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BERNSTEIN-TYPE APPROXIMATION USING
THE BETA-BINOMIAL DISTRIBUTION

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1. INTRODUCTION

The Bernstein polynomials are generally regarded as the most basic tools for the uniform approximation in the sense of Weierstrass of a continuous and real-valued function g on the closed interval $[0,1]$. The Bernstein polynomials are elegant linear positive operators. The Bernstein polynomials of order m are defined by the binomial distribution $p_m(k;t)$, for $k=0,1,\ldots,m$, where $t \in [0,1]$ is the domain of g. The convergence of the Bernstein polynomials to g is uniform, as $m \to \infty$. Multivariate versions of the Bernstein polynomials can be defined by products of independent binomial distributions. See Korovkin (1960), chapter 1, Davis (1963), chapter 7, Feller (1968), chapter 6, Feller (1971), chapter 7, Rivlin (1981), chapter 1, Cheney (1982), chapters 1 to 4, Lorentz (1986), DeVore and Lorentz (1993), chapter 10, Phillips (2003), chapter 7.

The Bernstein-type approximations of order m in Pallini (2005) improve on the degree of approximation of the Bernstein polynomials by considering a convenient approximation coefficient in linear kernels. The convergence of these Bernstein-type approximations is uniform, as $m \to \infty$.

Here, following Pallini (2005), we study the Bernstein-type approximation of order m, that can be defined by using the beta-binomial distribution. We obtain integral operators that approximate to a continuous and real-valued function g on any closed interval $D \subseteq R^1$. We also obtain their multivariate versions for a continuous and real-valued function g on any closed interval $D \subseteq R^q$. The convergence of these univariate and multivariate Bernstein-type approximations is uniform, as $m \to \infty$.

In section 2, we overview the univariate and the multivariate Bernstein polynomials. In section 3, we present some basic notions for the use of the beta-binomial distribution in approximation. In section 4, we propose the univariate and multivariate Bernstein-type approximations that can be obtained by the beta-binomial distribution. We study the uniform convergence and the degree of approximation. We also compare these Bernstein-type approximations with the Bernstein polynomials. In section 5, we study the Bernstein-type estimators for smooth functions of the population means. In section 6, we discuss the results of a simulation study on some examples of smooth functions of means. Finally, in section 7, we conclude the contribution with comments and remarks.

We refer to Barndorff-Nielsen and Cox (1989), chapter 4, and Sen and Singer (1993), chapter 3, for more details on the smooth functions of means and their application to classical inferential problems.

2. BERNSTEIN POLYNOMIALS

Let P_m be the space of polynomials $P(x)$ of degree at most m, for all real numbers x. Let g be a bounded and real-valued function defined on the closed interval $[0,1]$. The Bernstein polynomial $B_m(g;x)$ of order m for the function g is defined as

$$B_m(g;x) = \sum_{k=0}^{m} g(m^{-1}k)\binom{m}{k} x^k (1-x)^{m-k}, \tag{1}$$

where m is a positive integer number, and $x \in [0,1]$. See Lorentz (1986), chapter 1, and DeVore and Lorentz (1993), chapter 10. Point x in (1) is the population probability for the binomial distribution, where $x \in [0,1]$. It is seen that $B_m(g;x) \in \mathrm{P}_m$, for every $x \in [0,1]$. If $g(x)$ is continuous on $x \in [0,1]$, then we have that $B_m(g;x) \to g(x)$, as $m \to \infty$, uniformly at any point $x \in [0,1]$. The basic proofs of this uniform convergence can be found in Rivlin (1981), chapter 1, and Lorentz (1986), chapter 1. See also Korovkin (1960), chapters 1 to 4, Davis (1963), chapter 6, Feller (1971), chapter 7, and Cheney (1982), chapters 1 to 4, DeVore and Lorentz (1993), chapter 10, Phillips (2003), chapter 7.

Let g be a bounded and real-valued function defined on the closed q-dimensional cube $[0,1]^q$. We let $\mathrm{x} = (x_1,\ldots,x_q)^T$, where $\mathrm{x} \in [0,1]^q$. The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ for the function g is defined as

$$B_{\mathrm{m}}(g;x) = \sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q} g(m_1^{-1}v_1,\ldots,m_q^{-1}v_q)\binom{m_1}{v_1}\cdots\binom{m_q}{v_q} x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_q^{v_q}(1-x_q)^{m_q-v_q}, \tag{2}$$

where $\mathrm{m} = (m_1,\ldots,m_q)^T$ are positive integer numbers, and $\mathrm{x} \in [0,1]^q$. See Lorentz (1986), chapter 2, and DeVore and Lorentz (1993), chapter 1. Points $x_1,\ldots,x_q$ in (2) are the population probabilities for the product of q independent binomial distributions, where $\mathrm{x} \in [0,1]^q$. It is seen that the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x}) \in \mathrm{P}_m$, where $m = \sum_{i=1}^{q} m_i$ is the total degree in $B_{\mathrm{m}}(g;\mathrm{x})$, for every $\mathrm{x} \in [0,1]^q$. The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;x)$ converges to $g(\mathrm{x})$ uniformly, at any q-dimensional point of continuity $\mathrm{x} \in [0,1]^q$, as $m_i \to \infty$, where $i = 1,\ldots,q$. See also Pallini (2005).

3. THE BETA-BINOMIAL DISTRIBUTION

More accurate versions of the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, defined by (1) and (2), where $x \in [0,1]$ and $\mathrm{x} \in [0,1]^q$, can be obtained by the beta-binomial distribution, that is reviewed and studied in Wilcox (1981) and Johnson, Kemp and Kotz (2005), chapter 6.

We recall that the complete gamma function $\Gamma(a)$ is defined as $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$, where $a > 0$, and the complete beta function $B(a,b)$ is defined as $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, where $a > 0$ and $b > 0$. The factorial a, where a is a positive integer number is denoted by $a!$ and is defined as $a! = a(a-1)\cdots 1$. We have $\Gamma(a+1) = a\Gamma(a)$, and $\Gamma(a) = (a-1)!$, the factorial $(a-1)$, whereas a is a positive real number, and $B(a,b) = (\Gamma(a+b))^{-1}\Gamma(a)\Gamma(b)$. Finally, the standard beta distribution, with parameters $a > 0$ and $b > 0$, has probability density function (p.d.f.) $p(t;a,b) = \{B(a,b)\}^{-1}t^{a-1}(1-t)^{b-1}$, where $t \in (0,1)$. See Balakrishnan and Nevzorov (2003), chapters 16 and 20.

The beta-binomial random variable (r.v.) Y, with parameters m, $a>0$ and $b>0$, has p.d.f. $p_m(k;a,b)=\Pr[Y=k]$, that is defined as

$$p_m(k;a,b)=\int_0^1 \binom{m}{k} t^k(1-t)^{m-k}\{B(a,b)\}^{-1}t^{a-1}(1-t)^{b-1}dt$$
$$=\binom{m}{k}\{B(a,b)\}^{-1}\int_0^1 t^{a+k-1}(1-t)^{b+m-k-1}dt, \tag{3}$$

for every $k=0,1,\ldots,m$. In particular, we have $p_m(k;a,b)=\binom{m}{k}\{B(a,b)\}^{-1}B(a+k,b+m-k)$, for every $k=0,1,\ldots,m$. We also have $\sum_{k=0}^{m} p_m(k;a,b)=1$, with $p_m(k;a,b)>0$, for every $k=0,1,\ldots,m$.

We can rewrite the definition (3) as

$$p_m(k;a,b)=\{B(a,b)\}^{-1}\int_0^1 p_m(k;t)t^{a-1}(1-t)^{b-1}dt,$$

where $p_m(k;t)=\binom{m}{k}t^k(1-t)^{m-k}$ is the binomial p.d.f., with $\binom{m}{k}=(k!(m-k)!)^{-1}m!$, parameters m and t, $t\in[0,1]$, for every $k=0,1,\ldots,m$. Moments of the beta-binomial r.v. Y, are obtained by integrating the moments of the binomial p.d.f. $p_m(k;t)$, $t\in[0,1]$, $k=0,1,\ldots,m$, that are functions of t, $t\in[0,1]$, through the definition (3) of the beta-binomial p.d.f. $p_m(k;a,b)$, $k=0,1,\ldots,m$.

In particular, we recall that the first three moments about the origin, η_1', η_2', and η_3', of the binomial p.d.f. $p_m(k;t)$, with values $k=0,1,\ldots,m$, are $\eta_1'=mt$, $\eta_2'=m(m-1)t^2+mt$, and $\eta_3'=m(m-1)(m-2)t^3+3m(m-1)t^2+mt$, where $t\in[0,1]$. The first two moments about the origin, $\lambda_1'(a,b)\equiv\lambda_1'$ and $\lambda_2'(a,b)\equiv\lambda_2'$, of the beta p.d.f. $p(t;a,b)$, with values $t\in[0,1]$, are

$$\lambda_1'(a,b)=\{B(a,b)\}^{-1}B(a+1,b)$$
$$=(a+b)^{-1}a, \tag{4}$$

$$\lambda_2'(a,b)=\{(a+b)(a+b+1)\}^{-1}a(a+1), \tag{5}$$

and the third moment about the origin λ_3' is

$$\lambda_3'=\{(a+b)(a+b+1)(a+b+2)\}^{-1}a(a+1)(a+2). \tag{6}$$

See Balakrishnan and Nevzorov (2003), chapters 5 and 16, and Johnson, Kemp and Kotz (2005), chapter 3. Finally, the first three moments about the origin, μ_1', μ_2', and μ_3', of the beta-binomial p.d.f. $p_m(k;a,b)$, with values $k=0,1,\ldots,m$, are

$$\mu_1^{'} = m\lambda_1^{'}(a,b),$$

$$\mu_2^{'} = m(m-1)\lambda_2^{'}(a,b) + m\lambda_1^{'}(a,b),$$

$$\mu_3^{'} = m(m-1)(m-2)\lambda_3^{'} + 3m(m-1)\lambda_2^{'}(a,b) + m\lambda_1^{'}(a,b).$$

The variance of the beta-binomial p.d.f. $p_m(k;a,b)$, with values $k = 0,1,\ldots,m$, is

$$\mu_2 = (a+b)^{-2}(a+b+1)^{-1} mab(m+a+b).$$

From the third central moment $\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2(\mu_1^{'})^3$, it is seen that the beta-binomial p.d.f. $p_m(k;a,b)$, with values $k = 0,1,\ldots,m$, is negatively skewed for

$$b^3 + ab^2 + 3mb^2 + 2m^2 b < a^3 + a^2 b + 3ma^2 + 2m^2 a,$$

and is positively skewed for

$$b^3 + ab^2 + 3mb^2 + 2m^2 b > a^3 + a^2 b + 3ma^2 + 2m^2 a.$$

Examples of the beta-binomial p.d.f. $p_m(k;a,b)$, values $k = 0,1,\ldots,m$, are plotted in figure 1, for different values of the parameters a and b, with a reasonable $m = 20$.

