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# SATURATION AND SUPEREFFICIENCY SOME APPROXIMATIONS OF THE BERNSTEIN TYPE

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## 1. Introduction

The Bernstein polynomials are linear positive operators $B_m(g;x)$ that can be used for the uniform appproximation of continuous and real-valued functions $g(x)$, where $x \in [0,1]$, and $m$ is the parameter of size of the binomial distribution $bi(m,x)$, where $x \in [0,1]$. See Korovkin (1960), chapter 1, Davis (1963), chapter 6, Feller (1971), chapter 7, Cheney (1982), chapters 3 and 4, Lorentz (1986a), chapter 1, Rivlin (1981), chapter 1, DeVore and Lorentz (1993), chapter 1, and Phillips (2003), chapter 7. See also Rivlin and Shapiro (1961). Recently, Bernstein-type approximations $B_m^{(s)}(g;x)$, where $s > -1/2$ is an approximation coefficient, that improve on the speed of approximation of the Bernstein polynomials $B_m(g;x)$, $x \in [0,1]$, as $m \to \infty$, have been proposed and studied in Pallini (2005).

The sequence $\{B_m(g;x)\}$ of Bernstein polynomials is regarded as saturated, as $m \to \infty$, when the polynomial $B_m(g;x)$ has an 'optimal' degree of approximation, $\varphi_m(x)$, that is $\varphi_m(x) = m^{-1}x(1-x)$, for every $x \in [0,1]$. Better approximation results can be achieved for linear functions $g(x)$ of $x \in [0,1]$, with a null degree of approximation, $\varphi_m(x) = 0$, for every $x \in [0,1]$. The Bernstein-type approximation $B_m^{(s)}(g;x)$ of Pallini (2005) has an 'optimal' degree of approximation $\varphi_m^{(s)}(x) = m^{-2s-1}x(1-x)$, where $s > -1/2$, and $x \in [0,1]$. Accordingly, better approximation results, with a null degree of approximation $\varphi_m^{(s)}(x) = 0$, for every $x \in [0,1]$, can be achieved for linear functions $g(x)$ of $x \in [0,1]$.

Superefficiency is the technical setting in which the variance of estimation can be decreased by some factor to a value that turns out to be below the theoretical minimum of the variance. See Lehmann (1991), chapter 6, Schervish (1995), chapter 7, and Bickel, Klaassen, Ritov and Wellner (1998), chapter 6. The superefficiency of the Bernstein-type estimators will be valid for the actual variance of estimation or for any sample estimate of this variance of estimation. In any case, this superefficiency will directly depend on the parameter $m$ of size for the binomial $bi(m,x)$ distribution and its principal effect will vanish as $m$ increases, as $m \to \infty$. A superefficient Bernstein-type estimator of a smooth function $g(x)$ of the population means, where $x \in [0,1]$, is obtained by minimizing the 'optimal' degree of approximation for $g(x)$, where $x \in [0,1]$. These superefficient Bernstein-type estimators can typically be preferred to their natural counterparts.

In section 2, we overview the main features of the Bernstein-type approximations of Pallini (2005), including some descriptions of the Bernstein polynomials. In section 3, we study the saturation classes of the univariate and the multivariate Bernstein-type approximations of Pallini (2005). In section 4, we provide the basic asymptotics. In section 5, we study the superefficient estimation of smooth functions of population means. We also study the superefficient estimation of a multivariate mean. In section 6, we report on the results of a simulation experiment. Finally, in section 7, we conclude the contribution with some remarks.

We refer to Serfling (1980), Barndorff-Nielsen and Cox (1989), and Sen and Singer (1993), for asymptotics and the basic results of the classical theory of statistical inference.

## 2. BERNSTEIN-TYPE APPROXIMATIONS

### *2.1. Bernstein-type approximations*

Let $g(x)$ be a bounded, continuous, and real-valued function, where $x \in [0,1]$. The Bernstein-type approximation $B_m^{(s)}(g;x)$ of order $m$ for the function $g(x)$ of Pallini (2005) is defined as

$$B_m^{(s)}(g;x) = \sum_{v=0}^{m} g\left(m^{-s}(m^{-1}v - x) + x\right)\binom{m}{v} x^v (1-x)^{m-v}, \tag{1}$$

where $s > -1/2$ is an approximation coefficient, $m$ is a positive integer, and $x \in [0,1]$.

Let $B_m(g;x)$ be the Bernstein polynomial of order $m$, where $x \in [0,1]$. The Bernstein polynomial $B_m(g;x)$ is the special case $s = 0$ in (1), where $x \in [0,1]$.

### *2.2. Multivariate Bernstein-type approximations*

Let $g(\mathrm{x})$ be a bounded, continuous, and real-valued function, where $\mathrm{x} = (x_1,\ldots,x_k)^T \in [0,1]^k$. The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$ of order $m$ for the function $g(\mathrm{x})$ of Pallini (2005) is defined as

$$B_{\mathrm{m}}^{(s)}(g;\mathrm{x}) = \sum_{v_1=0}^{m_1}\cdots\sum_{v_k=0}^{m_k} g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}v_1 - x_1\right) + x_1 \\ \vdots \\ m_k^{-s}\left(m_k^{-1}v_k - x_k\right) + x_k \end{pmatrix}$$
$$\cdot\binom{m_1}{v_1}\cdots\binom{m_k}{v_k} x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_k^{v_k}(1-x_k)^{m_k-v_k}, \tag{2}$$

where $s > -1/2$ is an approximation coefficient, $\mathrm{m} = (m_1,\ldots,m_k)^T$ are positive integers, with $m = \sum_{i=1}^{k} m_i$, and $\mathrm{x} \in [0,1]^k$.

