X, x_0) \subseteq T(X, x_0)$. Hence, since $I(X, x_0)$ is open and $T(X, x_0)$ is closed we have $I(X, x_0) \subseteq Int(F(X, x_0))$ and $Cl(F(X, x_0)) \subseteq T(X, x_0)$. Since $Int(X) \subseteq X \subseteq Cl(X)$ we have $F(Int(X), x_0) \subseteq F(X, x_0) \subseteq F(Cl(X), x_0)$. The result then follows since $T(X, x_0) = T(Cl(X), x_0)$ and $I(X, x_0) = I(Int(X), x_0)$. □

In order to study optimality conditions, a key tool is the conical approximation of the intersection of two sets. With this aim, given $x_0 \in Cl(X \cap H) \subseteq Cl(X) \cap Cl(H)$ with $X \subseteq \Re^n$ and $H \subseteq \Re^n$ sets such that $X \cap H \neq \emptyset$, in [34] the following results are proved.

- $I(X \cap H, x_0) = I(X, x_0) \cap I(H, x_0)$;
- $F(X \cap H, x_0) = F(X, x_0) \cap F(H, x_0)$;
- $T(X \cap H, x_0) \subseteq T(X, x_0) \cap T(H, x_0)$.

The following example points out that the last inclusion relationship may be strict.

Example 7.3 Let $X = \{(x_1, x_2) \in \Re^2 : x_2 = x_1^2, x_1 \geq 0\}$, $H = \{(x_1, x_2) \in \Re^2 : x_2 = -x_1^2, x_1 \geq 0\}$ and let $x_0 = (0,0)$. It is $X \cap H = \{x_0\}$ so that $T(X \cap H, x_0) = \{(0,0)\}$, while $T(X, x_0) = T(H, x_0) = \{(x_1, x_2) \in \Re^2 : x_2 = 0, x_1 \geq 0\} \neq T(X \cap H, x_0)$.

The following further useful property can be easily proved.

Property 7.5 *Let $X \subseteq \Re^n$ and $H \subseteq \Re^n$ be such that $X \cap H \neq \emptyset$ and let $x_0 \in Cl(X \cap H)$. Then,*

$$\begin{aligned} I(X, x_0) \cap T(H, x_0) &\subseteq T(X \cap H, x_0) \\ F(X, x_0) \cap F(H, x_0) &\subseteq T(X \cap H, x_0) \end{aligned}$$

Proof The second inclusion relationship follows trivially since

$$F(X, x_0) \cap F(H, x_0) = F(X \cap H, x_0) \subseteq T(X \cap H, x_0)$$

Let us now prove the first one. If $I(X, x_0) \cap T(H, x_0) = \emptyset$ or $I(X, x_0) \cap T(H, x_0) = \{0\}$ the result is trivial. Assume now that there exists $t \neq 0$, $t \in I(X, x_0) \cap T(H, x_0)$; since $t \in T(H, x_0)$ then $\exists \{x_k\} \subset H$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to +\infty$, such that $t = \lim_{k \to +\infty} \lambda_k (x_k - x_0)$; since $t \in I(X, x_0)$ then $\exists \epsilon > 0$, $\exists \delta > 0$ such that $\|y - t\| < \delta$ imply $x_0 + \mu y \in X$ $\forall \mu \in (0, \epsilon)$. Since $\lambda_k \to +\infty$ and $t = \lim_{k \to +\infty} \lambda_k (x_k - x_0)$ there exists an integer $\bar{k} > 0$ such that $\forall k > \bar{k}$ it is $\|(\lambda_k (x_k - x_0)) - t\| < \delta$ and $\frac{1}{\lambda_k} < \epsilon$; hence, assuming $y = \lambda_k (x_k - x_0)$ and $\mu = \frac{1}{\lambda_k}$ we have:

$$x_k = x_0 + \frac{1}{\lambda_k} (\lambda_k (x_k - x_0)) \in X$$

Hence, $\forall k > \bar{k}$ we have $x_k \in X \cap H$ so that $t \in T(X \cap H, x_0)$ and the result is proved. □

In order to deepen on the previous property, notice that in general

$$\begin{aligned} Int(F(X, x_0)) \cap T(H, x_0) &\not\subseteq T(X \cap H, x_0), \\ Cl(I(X, x_0)) \cap T(H, x_0) &\not\subseteq T(X \cap H, x_0), \\ Cl(F(X, x_0)) \cap Cl(F(H, x_0)) &\not\subseteq T(X \cap H, x_0). \end{aligned}$$

as it is shown in the next examples.