The values of the parameters a and b, in the moments $\lambda_1^{'}(a,b)$ and $\lambda_2^{'}(a,b)$, given by (4) and (5), respectively, of the beta p.d.f. $p(t;a,b)$, with values $t \in (0,1)$, that yield a conveniently small quantity

$$\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b) = \{(a+b)(a+b+1)\}^{-1} ab, \tag{7}$$

can be regarded as constructive. More precisely, constructive values of a and b in (7) can directly help to improve the numerical performance of the Bernstein-type approximations that we are going to introduce in section 4. Constructive values of a and b in (7) can lower their uniform convergence rates, as $m \to \infty$.

The quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ given by (7) does not admit a minimizer, for $a > 0$ and $b > 0$. For further details and descriptions, see sections 6 and 7.

4. BERNSTEIN-TYPE APPROXIMATIONS

4.1. Bernstein-type approximations

Let g be a bounded and real-valued function defined on the closed interval $D \subseteq R^1$. The Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$ of order m for the function $g(x)$ is defined as

$$C_m^{(s)}(g;x,a,b) = \{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} g\left(m^{-s}\left(m^{-1}k - t\right) + x\right)\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\,dt, \tag{8}$$

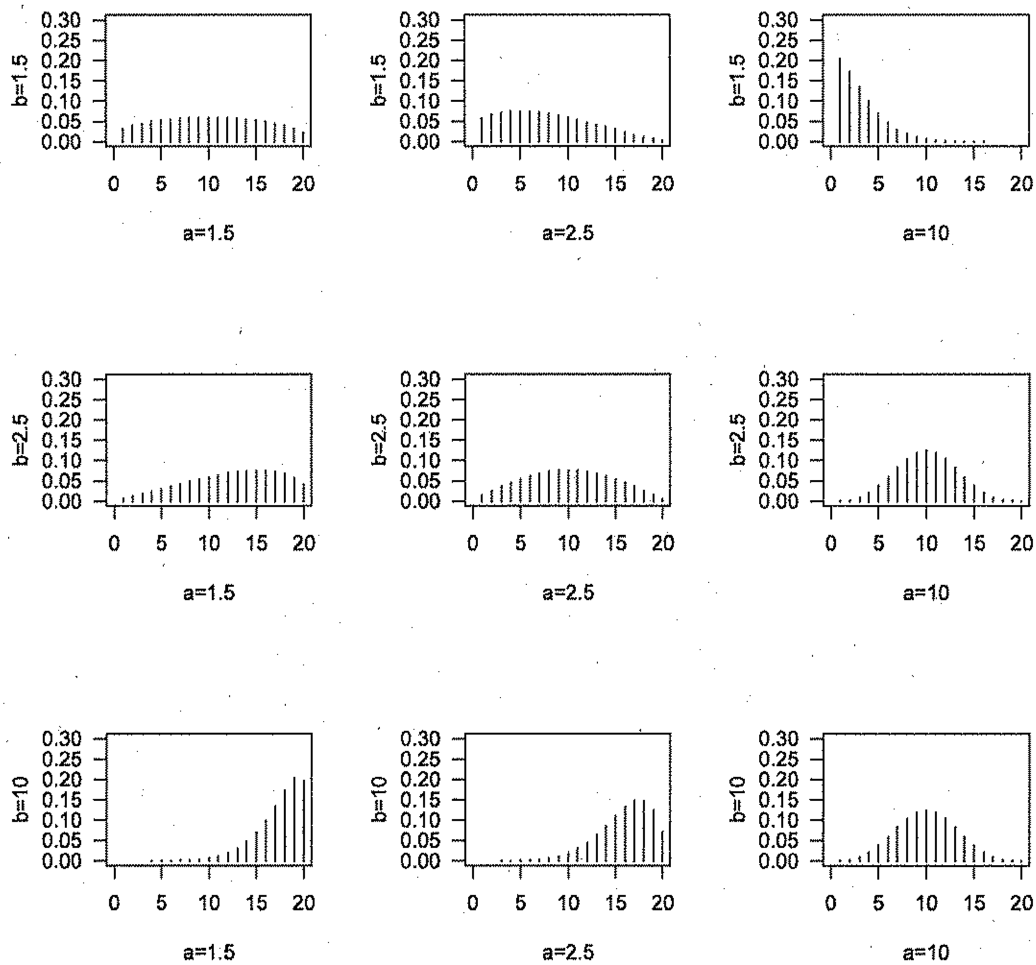

Figure 1: some beta-binomial probability distribution functions $p_m(k;a,b)$, for $k = 0,1,\ldots,m$, $m = 20$, where $a = 1.5$, $a = 2.5$, $a = 10$, and $b = 1.5$, $b = 2.5$, $b = 10$.

where $s > -1/2$ is fixed, m is a positive integer number, and $x \in D \subseteq R^1$. Properties of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, given by (8), $x \in D$, are outlined in Appendix 8.1.

If $g(x)$ is continuous on $x \in D$, where $s > -1/2$, then $C_m^{(s)}(g;x,a,b) \to g(x)$, as $m \to \infty$, uniformly at any point $x \in D$. In Appendix 8.2, we provide a proof of this uniform convergence. An alternative proof that applies uniform convergence of integral operators can be found in Appendix 8.4.

Let g be a bounded and real-valued function defined on the closed interval $D \subseteq R^q$. The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$ of order m for the function $g(\mathrm{x})$ is defined as

$$C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b) = \{B(a,b)\}^{-q} \int_0^1 \cdots \int_0^1 \sum_{k_1=0}^{m_1} \cdots \sum_{k_q=0}^{m_q} g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}k_1 - t_1\right) + x_1 \\ \vdots \\ m_q^{-s}\left(m_q^{-1}k_q - t_q\right) + x_q \end{pmatrix}$$

$$\cdot \binom{m_1}{k_1} \cdots \binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{m_1+b-k_1-1} \cdots t_q^{a+k_q-1}(1-t_q)^{m_q+b-k_q-1} dt_1 \cdots dt_q, \qquad (9)$$

where $s > 1/2$ is fixed, $\mathrm{m} = (m_1, \ldots, m_q)^T$ are positive integer numbers, $m = \sum_{i=1}^{q} m_i$ is the total degree, and $\mathrm{x} \in \mathrm{D} \subseteq R^q$. Properties of the Bernstein-type approximations $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), $\mathrm{x} \in \mathrm{D}$, are outlined in Appendix 8.1.

If $g(\mathrm{x})$ is continuous on $\mathrm{x} \in \mathrm{D}$, where $s > -1/2$, then $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b) \to g(\mathrm{x})$, as $m \to \infty$, uniformly at any q-dimensional point $\mathrm{x} \in \mathrm{D}$. In Appendix 8.2, we provide a proof of this uniform convergence. An alternative proof that applies uniform convergence of integral operators can be found in Appendix 8.4.

4.2. Degrees of approximation

Let $\omega(\delta)$ be the modulus of continuity of the real-valued function g, for every $\delta > 0$. The modulus of continuity $\omega(\delta)$ of the function $g(x)$, where $x \in [0,1]$, is defined as the maximum of $|g(x_0) - g(x)|$, for $|x_0 - x| < \delta$, where $x_0, x \in [0,1]$. If the function g is continuous, then $\omega(\delta) \to 0$, as $\delta \to 0$.

Setting $\delta = m^{-1/2}$, for every $x \in [0,1]$, it can be shown that the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, given by (8), has degree of approximation

$$\left| C_m^{(s)}(g;x,a,b) - g(x) \right| \le \left[1 + m^{-1} m^{-2s-1} \left\{ \lambda_1'(a,b) - \lambda_2'(a,b) \right\} \right] \omega\left(m^{-1/2}\right), \qquad (10)$$

where the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7). See Appendix 8.3.

We let $|\mathrm{x}| = \left(\sum_{i=1}^{q} x_i^2 \right)^{1/2}$, where $\mathrm{x} \in [0,1]^q$. The modulus of continuity $\omega(\delta)$ of the real-valued function $g(\mathrm{x})$, $\mathrm{x} \in [0,1]^q$, for every $\delta > 0$, is defined as the maximum of $|g(\mathrm{x}_0) - g(\mathrm{x})|$, for $|\mathrm{x}_0 - \mathrm{x}| < \delta$, where $\mathrm{x}_0, \mathrm{x} \in [0,1]^q$. If the function g is continuous, then $\omega(\delta) \to 0$, as $\delta \to 0$.

Setting $\delta = m^{-1/2}$, for every $\mathrm{x} \in [0,1]^q$, it can be shown that the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), has degree of approximation

$$\left|C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b) - g(\mathrm{x})\right| \le \left[1 + m^{-1}\left(\sum_{i=1}^{q} m_i^{-2s-1}\right)\left\{\lambda_1'(a,b) - \lambda_2'(a,b)\right\}\right]\omega\left(m^{-1/2}\right), \tag{11}$$

where $m = \sum_{i=1}^{q} m_i$, and the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7). See Appendix 8.3.

4.3. A comparison

For a convenient value of the approximation coefficient s, the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), where $s > -1/2$, can typically outperform the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, given by (1) and (2), for any function g to approximate, for every $x \in [0,1]$ and $\mathrm{x} \in [0,1]^q$, respectively.

Choosing a value of s, where $s > -1/2$, can only modify the coefficients in the degrees of approximation (10) and (11), without affecting their modulus of continuity $\omega\left(m^{-1/2}\right)$, for any fixed $m = \sum_{i=1}^{q} m_i$. Large values of s do not bring any advantage, with typical examples of application for the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined by (8) and (9), respectively, where $s > -1/2$, $x \in [0,1]$ and $\mathrm{x} \in [0,1]^q$. Convergence to unity of the coefficients that distinguish the degrees of approximation (10) and (11) is fast, as s increases.

In Figure 2, we compare the differences $B_m(g;x) - g(x)$ with $C_m^{(s)}(g;x,a,b) - g(x)$, where the Bernstein polynomial $B_m(g;x)$ is given by (1), the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$ is given by (8), and the function g is defined as $g(x) = x^3 + x^2 + x$, and $g(x) = x^2 + x$, for $x \in [0.25, 0.75]$, where $m = 4$, $a = 1.5$ and $b = 10$, and $s = -0.1, -0.005, 0.05, 0.5, 1.5$. We also compare the difference $B_{\mathrm{m}}(g;\mathrm{x}) - g(\mathrm{x})$ with $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b) - g(\mathrm{x})$, where the Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ is given by (2), the Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$ is given by (9), and the function g is $g(\mathrm{x}) = (x_2+1)^{-1}(x_1+1)$, for $\mathrm{x} = (x_1, x_2)^T$, $x_1 \in [0.25, 0.75]$, $x_2 \in [0.45, 0.85]$, where $m_1 = m_2 = 4$, $a = 1.5$, $b = 10$, and $s = -0.1, -0.005, 0.05, 0.5, 1.5$. The values $m = 4$ and $m_1 = m_2 = 4$ are very small, computationally and numerically. In any case, the numerical performances of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, $x \in [0.25, 0.75]$, and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, $\mathrm{x} = (x_1, x_2)^T$, $x_1 \in [0.25, 0.75]$, $x_2 \in [0.45, 0.85]$, are always very effective.

5. ESTIMATION OF SMOOTH FUNCTIONS OF MEANS

5.1. Bernstein-type estimators

The Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), where

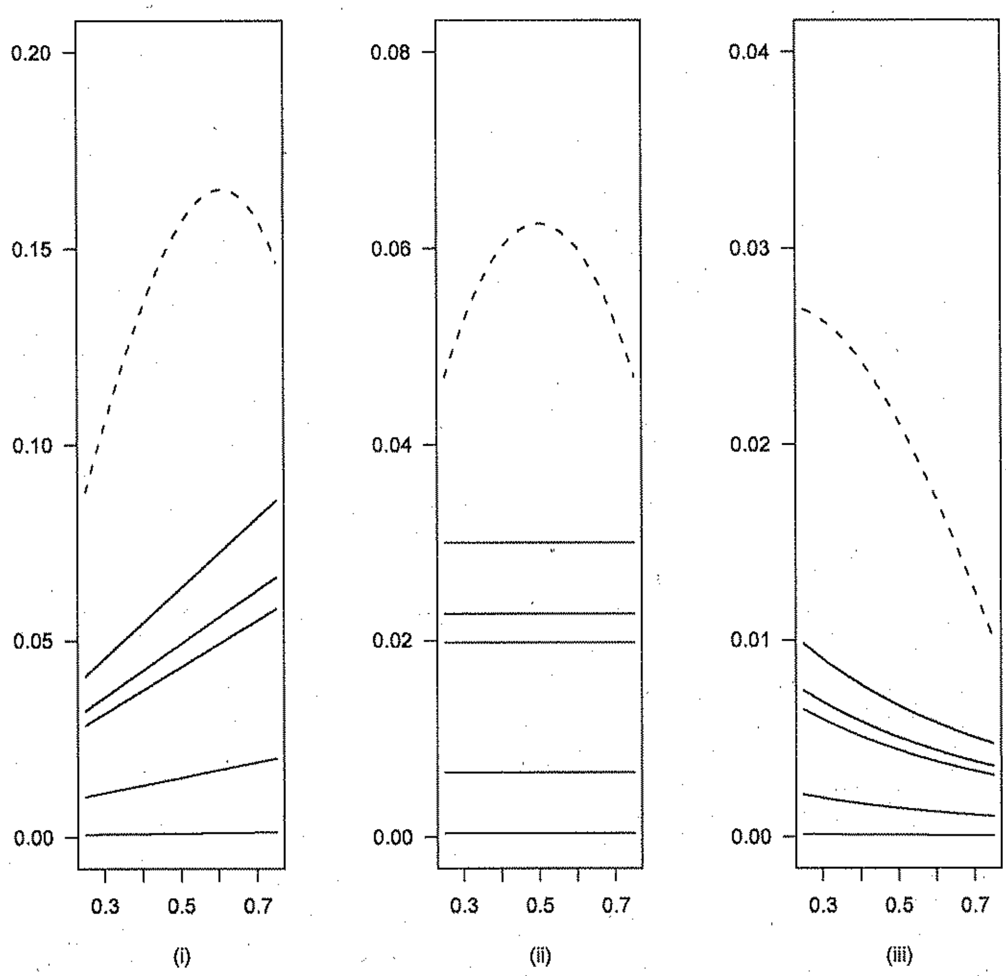

Figure 2: Differences $B_m(g;x)-g(x)$, (hatched line), and $C_m^{(s)}(g;x,a,b)-g(x)$, (solid line), for the smooth function $g(x)=x^3+x^2+x$, where $x\in[0.25,0.75]$, where $m=4$, $a=1.5$, and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (i)). Differences $B_m(g;x)-g(x)$, (hatched line), and $C_m^{(s)}(g;x,a,b)-g(x)$, (solid line), for $g(x)=x^2+x$, where $x\in[0.25,0.75]$, where $m=4$, $a=1.5$ and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (ii)). The difference $B_{\mathrm{m}}(g;\mathrm{x})-g(\mathrm{x})$, (hatched line), and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)-g(\mathrm{x})$, (solid line), for $g(\mathrm{x})=(x_2+1)^{-1}(x_1+1)$, where $x_1\in[0.25,0.75]$, $x_2\in[0.45,0.85]$, where $m_1=m_2=4$, $a=1.5$, and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (iii)).

$x \in D \subseteq R^1$ and $\mathrm{x} \in \mathrm{D} \subseteq \mathrm{R}^q$, can be used for estimating smooth functions of the population means in the statistical inference on a random sample of n independent and identically distributed (i.i.d.) observations.

Let X be a univariate random variable with values $x \in D$, distribution function F, and finite mean $\mu = E[X]$. We want to estimate a population characteristic $\theta = g(\mu)$, where g is a smooth function $g: D \to R^1$. The natural estimator of θ is $\hat{\theta} = g(\bar{x})$, where $\bar{x} = n^{-1}\sum_{j=1}^{n} X_j$ is the sample mean, calculated on a random sample of n i.i.d. observations X_j, $j = 1,\ldots,n$, of X. An alternative estimator of $\theta = g(\mu)$ is the Bernstein-type estimator $C_m^{(s)}(g;\bar{x},a,b)$,

$$C_m^{(s)}(g;\bar{x},a,b) = \{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} g\left(m^{-s}\left(m^{-1}k - t\right) + \bar{x}\right)\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\, dt. \tag{12}$$

where $s > -1/2$ is fixed. The Bernstein-type estimator (12) follows from the definition (8) of $C_m^{(s)}(g;x,a,b)$, $s > -1/2$, by substituting $x \in D$ with the sample mean $\bar{x}$, where $\bar{x}$ ranges in D.

Let X be a q-variate random variable with values $\mathrm{x} \in \mathrm{D}$, where $\mathrm{X} = (X_1,\ldots,X_q)^T$, with distribution function F, and finite q-variate mean $\mu = E[\mathrm{X}]$, $\mu = (\mu_1,\ldots,\mu_q)^T$. We want to estimate a population characteristic $\theta = g(\mu)$, where $g: \mathrm{D} \to R^1$. The natural estimator of θ is $\hat{\theta} = g(\bar{\mathrm{x}})$, where $\bar{\mathrm{x}} = (\bar{x}_1,\ldots,\bar{x}_q)^T$ is the q-variate sample mean on a random sample of n i.i.d. q-variate observations X_i, $i = 1,\ldots,n$, of X, $\bar{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, $i = 1,\ldots,q$. An alternative estimator of $\theta = g(\mu)$ is the multivariate Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b)$,

$$C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b) = \{B(a,b)\}^{-q} \int_0^1 \cdots \int_0^1 \sum_{k_1=0}^{m_1} \cdots \sum_{k_q=0}^{m_q} g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}k_1 - t_1\right) + \bar{x}_1 \\ \vdots \\ m_q^{-s}\left(m_q^{-1}k_q - t_q\right) + \bar{x}_q \end{pmatrix}$$
$$\cdot \binom{m_1}{k_1} \cdots \binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{b+m_1-k_1-1} \cdots t_q^{a+k_q-1}(1-t_q)^{b+m_q-k_q-1}\, dt_1 \cdots dt_q, \tag{13}$$

where $s > -1/2$ is fixed. The multivariate Bernstein-type estimator (13) follows the definition (9) of $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, $s > -1/2$, by substituting $\mathrm{x} \in \mathrm{D}$ with the sample mean $\bar{\mathrm{x}}$, where $\bar{\mathrm{x}}$ ranges in D.