Example 7.4 Let $X = X_1 \cup X_2 \subset \Re^2$ and $H \subset \Re^2$ where:

$$\begin{aligned} X_1 &= \{(x_1, x_2) \in \Re^2 : x_2 \geq 2x_1^2\} \\ X_2 &= \{(x_1, x_2) \in \Re^2 : x_2 \leq 0\} \\ H &= \{(x_1, x_2) \in \Re^2 : x_2 - x_1^2 = 0\} \end{aligned}$$

Defining $x_0 = (0,0)$, it is $X \cap H = \{x_0\}$ so that $T(X \cap H, x_0) = \{0\}$. On the other hand, it is $T(H, x_0) = \{(x_1, x_2) \in \Re^2 : x_2 = 0\}$, $F(X, x_0) = T(X, x_0) = \Re^2$ and $I(X, x_0) = \{(x_1, x_2) \in \Re^2 : x_2 \neq 0\}$, so that $Cl(I(X, x_0)) = Int(F(X, x_0)) = \Re^2$. As a consequence it results:

$$\begin{aligned} Int(F(X, x_0)) \cap T(H, x_0) = T(H, x_0) &\not\subseteq \{0\} = T(X \cap H, x_0) \\ Cl(I(X, x_0)) \cap T(H, x_0) = T(H, x_0) &\not\subseteq \{0\} = T(X \cap H, x_0) \end{aligned}$$

Example 7.5 Consider the following convex subsets of $\Re^2$:

$$\begin{aligned} X &= \{(x_1, x_2) \in \Re^2 : x_2 \geq x_1^2 \}, \\ H &= \{(x_1, x_2) \in \Re^2 : x_2 = 0\} \end{aligned}$$

Defining $x_0 = (0,0)$, it is $X \cap H = \{x_0\}$ so that $T(X \cap H, x_0) = \{0\}$. On the other hand, it is $F(H, x_0) = T(H, x_0) = H$ and $F(X, x_0) = \{(x_1, x_2) \in \Re^2 : x_2 > 0\}$. As a consequence it results:

$$Cl(F(X, x_0)) \cap Cl(F(H, x_0)) = H \not\subseteq \{0\} = T(X \cap H, x_0)$$

To conclude the discussion, it is worth providing the following result proved in [6]:

$$\text{if } x_0 \in Int(X) \text{ then } T(X \cap H, x_0) = T(H, x_0)$$

Finally, let us recall the following properties (see for all [33, 34]) dealing for locally convex sets ([5]).

Property 7.6 *Let $X \subseteq \Re^n$ be a locally convex set at $x_0 \in Cl(X)$. Then,*

- $I(X, x_0) = cone(Int(X), x_0)$;
- $F(X, x_0) = cone(X, x_0)$;
- $T(X, x_0) = Cl(cone(X, x_0))$.

All these conical approximations are convex. Furthermore, if $Int(X) \neq \emptyset$ then,

- $I(X, x_0) = Int(T(X, x_0))$;
- $T(X, x_0) = Cl(I(X, x_0))$.

[5] $X \subseteq \Re^n$ is a locally convex set at x_0 if $\exists I_{x_0}$, arbitrary open ball about x_0, such that $X \cap I_{x_0}$ is convex.

7.3 Generalized Lyusternik Theorem

In optimization theory it is sometimes useful to study conical approximations of the points verifying some equality constraints.

Definition 7.5 Let $h : A \to \Re^p$, with $A \subseteq \Re^n$ open set, be an Hadamard differentiable function and let $H = \{x \in A : h(x) = 0\}$. Given a point $x_0 \in H$, the *linearizing cone to* H *at* x_0, denoted with $L(H, x_0)$, is defined as follows:

$$L(H, x_0) = \{0\} \cup \{d \in \Re^n \setminus \{0\} : h'_H(x_0, d) = 0\}$$

The following fundamental preliminary result holds.