5.2. Orders of error in probability of the Bernstein-type estimators

We know that $\bar{x} = \mu + O_p\left(n^{-1/2}\right)$, as $n \to \infty$. We also know that

$$g(\bar{x}) = g(\mu) + O_p\left(n^{-1/2}\right),$$

as $n \to \infty$. It is shown that the Bernstein-type estimator $C_m^{(s)}(g;\bar{x},a,b)$, given by (12), for $s > -1/2$, is a consistent estimator of $g(\mu)$, as $m \to \infty$ and $n \to \infty$. In particular, it is shown that

$$C_m^{(s)}(g;\bar{x},a,b) = g(\mu) + O\left(m^{-2s-1}\right) + O_p\left(n^{-1/2}\right), \tag{14}$$

for $s > -1/2$, as $m \to \infty$ and $n \to \infty$. See Appendix 8.5.

We know that $\bar{\mathrm{x}} = \mu + \mathrm{O}_\mathrm{p}\left(n^{-1/2}\right)$, where $\bar{x}_i = \mu_i + O_p\left(n^{-1/2}\right)$, $i = 1,\ldots,q$, as $n \to \infty$. We also know that

$$g(\bar{\mathrm{x}}) = g(\mu) + O_p\left(n^{-1/2}\right),$$

as $n \to \infty$. It is shown that the multivariate Bernstein-type estimator $C_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}},a,b)$, given by (13), where $\mathrm{m} = (m_1,\ldots,m_q)^T$, for $s > -1/2$, is a consistent estimator of $g(\mu)$, as $m_i \to \infty$, $i = 1,\ldots,q$, and $n \to \infty$. In particular, it is shown that

$$C_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}},a,b) = g(\mu) + \sum_{i=1}^{q} O\left(m_i^{-2s-1}\right) + O_p\left(n^{-1/2}\right), \tag{15}$$

for $s > -1/2$, as $m_i \to \infty$, $i = 1,\ldots,q$, and $n \to \infty$. See Appendix 8.5.

5.3. Asymptotic normality of Bernstein-type estimators

The Bernstein-type estimator $C_m^{(s)}(g;\bar{x},a,b)$ is defined by (12), where $s > -1/2$, and m is a positive integer. We denote by σ^2 the asymptotic variance of $n^{1/2}g(\bar{x})$, as $n \to \infty$. That is,

$$\sigma^2 = \{g'(\mu)\}^2 E\left[(X-\mu)^2\right],$$

where $g'(x) = (dx)^{-1}dg(x)$, and $x \in D$. The distribution of the Bernstein-type estimator $C_m^{(s)}(g;\bar{x},a,b)$ is asymptotically normal,

$$n^{1/2}\left\{C_m^{(s)}(g;\bar{x},a,b) - g(\mu)\right\} \xrightarrow{d} N(0,\sigma^2), \tag{16}$$

for $s > -1/2$, as $m \to \infty$ and $n \to \infty$. See Appendix 8.6.

The Bernstein-type estimator $C_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}},a,b)$ is defined by (13), where $s > -1/2$, and $\mathrm{m} = (m_1,\ldots,m_q)^T$ are positive integers. We denote by σ^2 the asymptotic variance of $n^{1/2}g(\bar{\mathrm{x}})$, as $n \to \infty$. That is,

$$\sigma^2 = \sum_{i=1}^{q}\sum_{j=1}^{q} (\partial x_i)^{-1}\partial g(x_1,\ldots,x_i,\ldots,x_q)\Big|_{\mathrm{x}=\mu} (\partial x_j)^{-1}\partial g(x_1,\ldots,x_j,\ldots,x_q)\Big|_{\mathrm{x}=\mu}$$
$$\cdot E\left[(X_i-\mu)(X_j-\mu)\right].$$

The distribution of the Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$ is asymptotically normal,

$$n^{1/2}\{C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)-g(\mu)\}\xrightarrow{d}N(0,\sigma^2), \tag{17}$$

for $s>-1/2$, as $m_i\to\infty$, $i=1,\ldots,q$, and $n\to\infty$. See Appendix 8.6.

6. SIMULATION STUDY

Following subsection 4.3, we report on a small Monte Carlo experiment concerning with the empirical behaviour of the Bernstein-type estimators $C_m^{(s)}(g;\overline{x},a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (12) and (13). We applied the Bernstein-type estimators $C_m^{(s)}(g;\overline{x},a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (12) and (13), to the approximation of the smooth functions of means $g(\overline{x})=\overline{x}^3+\overline{x}^2+\overline{x}$, $g(\overline{x})=\overline{x}^2+\overline{x}$, where $\overline{x}=n^{-1}\sum_{j=1}^{n}X_j$, and $g(\overline{\mathrm{x}})=(\overline{x}_2+1)^{-1}(\overline{x}_1+1)$, where $\overline{\mathrm{x}}=(\overline{x}_1,\overline{x}_2)^T$, $\overline{x}_i=n^{-1}\sum_{j=1}^{n}X_{ij}$, $i=1,2$. Random samples of different size n, of i.i.d. observations, were always drawn from the univariate folded normal distribution $|N(0,1)|$ and from the bivariate folded normal distribution with independent marginals $|N(0,1)|$. We always considered the values $a=1.5$ and $b=10$. We denote by $\hat{\sigma}_n^2$ both the Monte Carlo variance of the Bernstein-type estimator $C_m^{(s)}(g;\overline{x},a,b)$, given by (12), and the Monte Carlo variance of the Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (13).

From the definition (12) of $C_m^{(s)}(g;\overline{x},a,b)$, we have the Bernstein-type estimator

$$C_m^{(s)}(\overline{x}^3+\overline{x}^2+\overline{x};\overline{x},a,b)=\overline{x}^3+\overline{x}^2+\overline{x}+m^{-2s-1}(3\overline{x}+1)\{\lambda_1'(a,b)-\lambda_2'(a,b)\}$$
$$+m^{-3s-2}\{2\lambda_3'-3\lambda_2'(a,b)+\lambda_1'(a,b)\},$$

where $s>-1/2$, and the moments $\lambda_1'(a,b)$, $\lambda_2'(a,b)$, and λ_3' are given by (4), (5), and (6), respectively.

In Figure 3, we show the empirical behaviour of the difference

$$C_m^{(s)}(\overline{x}^3+\overline{x}^2+\overline{x};\overline{x},a,b)-\overline{x}^3-\overline{x}^2-\overline{x},$$

for $m=3,4,4,5$, $s=0.5,0.5,0.6,2$, and the sample sizes $n=4,6,10,16$. We had the Monte Carlo variances $\hat{\sigma}_4^2=1.844476$, $\hat{\sigma}_6^2=1.145010$, $\hat{\sigma}_{10}^2=0.634065$, and $\hat{\sigma}_{16}^2=0.581082$.

From the definition (12) of $C_m^{(s)}(g;\overline{x},a,b)$, we have the Bernstein-type estimator

$$C_m^{(s)}(\overline{x}^2+\overline{x};\overline{x},a,b)=\overline{x}^2+\overline{x}+m^{-2s-1}\{\lambda_1'(a,b)-\lambda_2'(a,b)\},$$

where $s>-1/2$, and the quantity $\lambda_1'(a,b)-\lambda_2'(a,b)$ is given by (7). We have a constant difference

$$C_m^{(s)}(\overline{x}^2+\overline{x};\overline{x},a,b)-\overline{x}^2-\overline{x},$$

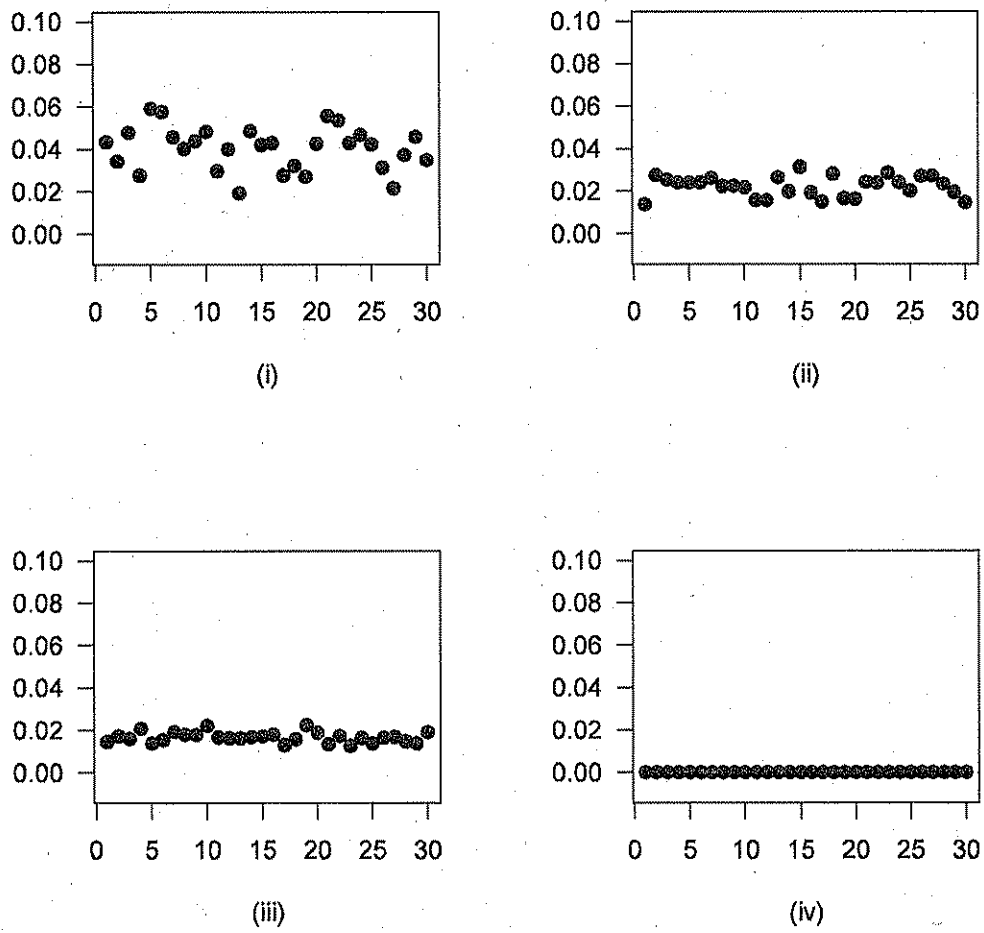

Figure 3: Differences $C_m^{(s)}(\bar{x}^3+\bar{x}^2+\bar{x};\bar{x},a,b)-\bar{x}^3-\bar{x}^2-\bar{x}$, where $a=1.5$ and $b=10$, for random samples of size n, from the folded normal distribution; $s=0.5$, $m=3$, and $n=4$, in panel (i), $s=0.5$, $m=4$, and $n=6$, in panel (ii), $s=0.6$, $m=4$, and $n=10$, in panel (iii), $s=2$, $m=5$, and $n=16$, in panel (iv).

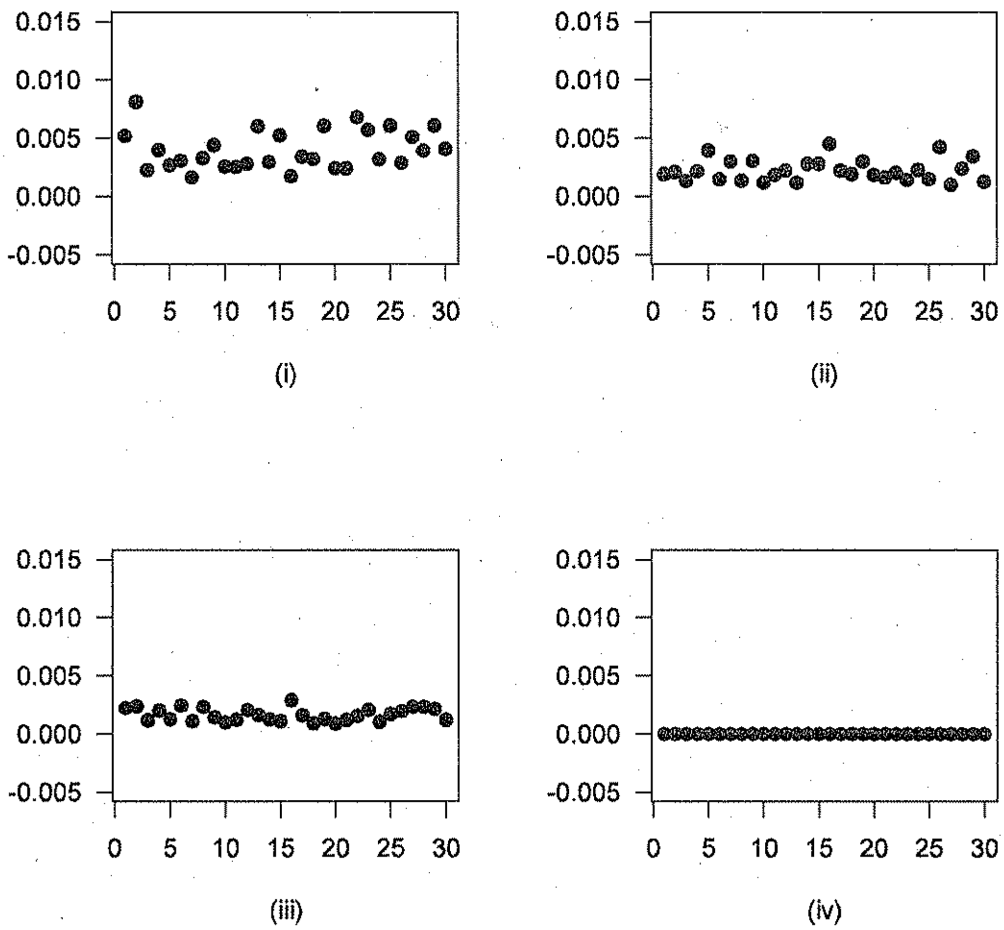

Figure 4: Differences $\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)-(\bar{x}_2+1)^{-1}(\bar{x}_1+1)$, where $a=1.5$ and $b=10$, for random samples of size n, from the bivariate folded normal distribution; $s=0.5$, $m_1=m_2=3$, and $n=4$, in panel (i), $s=0.5$, $m_1=m_2=4$, and $n=6$, in panel (ii), $s=0.6$, $m_1=m_2=4$, and $n=10$, in panel (iii), $s=2$, $m_1=m_2=5$, and $n=16$, in panel (iv).

with this example. We had the value $C_m^{(s)}(\bar{x}^2+\bar{x};\bar{x},a,b)-\bar{x}^2-\bar{x}=0.011594$, for $m=3$, $s=0.5$, and the sample size $n=4$, $C_m^{(s)}(\bar{x}^2+\bar{x};\bar{x},a,b)-\bar{x}^2-\bar{x}=0.006522$, for $m=4$, $s=0.5$, and $n=6$, $C_m^{(s)}(\bar{x}^2+\bar{x};\bar{x},a,b)-\bar{x}^2-\bar{x}=0.004943$, for $m=4$, $s=0.6$, and $n=10$, $C_m^{(s)}(\bar{x}^2+\bar{x};\bar{x},a,b)-\bar{x}^2-\bar{x}$ $=0.000339$, for $m=5$, $s=2$, and $n=16$.

From the definition (13) of $C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b)$, in order to approximate the integral in the Bernstein-type estimator $C_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)$, we obtained $\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)$,

$$\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)=(\bar{x}_2+1)^{-1}(\bar{x}_1+1) \\ +m_2^{-2s-1}(\bar{x}_2+1)^{-3}(\bar{x}_1+1)\left\{\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)\right\},$$

where $s>-1/2$, and the quantity $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$ is given by (7), such that

$$\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)=C_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)+O\left(m_2^{-3s-2}\right),$$

as $m_1\to\infty$ and $m_2\to\infty$. The Bernstein-type estimator $\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)$ was obtained by calculating the integral in $C_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)$ with the three-term Taylor expansion of the denominator $(\bar{x}_2+1)^{-1}$. See Wong (2001), chapter 5, for further details about this procedure.

In Figure 4, we show the empirical behaviour of the difference

$$\widetilde{C}_{\mathrm{m}}^{(s)}\left((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b\right)-(\bar{x}_2+1)^{-1}(\bar{x}_1+1),$$

for $m_1=m_2=3,4,4,5$, $s=0.5,0.5,0.6,2$, and the sample sizes $n=4,6,10,16$. We had the Monte Carlo variances $\hat{\sigma}_4^2=0.050593$, $\hat{\sigma}_6^2=0.031106$, $\hat{\sigma}_{10}^2=0.024603$, and $\hat{\sigma}_{16}^2=0.015765$.

7. CONCLUDING REMARKS

1). The quantity $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), is crucial for the numerical performance of the univariate Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, defined as (8), where $s>-1/2$, and $x\in D\subseteq R^1$, and for the numerical performance of the multivariate Bernstein-type approximations $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined as (9), where $s>-1/2$, and $\mathrm{x}\in\mathrm{D}\subseteq R^q$.

The function $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), does not admit a minimizer, for $a>0$ and $b>0$. See Chong and Żak (1996), chapter 6.

Space curves $(a(t),b(t),t)$, where $t\in E$, and $E\subseteq R^1$, can be easily drawn in order to determine specific degrees of approximation. See Montiel and Ros (2005), chapter 1.

In Figure 5, for the function $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), we plot the slices $\lambda_1^{'}(0.75,b)-\lambda_2^{'}(0.75,b)$, $\lambda_1^{'}(1.5,b)-\lambda_2^{'}(1.5,b)$, and $\lambda_1^{'}(3.5,b)-\lambda_2^{'}(3.5,b)$, where $0.01\le b\le 40.5$. The slice $\lambda_1^{'}(1.5,b)-\lambda_2^{'}(1.5,b)$, $0.01\le b\le 40.5$, refers to the simulation parameters used in section 6. Specifically, in section 6, we used the value $\lambda_1^{'}(1.5,10)-\lambda_2^{'}(1.5,10)=0.104348$.