Property 7.7 *Let* $h : A \to \Re^p$, *with* $A \subseteq \Re^n$ *open set, be an Hadamard differentiable function and let* $H = \{x \in A : h(x) = 0\}$. *Given a point* $x_0 \in H$ *it results*

$$T(H, x_0) \subseteq L(H, x_0)$$

Proof Let $d \in T(H, x_0)$ and let us prove that $d \in L(H, x_0)$. If $d = 0$ the result is trivial. Assume now $d \neq 0$; then, $\exists \{x_k\} \subset H$, $x_k \to x_0$, $\exists \{\lambda_k\} \subset \Re^{++}$, $\lambda_k \to 0^+$ such that $d = \lim_{k \to +\infty} \frac{x_k - x_0}{\lambda_k}$. Let us define $v_k = \frac{x_k - x_0}{\lambda_k}$, so that $v_k \to d$ and $x_k = x_0 + \lambda_k v_k$. It is:

$$h'_H(x_0, d) = \lim_{\lambda_k \to 0^+, v_k \to d} \frac{h(x_0 + \lambda_k v_k) - h(x_0)}{\lambda_k} = \lim_{k \to +\infty} \frac{h(x_k) - h(x_0)}{\lambda_k} = \lim_{k \to +\infty} 0 = 0$$

since $h(x_0) = 0$ and for all k it is $h(x_k) = 0$ and $\lambda_k > 0$. The result is then proved. □

In order to study optimality conditions, it is helpful to determine suitable assumptions guaranteeing that $T(H, x_0) = L(H, x_0)$. In this light, a very well known result is the so called *Lyusternik Theorem* (see for all [44, 48]).

Theorem 7.1 [44] *Let* $h : X \to \Re^p$, $X \subseteq \Re^n$, *be a given mapping and let* $x_0 \in H = \{x \in \Re^n : h(x) = 0\}$. *Let also h be Fréchet differentiable on a neighbourhood of* x_0, *let* $J_h(x)$ *be continuous at* x_0 *and let* $J_h(x_0)$ *be surjective. Then it follows:*

$$T(H, x_0) = Ker(J_h(x_0)) = L(H, x_0)$$

The Lyusternik theorem have been recently generalized in [46].

Theorem 7.2 [46] *Let us suppose the following:*

- *function* $h : \Re^n \to \Re^p$ *is continuous on a neighborhood of* x_0 *and Frechét differentiable at* x_0,
- $X \subset \Re^n$ *is a convex set and* $x_0 \in X \cap H$, *where* $H = \{x \in \Re^n : h(x) = 0\}$,
- *the following regularity condition holds:* $J_h(x_0)[T(X, x_0)] = \Re^p$.

Then $T(X \cap H, x_0) = Ker(J_h(x_0)) \cap T(X, x_0)$.

The following corollary follows directly from Property 7.2 just assuming $X = \Re^n$.

Corollary 7.1 *Let us suppose the following:*

- $h : \Re^n \to \Re^p$ *is continuous on a neighborhood of* x_0 *and Frechét differentiable at* x_0,
- $x_0 \in H = \{x \in \Re^n : h(x) = 0\}$,
- $J_h(x_0)$ *is surjective* ($J_h(x_0)[\Re^n] = \Re^p$), *so that the gradients* $\nabla h_1(x_0), \ldots, \nabla h_p(x_0)$ *are linearly independent.*

Then $T(H, x_0) = Ker(J_h(x_0)) = L(H, x_0)$.

7.4 Generalized Concavity

Generalized convexity/concavity has been widely studied in the recent literature due to its usefulness in applicative problems and in optimization. Both scalar and multiobjective generalized concave functions have been defined and studied (see for example [5, 40]). In the following the classes of generalized concave functions used in this paper have been recalled.

Definition 7.6 A differentiable scalar function $f : A \to \Re$, $A \subseteq \Re^n$, is said to be quasiconcave if:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T(x_1 - x_2) \geq 0 \qquad \forall x_1, x_2 \in A,\ x_1 \neq x_2,$$

it is said to be pseudoconcave if:

$$f(x_1) > f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T(x_1 - x_2) > 0 \qquad \forall x_1, x_2 \in A,$$

while it is said to be strictly pseudoconcave if:

$$f(x_1) \geq f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T(x_1 - x_2) > 0 \qquad \forall x_1, x_2 \in A,\ x_1 \neq x_2.$$

Definition 7.7 Let $C \subset \Re^s$ be a closed convex pointed cone with nonempty interior, and let $C^* \subseteq C$ be a cone such that $C^* = C$ or $Int(C) \subseteq C^* \subseteq C \setminus \{0\}$. A differentiable vector valued function $f : A \to \Re^s$, $A \subseteq \Re^n$, is said to be C^*-quasiconcave if:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in C \qquad \forall x_1, x_2 \in A,\ x_1 \neq x_2,$$

while it is said to be C^*-pseudoconcave if:

$$f(x_1) \in f(x_2) + C^* \quad \Rightarrow \quad J_f(x_2)(x_1 - x_2) \in Int(C) \qquad \forall x_1, x_2 \in A,\ x_1 \neq x_2.$$

References

[1] Aghezzaf, B. and Hachimi, M. (2000), Generalized invexity and Duality in Multiobjective Programming Problems, *Journal of Global Optimization*, Vol. 188, pp. 91-101.

[2] Aghezzaf, B. and Hachimi, M. (2001), Sufficiency and Duality in Multiobjective Programming Involving Generalized (F, ρ)–convexity, *Journal of Mathematical Analysis and Applications*, Vol. 258, pp. 617-628.

[3] Averbukh V.I. and O.G. Smolyanov (1967), The theory of differentiation in linear topological spaces, *Russian Mathematical Surveys*, vol.22, pp.201-258.

[4] Averbukh V.I. and O.G. Smolyanov (1968), The various definitions of the derivative in linear topological spaces, *Russian Mathematical Surveys*, vol.23, pp.67-113.

[5] Avriel M., Diewert W.E., Schaible S. and Zang I. (1988), *Generalized Concavity*, Mathematical Concepts and Methods in Science and Engineering, Vol.36, Plenum Press, New York.

[6] Bazaraa M.S. and C.M. Shetty (1976), *Foundations of Optimization*, Lecture Notes in Economics and Mathematical Systems, vol.122, Springer-Verlag.

[7] Bazaraa M.S., Sherali H.D. and C.M. Shetty (1993), *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, New York.

[8] Bhatia, D. and Jain, P. (1994), Generalized (F,ρ)-convexity and duality for non smooth multi-objective programs, *Optimization*, Vol. 31, pp. 153-164.

[9] Bathia, D. and Garg, P.K. (1998), Duality for non-smooth nonlinear fractional multiobjective programs via (F,ρ)-convexity, Optimization, Vol. 43, pp. 185-197.

[10] Bector, C.R., Bector, M.K., Gill, A. and Singh, C. (1994), Duality for Vector Valued B-invex Programming, in *Generalized Convexity*, edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, Vol. 405, Springer-Verlag, Berlin, pp. 358-373.

[11] Bector, C.R. (1996), Wolfe-Type Duality involving (B,η)-invex Functions for a Minmax Programming Problem, *Journal of Mathematical Analysis and Application*, Vol. 201, pp. 114-127.

[12] Cambini A. and L. Martein (1992), Tangent cones in optimization, in *Generalized Concavity for Economic Applications*, edited by P. Mazzoleni, Tecnoprint, Bologna, pp.29-39.

[13] Cambini A. and L. Martein (1994), Generalized concavity and optimality conditions in vector and scalar optimization, in *Generalized Convexity*, edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, vol.405, Springer-Verlag, Berlin, pp.337-357.

[14] Cambini A. and L. Martein (1995), Second order necessary optimality conditions in the image space: preliminary results, in *Scalar and Vector Optimization in Economic and Financial Problems*, edited by E. Castagnoli and G. Giorgi, pp.27-38.

[15] Cambini A., Martein L. and R. Cambini (1997), A new approach to second order optimality conditions in vector optimization, in *Advances in Multiple Objective and Goal Programming*, edited by R. Caballero, F. Ruiz and R. Steuer, Lecture Notes in Economics and Mathematical Systems, vol.455, Springer-Verlag, Berlin, pp.219-227.

[16] Cambini A., Komlósi S. and L. Martein (1998), Recent developments in second order necessary optimality conditions, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, vol.27, Kluwer Academic Publishers, Dordrecht, pp.347-356.

[17] Cambini R. (1996), Generalized concavity and optimality conditions in vector optimization, in *Operations research and its applications*, edited by D.Z. Du, X.S. Zhang and K. Cheng, Lecture Notes in Operations Research, vol.2, World Publishing Corporation, Beijing, pp.172-180.

[18] Cambini, R. (1996), Some new classes of generalized concave vector-valued functions, *Optimization*, Vol. 36, pp. 11-24.

[19] Cambini R. (1998), Second order optimality conditions in Multiobjective Programming, *Optimization*, vol.44, pp.139-160.

[20] Cambini R. (1998), Minimum Principle Type Necessary Optimality Conditions: Preliminary Results, in *Convessitá e Calcolo Parallelo*, edited by G. Giorgi and F.A. Rossi, Libreria Universitaria Editrice, Verona, pp.65-79.