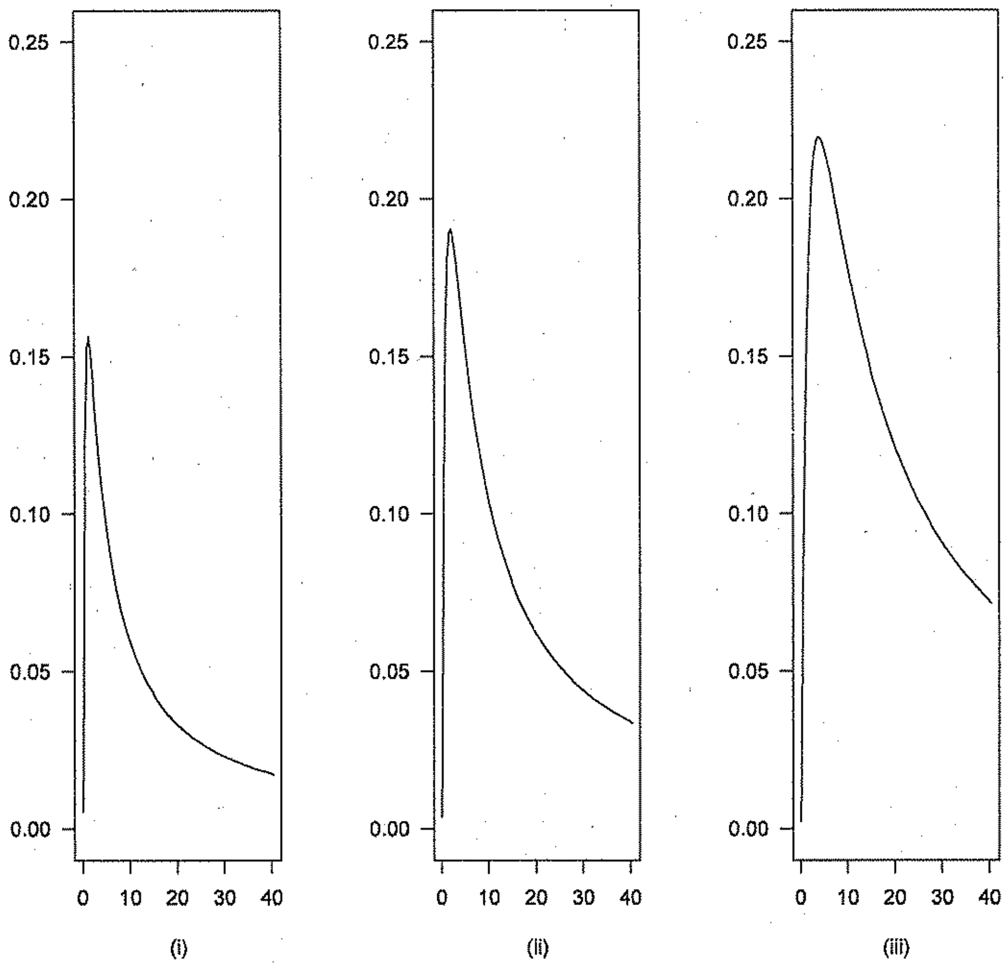

Figure 5: slices of the function $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), where $a>0$, and $b>0$. Slice $\lambda_1^{'}(0.75,b)-\lambda_2^{'}(0.75,b)$, where $0.01\leq b\leq 40.5$, in panel (i), slice $\lambda_1^{'}(1.5,b)-\lambda_2^{'}(1.5,b)$, where $0.01\leq b\leq 40.5$, in panel (ii), slice $\lambda_1^{'}(3.5,b)-\lambda_2^{'}(3.5,b)$, where $0.01\leq b\leq 40.5$, in panel (iii).

The degrees (10) and (11) of approximation of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D\subseteq R^1$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$, are better than the degrees of approximation of the Bernstein-type approximations, that are proposed in Pallini (2005), for values of a and b such that $\lambda_1'(a,b)-\lambda_2'(a,b)<1/4$.

2). More efficient results for the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined as (8) and (9), respectively, where $s>-1/2$, $x\in D\subseteq R^1$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$, can be obtained by over-skewing the beta p.d.f. $beta(a,b)$, with the moments $\lambda_1'(a,b)$ and $\lambda_2'(a,b)$, given by (4) and (5).

We can over-skew the beta p.d.f. $beta(a,b)$, by an additional parameter τ, with values $\tau>0$, by determining the beta p.d.f. $beta(a,b\tau)$. The beta p.d.f. $beta(a,b\tau)$ is negatively skewed, for $\tau<b^{-1}a$, and is positively skewed, for $\tau>b^{-1}a$.

From the definition (7) of $\lambda_1'(a,b)-\lambda_2'(a,b)$, under the condition $a^2\tau+a\tau+b^2\tau-b^2\tau^2\le a^2+a$, it is seen that the value of $\lambda_1'(a,b\tau)-\lambda_2'(a,b\tau)$ is smaller then the value of $\lambda_1'(a,b)-\lambda_2'(a,b)$.

3). We recall that the most special cases of the beta p.d.f. $beta(a,b)$ are the arcsine distribution, the power distribution and the uniform distribution. See Balakrishnan and Nevzorov (2003), chapter 16.

4). Rosenberg (1967) studies an application of the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, given by (2), to the Monte Carlo evaluation of an integral. The same application can be organized for the multivariate Bernstein-type approximation in Pallini (2005) and for the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, $\mathrm{m}=(m_1,\ldots,m_q)^T$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$.

Most importantly, straightforward versions of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D$, $\mathrm{x}\in\mathrm{D}$, are both multivariate approximations for functions and approximate multivariate integrals of functions. Focussing on $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, $\mathrm{x}\in\mathrm{D}$, let us suppose that we are interested in the evaluation of an integral $\int_{\mathrm{D}}\mathrm{g(x)dx}$, where $\mathrm{D}\subseteq R^q$. In particular, we can start from an approximate integration rule of the form

$$C_{\mathrm{m}}^{(s)}(h_{\mathrm{m}}^{(s)};\mathrm{x},a,b)=\{B(a,b)\}^{-q}\int_0^1\cdots\int_0^1\sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q}h_{\mathrm{m}}^{(s)}\begin{pmatrix} m_1^{-s}\left(m_1^{-1}k_1-t_1\right)+x_1\\ \vdots\\ m_q^{-s}\left(m_q^{-1}k_q-t_q\right)+x_q\end{pmatrix}$$
$$\cdot\binom{m_1}{k_1}\cdots\binom{m_q}{k_q}t_1^{a+k_1-1}(1-t_1)^{m_1+b-k_1-1}\cdots t_q^{a+k_q-1}(1-t_q)^{m_q+b-k_q-1}dt_1\cdots dt_q,$$

where $s>-1/2$, $h_{\mathrm{m}}^{(s)}:[0,1]^q\to R^q$, and then apply a procedure for a more efficient integration rule. See Wong (2001), and Hanselman and Littlefield (2005), chapter 24.

5). In the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D$ and $\mathrm{x}\in\mathrm{D}$, the linear kernels $m^{-s}\left(m^{-1}k-t\right)+x$, the linear kernels $m^{-s}\left(m^{-1}v-x\right)+x$ and $m_i^{-s}\left(m_i^{-1}v_i-x_i\right)+x_i$ can be substituted by nonlinear kernels, where $k=0,1,\ldots,m$, $k_i=0,1,\ldots,m_i$, $i=1,\ldots,q$, and $x\in D$, $\mathrm{x}=(x_1,\ldots,x_q)^{\mathrm{T}}\in\mathrm{D}$, respectively.

6). The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$, can be generalized by using a different approximation coefficient for each component. That is, we can use $\mathrm{s}=(s_1,\ldots,s_q)^T$ in the Bernstein-type approximation $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, where $\mathrm{x}\in \mathrm{D}\subseteq R^q$. Another generalization of the Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $\mathrm{x}\in \mathrm{D}\subseteq R^q$, can be based on q different beta-binomial p.d.f.'s, $p_m(k;a_i,b_i)$, that can be defined from (3), for every $i=1,\ldots,q$.

7). Variants of the Bernstein polynomials that are discussed and studied in DeVore and Lorentz (1993), chapter 10, can also be regarded as extensions to the use of the binomial p.d.f. in Bernstein-like approximations. Extensions to the beta-binomial p.d.f. $p_m(k;a,b)$, given by (3), where $k=0,1,\ldots,m$, are discussed and studied in Wilcox (1981).

8. APPENDIX

8.1. Basic properties of the Bernstein-type approximations (8) and (9)

The Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D$ and $\mathrm{x}\in \mathrm{D}$, respectively, are linear positive operators. Let γ_1 and γ_2 be finite constants. Let g, g_1, and g_2 be functions, $g(x)$, $g_1(x)$, and $g_2(x)$, $x\in D$. We have

$$C_m^{(s)}(\gamma_1+\gamma_2 g;x,a,b)=\gamma_1+\gamma_2 C_m^{(s)}(g;x,a,b),$$

$$C_m^{(s)}(g_1+g_2;x,a,b)=C_m^{(s)}(g_1;x,a,b)+C_m^{(s)}(g_2;x,a,b),$$

$x\in D$. If $g_1(x)\le g_2(x)$, for all $x\in D$, we have

$$C_m^{(s)}(g_1;x,a,b)\le C_m^{(s)}(g_2;x,a,b),$$

$x\in D$. Multivariate versions of these properties hold for $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, given by (9), where $\mathrm{g}(\mathrm{x})$, $s>-1/2$, and $\mathrm{x}\in \mathrm{D}$.

8.2. Uniform convergence of the Bernstein-type approximations (8) and (9)

The uniform norm $\|g\|$ of the function $g(x)$, where $x\in D$, is defined as $\|g\|=\max_{x\in D}|g(x)|$. The Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, where $x\in D$, is given by (8). We want to show that, given a constant $\varepsilon>0$, there exists a positive integer m_0, such that

$$\left|C_{m_0}^{(s)}(g;x,a,b)-g(x)\right|<\varepsilon, \tag{18}$$

for every $x\in D$.

For every $x\in D$, the Bernstein-type approximation $C_m^{(s)}(1;x,a,b)$ is

$$C_m^{(s)}(1;x,a,b) = \int_0^1 \sum_{k=0}^{m} \binom{m}{k} t^k (1-t)^{m-k} \{B(a,b)\}^{-1} t^{a-1}(1-t)^{b-1}\, dt$$
$$= \{B(a,b)\}^{-1} \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$$
$$= 1. \tag{19}$$

We define the function $\mu_1(x)$ as $\mu_1(x) = x$, and the function $\mu_2(x)$ as $\mu_2(x) = x^2$. The Bernstein-type approximation $C_m^{(s)}(\mu_1(x);x,a,b)$ is

$$C_m^{(s)}(\mu_1(x);x,a,b) = \int_0^1 \sum_{k=0}^{m} \{m^{-s}(m^{-1}k - t) + x\} \binom{m}{k} t^k (1-t)^{m-k} \{B(a,b)\}^{-1} t^{a-1}(1-t)^{b-1}\, dt$$
$$= \{B(a,b)\}^{-1} \int_0^1 x t^{a-1}(1-t)^{b-1}\, dt$$
$$= x. \tag{20}$$

The Bernstein-type approximation $C_m^{(s)}(\mu_2(x);x,a,b)$ is

$$C_m^{(s)}(\mu_2(x);x,a,b) = \int_0^1 \sum_{k=0}^{m} \{m^{-s}(m^{-1}k - t) + x\}^2 \binom{m}{k} t^k (1-t)^{m-k} \{B(a,b)\}^{-1} t^{a-1}(1-t)^{b-1}\, dt$$
$$= \{B(\alpha,\beta)\}^{-1} \int_0^1 \{m^{-2s-1} t(1-t) + x^2\} \binom{m}{k} t^{a-1}(1-t)^{b-1}\, dt$$
$$= m^{-2s-1}\{\lambda_1'(a,b) - \lambda_2'(a,b)\} + x^2, \tag{21}$$

where the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7).

Suppose that $\|g\| = M$. We take $x_0 \in D$. We have

$$-2M \le g(x_0) - g(x) \le 2M, \tag{22}$$

where $x_0, x \in D$. The function g is continuous; given $\varepsilon_1 > 0$, there exists a constant $\delta > 0$, such that

$$-\varepsilon_1 < g(x_0) - g(x) < \varepsilon_1, \tag{23}$$

for $|x_0 - x| < \delta$, and $x_0, x \in D$. From (22) and (23), it follows that

$$-\varepsilon_1 - 2M \le g(x_0) - g(x) \le \varepsilon_1 + 2M,$$

and then

$$-\varepsilon_1 - 2M\delta^{-2}(x_0 - x)^2 \le g(x_0) - g(x) \le \varepsilon_1 + 2M\delta^{-2}(x_0 - x)^2, \tag{24}$$

for $x_0, x \in D$. In fact, if $|x_0 - x| < \delta$, (22) implies (24), $x_0, x \in D$. If $|x_0 - x| \ge \delta$, then $\delta^{-2}(x_0 - x)^2 \ge 1$ and (23) imply (24), $x_0, x \in D$. Following Appendix 8.1, (24) becomes

$$\begin{aligned} -\varepsilon_1 - 2M\delta^{-2}C_m^{(s)}\left((x_0 - x)^2; x, a, b\right) &\le C_m^{(s)}\left(g; x, a, b\right) - g(x) \\ &\le \varepsilon_1 + 2M\delta^{-2}C_m^{(s)}\left((x_0 - x)^2; x, a, b\right), \end{aligned} \tag{25}$$

for $x_0, x \in D$. We observe that $(x_0 - x)^2 = x_0^2 - 2x_0 x + x^2$, $x_0, x \in D$. Hence, the Bernstein-type approximations $C_m^{(s)}(1; x, a, b)$, $C_m^{(s)}(\mu_1(x_0); x, a, b)$, and $C_m^{(s)}(\mu_2(x_0); x, a, b)$, that can be obtained as (19), (20) and (21), imply that

$$\begin{aligned} C_m^{(s)}\left((x_0 - x)^2; x, a, b\right) &= C_m^{(s)}\left(x_0^2; x, a, b\right) - 2xC_m^{(s)}(x_0; x, a, b) + x^2 C_m^{(s)}(1; x, a, b) \\ &= m^{-2s-1}\left\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\right\} + x^2 - 2x^2 + x^2 \\ &= m^{-2s-1}\left\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\right\}, \end{aligned} \tag{26}$$

for $x \in D$, where the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). We have

$$C_m^{(s)}\left((x_0 - x)^2; x, a, b\right) = O\left(m^{-2s-1}\right),$$

as $m \to \infty$, $x \in D$. Finally, we have

$$\left|C_m^{(s)}(g; x, a, b) - g(x)\right| \le \varepsilon_1 + 2M\delta^{-2}m^{-2s-1}\left\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\right\},$$

$x \in D$, where the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). Setting $\varepsilon_1 = \varepsilon / 2$, for any

$$m_0 > \left[4M\delta^{-2}\varepsilon^{-1}\left\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\right\}\right]^{1/(2s+1)},$$

for $s > -1/2$, where the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7), the uniform convergence (18) is proved.

The condition $s > -1/2$ is required for the uniform convergence. The convergence $C_m^{(s)}(g; x, a, b) \to g(x)$, for $s > -1/2$, is uniform, at any point of continuity $x \in D$, as $m \to \infty$, in the sense that the upper bound (26) for the uniform norm does not depend on x, $x \in D$.

The multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b)$, where $s > -1/2$, and $\mathrm{x} \in \mathrm{D}$, is given by (9). We observe that q is fixed and does not depend on m. Considering the uniform norm $\|g\|$ of the function $g(\mathrm{x})$, $\mathrm{x} \in \mathrm{D}$, defined as $\|g\| = \max_{\mathrm{x} \in \mathrm{D}} |g(\mathrm{x})|$, we want to show that, given a constant $\varepsilon > 0$, there exist positive integers $\mathrm{m}_0 = (m_{01}, \ldots, m_{0q})^T$, such that

$$\left|C_{\mathrm{m}_0}^{(s)}(g; \mathrm{x}, a, b) - g(\mathrm{x})\right| < \varepsilon, \tag{27}$$

for every $\mathrm{x} \in \mathrm{D}$.

For every $\mathrm{x} \in \mathrm{D}$, the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(1;\mathrm{x},a,b)$ is

$$\begin{aligned} C_{\mathrm{m}}^{(s)}(1;\mathrm{x},a,b) &= \int_0^1 \cdots \int_0^1 \sum_{k_1=0}^{m_1} \cdots \sum_{k_q=0}^{m_q} \binom{m_1}{k_1} t_1^{k_1}(1-t_1)^{m_1-k_1}\{B(a,b)\}^{-1} t_1^{a-1}(1-t_1)^{b-1} \\ &\quad \cdots \binom{m_q}{k_q} t_q^{k_q}(1-t_q)^{m_q-k_q}\{B(a,b)\}^{-1} t_q^{a-1}(1-t_q)^{b-1} dt_1 \cdots dt_q \\ &= 1, \end{aligned} \tag{28}$$

where $s > -1/2$. We define the functions $\mu_1(\mathrm{x}) = \sum_{i=1}^{q} x_i$ and $\mu_2(\mathrm{x}) = \sum_{i=1}^{q} x_i^2$. The multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(\mu_1(\mathrm{x});\mathrm{x},a,b)$ is

$$C_{\mathrm{m}}^{(s)}(\mu_1(\mathrm{x});\mathrm{x},a,b) = \sum_{i=1}^{q} x_i, \tag{29}$$

and the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(\mu_2(\mathrm{x});\mathrm{x},a,b)$ is

$$C_{\mathrm{m}}^{(s)}(\mu_2(\mathrm{x});\mathrm{x},a,b) = \left(\sum_{i=1}^{q} m_i^{-2s+1}\right)\left\{\lambda_1'(a,b) - \lambda_2'(a,b)\right\} + \sum_{i=1}^{q} x_i^2. \tag{30}$$

where the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7).

Suppose that $\|g\| = M$. We take $\mathrm{x}_0 = (x_{01},\ldots,x_{0q})^T$, where $\mathrm{x}_0 \in \mathrm{D}$. We observe that

$$(|\,\mathrm{x}_0 - \mathrm{x}\,|)^2 = \sum_{i=1}^{q} \left(x_{0i}^2 + x_i^2 - 2x_{0i}x_i\right),$$

$\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$. The uniform convergence (27) follows from the result

$$C_{\mathrm{m}}^{(s)}\left((|\,\mathrm{x}_0 - \mathrm{x}\,|)^2;\mathrm{x},a,b\right) = \sum_{i=1}^{q} O\left(m_i^{-2s-1}\right),$$

$\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$, as $m_i \to \infty$, for $s > -1/2$, where $i = 1,\ldots,q$, $\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$. Under the condition $s > -1/2$, the convergence $C_{\mathrm{m}}^{(s)}(\mathrm{g};\mathrm{x},a,b) \to g(\mathrm{x})$ is uniform at any point of continuity $\mathrm{x} \in \mathrm{D}$, as $m_i \to \infty$, where $i = 1,\ldots,q$.

8.3. Degrees of approximation (10) and (11)

For every $\delta > 0$, we denote by $\lambda(x_0, x; \delta)$ the maximum integer less than or equal to $\delta^{-1}|x_0 - x|$, where $x_0, x \in [0,1]$. We recall the definition of modulus of continuity $\omega(\delta)$, where $\delta > 0$. We have

$$|g(x_0) - g(x)| \le \omega(\delta)\{1 + \lambda(x_0, x; \delta)\},$$

$x_0, x \in [0,1]$.

The Bernstein-type approximation $C_m^{(s)}(g; x, a, b)$ is given by (8), where $s > -1/2$, and $x \in [0,1]$. Putting $x_0 = m^{-s}(m^{-1}k - t) + x$, for every $k = 0, 1, \ldots, m$, then, we have that

$$\begin{aligned}
\left| C_m^{(s)}(g; x, a, b) - g(x) \right| &\le \{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} \left| g\left(m^{-s}\left(m^{-1}v - t\right) + x\right) - g(x) \right| \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1} dt \\
&\le \omega(\delta)\{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} \{1 + \lambda(x_0, x; \delta)\} \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1} dt \\
&\le \omega(\delta)\{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} \left\{1 + \delta^{-1} \left| m^{-s}\left(m^{-1}k - t\right) \right| \right\} \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1} dt \\
&\le \omega(\delta)\{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} \left\{1 + \delta^{-2} m^{-2s-2}(k - mt)^2\right\} \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1} dt,
\end{aligned}$$

$x \in [0,1]$. It follows that

$$\left| C_m^{(s)}(g; x, a, b) - g(x) \right| \le \omega(\delta)\left[1 + \delta^{-2} m^{-2s-1}\left\{\lambda_1'(a,b) - \lambda_2'(a,b)\right\}\right],$$

$x \in [0,1]$. Setting $\delta = m^{-1/2}$, we finally have the degree of approximation (10).

For every $\delta > 0$, we denote by $\lambda(\mathrm{x}_0, \mathrm{x}; \delta)$ the maximum integer less than or equal to $\delta^{-1}|\mathrm{x}_0 - \mathrm{x}|$, where $|\mathrm{x}_0 - \mathrm{x}| = \left(\sum_{i=1}^{q} (x_{0i} - x_i)^2\right)^{1/2}$, and $\mathrm{x}_0, \mathrm{x} \in [0,1]^q$. We have

$$|g(\mathrm{x}_0) - g(\mathrm{x})| \le \omega(\delta)\{1 + \lambda(\mathrm{x}_0, \mathrm{x}; \delta)\},$$

where $\omega(\delta)$ is the modulus of continuity, $\delta > 0$, and $\mathrm{x}_0, \mathrm{x} \in [0,1]^q$.

The multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b)$ is given by (9), where $s > -1/2$, and $\mathrm{x} \in [0,1]^q$. We have

$$\left| C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b) - g(\mathrm{x}) \right| \le \{B(a,b)\}^{-q} \int_0^1 \cdots \int_0^1 \sum_{k_1=0}^{m_1} \cdots \sum_{k_q=0}^{m_q} \left| g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}k_1 - t_1\right) + x_1 \\ \vdots \\ m_q^{-s}\left(m_q^{-1}t_q - t_q\right) + x_q \end{pmatrix} - g\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} \right|$$

$$\cdot\binom{m_1}{k_1}\cdots\binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{b+m_1-k_1-1}\cdots t_q^{a+k_q-1}(1-t_q)^{b+m_q-k_q-1}\,dt_1\cdots dt_q$$

$$\le \omega(\delta)\left\{B(a,b)\right\}^{-q}\int_0^1\cdots\int_0^1\sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q}\left\{1+\lambda(\mathrm{x}_0,\mathrm{x};\delta)\right\}$$

$$\cdot\binom{m_1}{k_1}\cdots\binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{b+m_1-k_1-1}\cdots t_q^{a+k_q-1}(1-t_q)^{b+m_q-k_q-1}\,dt_1\cdots dt_q\,,$$

$\mathrm{x}\in[0,1]^q$. Thus, we have

$$\left|C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)-g(\mathrm{x})\right|\le\omega(\delta)\left[1+\delta^{-2}\sum_{i=1}^{q}m_i^{-2s-1}\left\{\lambda_1'(a,b)-\lambda_2'(a,b)\right\}\right],$$

$\mathrm{x}\in[0,1]^q$. Setting $\delta=m^{-1/2}$, where $m=\sum_{i=1}^{q}m_i$, we finally have the degree of approximation (11).

8.4. Uniform convergence of integral operators

We suppose that $g(x)\neq 0$, for every $x\in D$. We can define the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, given by (8), as the integral operator

$$C_m^{(s)}(g;x,a,b)=\int_0^1 h_m(t,x)g(t)\,dt\,, \qquad (31)$$

with the kernel

$$h_m(t,x)=\left\{g(x)\right\}^{-1}\left\{B(a,b)\right\}^{-1}\sum_{k=0}^{m}g\left(m^{-s}(m^{-1}k-t)+x\right)\binom{m}{k}t^{a+k-1}(1-t)^{b+m-k-1}\,,$$

where $x\in D$. The definition (31) is equivalent to the problem of approximating $\left\{g(x)\right\}^2$ with $C_m^{(s)}(g;x,a,b)$, where $x\in D$. We need that

$$\int_0^1 h_m(t,x)\,dt\to 1\,, \qquad (32)$$

uniformly at any point of continuity $x\in D$, as $m\to\infty$. We also need that

$$\int_E \left|h_m(t,x)\right|dt\to 0\,, \qquad (33)$$

where

$$E=\left\{t:\left|m^{-s}\left(m^{-1}k-t\right)+x-x\right|\geq\delta\right\}=\left\{t:\left|m^{-s}\left(m^{-1}k-t\right)\right|\geq\delta\right\},$$

for every $k=0,1,\ldots,m$, and $t\in(0,1)$, uniformly at any point of continuity $x\in D$, as $m\to\infty$. Following Appendix 8.2, the uniform convergence (32) can be shown by Taylor expanding $g\left(m^{-s}\left(m^{-1}k-t\right)+x\right)$ around x, for every $k=0,1,\ldots,m$, where $t\in(0,1)$, and $x\in D$, and by supposing that the function $\{g(x)\}^{-1}g''(x)$ can be bounded by a constant N_1 that does not depend on x, $\{g(x)\}^{-1}g''(x)\leq N_1$, where $x\in D$. Focussing on the convergence in probability, it is seen that the condition (33) is fulfilled. Condition (33) also implies that

$$\int_E\left|h_m(t,x)\right|g(t)dt\to 0,$$

uniformly at any point of continuity $x\in D$, as $m\to\infty$. Finally, we let

$$\int_0^1\left|h_m(t,x)\right|dt\leq N_2<+\infty, \tag{34}$$

where N_2 does not depend on x, and $x\in D$. Under (32) and (33), and (34), the convergence $C_m^{(s)}(g;x,a,b)\to g(x)$, for $s>-1/2$, can be shown to be uniform, at any point of continuity $x\in D$, as $m\to\infty$ (cf. DeVore and Lorentz (1993), chapter 1).

We recall that the Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$ is defined by (9), where $\mathrm{x}\in\mathrm{D}$. Similarly, following Appendix 8.2, we can show uniform convergence of the integral operator

$$C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)=\int_0^1\cdots\int_0^1 h_{m_1}(t_1,x_1)\cdots h_{m_q}(t_q,x_q)g(t_1)\cdots g(t_q)\,dt_1\cdots dt_q,$$

where

$$h_{m_i}(t_i,x_i)=\left\{g(x_i)\right\}^{-1}\left\{B(a,b)\right\}^{-1}\sum_{k_i=0}^{m_i}g\left(m_i^{-s}\left(m_i^{-1}k_i-t_i\right)+x_i\right)\binom{m_i}{k_i}t_i^{a+k-1}(1-t_i)^{b+m-k-1},$$

for every $i=1,\ldots,q$.

Under the condition $s>-1/2$, we have the uniform convergence $C_{\mathrm{m}}^{(s)}(\mathrm{g};\mathrm{x},a,b)\to g(\mathrm{x})$ at any point of continuity $\mathrm{x}\in\mathrm{D}$, as $m_i\to\infty$, where $i=1,\ldots,q$ (cf. DeVore and Lorentz (1993), chapter 1).

8.5. Orders of error in probability in (14) and (15)

The Bernstein-type approximation $C_m^{(s)}(g;\bar{x},a,b)$ is given by (12), where $s>-1/2$. Let $g'(x)=(dx)^{-1}dg(x)$ and $g''(x)=(dx)^{-2}d^2g(x)$ be the first two derivatives of the function $g(x)$, where $x\in[0,1]$. We recall that the quantity $\lambda_1'(a,b)-\lambda_2'(a,b)$ is given by (7). By Taylor expanding the function $g\left(m^{-s}\left(m^{-1}k-t\right)+\bar{x}\right)$ around μ, for every $k=0,1,\ldots,m$, we have

$$
\begin{aligned}
C_m^{(s)}(g;\bar{x},a,b) = g(\mu) & \\
&+ g'(\mu)(\bar{x}-\mu) \\
&+ 2^{-1} g''(\mu) m^{-2s-1}\{\lambda_1'(a,b) - \lambda_2'(a,b)\} \\
&+ 2^{-1} g''(\mu)(\bar{x}-\mu)^2 \\
&+ \cdots \\
&= g(\mu) + O_p(n^{-1/2}) + O(m^{-2s-1}),
\end{aligned}
$$

where $s > -1/2$, as $m \to \infty$, and $n \to \infty$. Order $O(m^{-2s-1}) + O_p(n^{-1/2})$ of error in probability in (14), as $m \to \infty$, and $n \to \infty$, is thus proved.

The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b)$ is given by (13), where $s > -1/2$. By Taylor expanding the function

$$
g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}k_1 - t_1\right) + \bar{x}_1 \\ \vdots \\ m_q^{-s}\left(m_q^{-1}k_q - t_q\right) + \bar{x}_q \end{pmatrix}
$$

around $\mu = (\mu_1, \ldots, \mu_q)^T$, for every $k_i = 1, \ldots, m_i$, $i = 1, \ldots, q$, we can prove the order $\sum_{i=1}^{q} O(m_i^{-2s-1}) + O_p(n^{-1/2})$ of error in probability in (15), $s > -1/2$, as $m_i \to \infty$, where $i = 1, \ldots, q$, and $n \to \infty$.

8.6. Asymptotic normality in (16) and (17)

Following (14), we have that $n^{1/2}\{C_m^{(s)}(g;\bar{x},a,b) - g(\mu)\}$, where $s > -1/2$, is asymptotically equivalent to $n^{1/2}\{g(\bar{x}) - g(\mu)\}$, as $m \to \infty$ and $n \to \infty$. An application of the Central Limit Theorem (cf. Sen and Singer (1993), chapter 3) then shows the asymptotic normality in (16), as $m \to \infty$, and $n \to \infty$.

Following (15), we have $n^{1/2}\{C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b) - g(\mu)\}$, where $s > -1/2$, $\mathrm{m} = (m_1, \ldots, m_q)^T$, is asymptotically equivalent to $n^{1/2}\{g(\bar{\mathrm{x}}) - g(\mu)\}$, as $m_i \to \infty$, where $i = 1, \ldots, q$, and $n \to \infty$. An application of the Central Limit Theorem (cf. Sen and Singer (1993), chapter 3) then shows the asymptotic normality in (17), as $m_i \to \infty$, where $i = 1, \ldots, q$, and $n \to \infty$.

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RIASSUNTO

Approssimazioni del tipo di Bernstein utilizzando la distribuzione beta-binomiale

Viene proposta e studiata una approssimazione del tipo di Bernstein utilizzando la distribuzione beta-binomiale. Vengono studiate approssimazioni del tipo di Bernstein sia univariate che multivariate.

Vengono studiate la convergenza uniforme ed il grado di approssimazione. Vengono anche proposti e studiati stimatori del tipo di Bernstein per funzioni regolari di medie nella popolazione.

SUMMARY

Bernstein-type approximations using the beta-binomial distribution

The Bernstein-type approximation using the beta-binomial distribution is proposed and studied. Both univariate and multivariate Bernstein-type approximations are studied. The uniform convergence and the degree of approximation are studied. The Bernstein-type estimators of smooth functions of population means are also proposed and studied.