[21] Cambini R. (1999), Minimum principle type necessary optimality conditions with equality constraints: preliminary results, in *Generalized Convexity and Optimization for Economic and Financial Decisions*, edited by G. Giorgi and F.A. Rossi, Pitagora Editrice, Bologna, pp.63-79.

[22] Cambini R. (2001), Necessary Optimality Conditions in Vector Optimization, Report n.212, Department of Statistics and Applied Mathematics, University of Pisa.

[23] Cambini R. (2002), Developments on U-regularity conditions, in *Optimization in Economics, Finance and Industry*, edited by G.P. Crespi, G. Giorgi, A. Guerraggio, D. La Torre, E. Miglierina and M. Rocca, Datanova Editrice, Milano, pp.103-117.

[24] Cambini R. (2003), Multiobjective Problems with Set Constraints, in *Recent Advances in Optimization*, edited by G.P. Crespi, A. Guerraggio, E. Miglierina and M. Rocca, Datanova Editrice, Milano, pp.1-20.

[25] Cambini R. and L. Carosi (2003), Mixed type duality for multiobjective optimization problems with set constraints, Report n.241, Department of Statistics and Applied Mathematics, University of Pisa.

[26] Cambini, R. and Komlósi, S. (1998), On the Scalarization of Pseudoconcavity and Pseudomonotonicity Concepts for Vector Valued Functions, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, Vol. 27, Kluwer Academic Publishers, Dordrecht, pp. 277-290.

[27] Cambini, R. and Komlósi S. (2000), On Polar Generalized Monotonicity in Vector Optimization, *Optimization*, Vol. 47, pp. 111-121.

[28] Cambini R. and Martein L. (2001), First and Second Order Characterizations of a Class of Pseudoconcave Vector Functions, in *Generalized Convexity and Generalized Monotonicity*, edited by N. Hadjisavvas, J.-E. Martinez-Legaz and J.P Penot, Lecture Notes in Economics and Mathematical Systems, Vol.502, Springer, Berlin, pp.144-158.

[29] Canon M., Cullum C. and E. Polak (1966), Constrained Minimization Problems in Finite-dimensional Spaces, *SIAM Journal on Control*, vol.4, pp.528-547.

[30] Caprari, E. (2003), ρ-invex functions and (F, ρ)-convex functions: properties and equivalencies, Optimization, Vol. 52, No. 1, pp.65-74.

[31] Demyanov V.F. and A.M. Rubinov (1995), *Constructive Nonsmooth Analysis*, Peter-Lang, Berlin.

[32] Egudo, R. R. (1989), Efficiency and generalized convex duality for multiobjective programs, *Journal of Mathematical Analysis and Applications*, Vol. 138, pp. 84-94.

[33] Giorgi G. and A. Guerraggio (1992), On the notion of Tangent Cone in Mathematical Programming, *Optimization*, vol.25, pp.11-23.

[34] Giorgi G. and A. Guerraggio (1992), Approssimazioni Coniche Locali: Proprietá Algebriche e Topologiche, Technical Report n.14, Istituto di Metodi Quantitativi, Bocconi University of Milan (Italy).

[35] Giorgi, G. and Guerraggio, A. (1994), First order generalized optimality conditions for programming problems with a set constraint, in *Generalized Convexity*, edited by S. Komlósi, T. Rapcsák and S. Schaible, Lecture Notes in Economics and Mathematical Systems, Vol. 405, Springer-Verlag, Berlin, pp. 171-185.

[36] Giorgi, G. and Mititelu S. (1993), Convexités généralisées et propriétés, *Revue Roumaine de mathématiques pures et appliquées*, Vol. 38, pp. 125-142.

[37] Giorgi, G. and Guerraggio, A. (1998), The notion of invexity in vector optimization: smooth and nonsmooth case, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, edited by J.-P. Crouzeix, J.-E. Martinez-Legaz and M. Volle, Nonconvex Optimization and Its Applications, Vol. 27, Kluwer Academic Publishers, Dordrecht, pp. 389-405.

[38] Göpfert, A. and Tammer, C. (2002), Theory of Vector Optimization, in *Multiple Criteria Optimization*, edited by M. Ehrgott and X. Gandibleux, International Series in Operations Research and Management Science, Vol. 52, Kluwer Academic Publishers, Boston.

[39] Gulati, T. R. and Islam, M.A. (1994), Sufficiency and Duality in Multiobjective Programming Involving Generalized F-convexity, *Journal of Mathematical Analysis and Applications*, Vol. 183, pp. 181-195.

[40] Hadjisavvas N., Komlsi S. and S. Schaible (Eds.) (2005), *Handbook of Generalized Convexity and Generalized Monotonicity*, Nonconvex Optimization and its Applications, vol.76, Springer.

[41] Halkin H. (1966), A Maximum Principle of the Pontryagin Type for Systems Described by Nonlinear Difference Equations, *SIAM Journal on Control*, vol.4, pp.90-111.

[42] Hanson, M.A. and Mond, B. (1987), Necessary and Sufficiency Conditions in Constrained Optimization, *Mathematical Programming*, Vol. 37, pp. 51-58.

[43] Hestenes M.R. (1975), *Optimization Theory: The Finite Dimensional Case*, John Wiley & Sons, New York.

[44] Jahn J. (1994), *Introduction to the Theory of Nonlinear Optimization*, Springer-Verlag, Berlin.

[45] Jeyakumar, V. (1986), ρ-Convexity and second order duality, *Utilitas Mathematica*, Vol. 29, pp. 71-85.

[46] Jiménez, B. and Novo, V. (2002), A finite dimensional extension of Lyusternik theorem with applications to multiobjective optimization, *Journal of Mathematical Analysis and Applications*, vol. 270, pp. 340-356.

[47] Kaul, R.N., Suneja, S. K., and Srivastava, M. K. (1994), Optimality criteria and duality in multiple-objective optimization involving generalized invexity, *Journal of Optimization Theory and Applications*, Vol. 80, pp. 465-482.

[48] Lyusternik L.A. (1934), Conditional extrema of functionals, *Math. Sb.*, vol.41, pp.390-401.

[49] Mangasarian O.L. (1969), *Nonlinear Programming*, McGraw-Hill, New York.

[50] Mangasarian O.L. and S. Fromovitz (1967), A Maximum Principle in Mathematical Programming, in *Mathematical Theory of Control*, edited by A.V. Balakrishnan and L.W. Neustadt, Academic Press Inc., New York, pp.85-95.

[51] Massam H. and S. Zlobec (1974), Various definitions of the derivative in mathematical programming, *Mathematical Programming*, vol.7, pp.144-161.

[52] Mishra, S.K. (1996), On Sufficiency and Duality for Generalized Quasiconvex Nonsmooth Programs, *Optimization*, Vol. 38, pp. 223-235.

[53] Mond, B. and Weir, T. (1981), Generalized concavity and duality, in *Generalized Concavity and Duality in Optimization and Economics*, edited by S. Schaible and W.T. Ziemba, Academic Press, New York, pp. 263-279.

[54] Mukherjee, R.N. and Ch. Purnachandra Rao (2000), Mixed Type Duality for Multiobjective Variational Problems, *Journal of Mathematical Analysis and Applications*, Vol. 252, pp. 571-586.

[55] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V. and E.F. Mishchenko (1962), *The Mathematical Theory of Optimal Processes*, John Wiley and Sons Inc., New York.

[56] Preda, V. (1992), On Efficiency and Duality in Multiobjective Programs, *Journal of Mathematical Analysis and Applications*, Vol. 166, pp. 365-377.

[57] Rueda, N.G., Hanson, M.A. and Singh C. (1995), Optimality and Duality with Generalized Convexity, *Journal of Optimization Theory and Application*, Vol. 86, pp. 491-500.

[58] Tanino, T. and Sawaragi, Y. (1979), Duality Theory in Multiobjective Programming, *Journal of Optimization Theory and Application*, Vol. 27, pp. 509-529.

[59] Sawaragi Y., Nakayama H. and T. Tanino (1985), *Theory of Multiobjective Optimization*, Academic Press, London.

[60] Shapiro A. (1990), On concepts of directional differentiability, *Journal of Optimization Theory and Applications*, vol.66, n.3, pp.477-487.

[61] Vial, J.-P. (1982), Strong convexity of sets and functions, *Journal of Mathematical Economics*, Vol.9, pp.187-205.

[62] Weir, T., Mond, B. and Craven B.D. (1986), On duality for weakly minimized vector-valued optimization problems, *Optimization*, Vol. 17, pp. 711-721.

[63] Xu, Z. (1996), Mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Applications*, Vol. 198, pp. 621-635.