The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ can be obtained by setting $s = 0$ in (2), where $\mathrm{x} \in [0,1]^k$.

### *2.3. Uniform convergenece*

For the Bernstein-type approximation $B_m^{(s)}(g;x)$, $x \in [0,1]$, given by (1), where $s > -1/2$, Pallini (2005) shows that $B_m^{(s)}(g;x) \to g(x)$, as $m \to \infty$, uniformly at any point of continuity $x \in [0,1]$.

For the multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, $\mathrm{m} = (m_1,\ldots,m_k)^T$, $\mathrm{x} \in [0,1]^k$, given by (2), where $s > -1/2$, Pallini (2005) also shows that $B_{\mathrm{m}}^{(s)}(g;\mathrm{x}) \to g(\mathrm{x})$, as $m_i \to \infty$, $i = 1,\ldots,k$, uniformly at any

$k$-dimensional point of continuity $\mathrm{x} = (x_1, \ldots, x_k)^T \in [0,1]^k$

3. SATURATION CLASSES

The saturation class of the Bernstein-type approximations $B_m^{(s)}(g;x)$, given by (1), where $s > -1/2$, and $x \in [0,1]$, consists of all bounded and continuous functions $g(x)$, $x \in [0,1]$, with derivative $g'(x) = (dx)^{-1} dg(x)$ that satisfies a Lipschitz condition $g' \in Lip_Q 1$, where $Q > 0$ is a finite constant, and $x \in [0,1]$. See Appendixes 8.1 and 8.2.

The condition $g' \in Lip_Q 1$ is equivalent to

$$\left| B_m^{(s)}(g;x) - g(x) \right| \leq \frac{1}{2} Q m^{-2s-1} x(1-x), \tag{3}$$

where $s > -1/2$, and $x \in [0,1]$. The degree of approximation $\varphi_m^{(s)}(x) = m^{-2s-1} x(1-x)$, where $s > -1/2$, and $x \in [0,1]$, is 'optimal' in the sense that a smaller degree can only be achieved for very special functions, for the linear functions $g(x)$ of $x \in [0,1]$. In particular, if $g(x)$ is linear, $x \in [0,1]$, then $\varphi_m^{(s)}(x) = 0$, where $s > -1/2$, and $x \in [0,1]$. See Appendix 8.2.

The saturation class of the multivariate Bernstein-type approximations $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (2), where $s > -1/2$, $\mathrm{x} = (x_1, \ldots, x_k)^T \in [0,1]^k$, consists of all bounded and continuous functions $g(\mathrm{x})$, $\mathrm{x} \in [0,1]^k$, with derivative $\sum_{i=1}^{k} (\partial x_i)^{-1} \partial g(x_1, \ldots, x_i, \ldots, x_k) u_{j_i} = grad\, g(x_1, \ldots, x_i, \ldots, x_k)\vec{u}$ that satisfies a Lipschitz condition $grad\, g(x_1, \ldots, x_i, \ldots, x_k)\vec{u} \in Lip_Q 1$, where $Q > 0$ is a finite constant, and $\mathrm{x} \in [0,1]^k$. See Appendixes 8.1. and 8.2.

The Lipschitz condition $grad\, g(x_1, \ldots, x_i, \ldots, x_k)\vec{u} \in Lip_Q 1$ is equivalent to

$$\left| B_{\mathrm{m}}^{(s)}(g;\mathrm{x}) - g(\mathrm{x}) \right| \leq \frac{1}{2} Q \sum_{i=1}^{k} m_i^{-2s-1} x_i (1-x_i), \tag{4}$$

where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$. The degree of approximation $\varphi_{\mathrm{m}}^{(s)}(\mathrm{x}) = \sum_{i=1}^{k} m_i^{-2s-1} x_i (1-x_i)$, where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$, is 'optimal' in the sense that a smaller degree can only be achieved for the linear functions $g(\mathrm{x})$ of $\mathrm{x} \in [0,1]^k$. In particular, if $g(\mathrm{x})$ is linear, $\mathrm{x} \in [0,1]^k$, then $\varphi_{\mathrm{m}}^{(s)}(\mathrm{x}) = 0$, where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$. See Appendix 8.2.

4. BASIC ASYMPTOTICS

*4.1. Bernstein-type estimators*

Let $X$ be a univariate random variable with values $x \in [0,1]$, distribution function $F$, and finite mean $\mu = E[X]$. We want to estimate a population characteristic $\theta = g(\mu)$, where $g(x)$ is a real-valued, bounded, continuous function of $x \in [0,1]$, $g:[0,1] \to \Re^1$. The natural estimator of $\theta$ is $\hat{\theta} = g(\bar{x})$, where $\bar{x} = n^{-1}\sum_{j=1}^{n} X_j$ is the sample mean of $n$ i.i.d. observations of X. Following Pallini (2005), the Bernstein-type estimator $B_m^{(s)}(g;x)$ of $\theta = g(\mu)$ is defined as

$$B_m^{(s)}(g;\bar{x}) = \sum_{\nu=0}^{m} g\left(m^{-s}(m^{-1}\nu - \bar{x}) + \bar{x}\right)\binom{m}{\nu}\bar{x}^{\nu}(1-\bar{x})^{m-\nu}, \tag{5}$$

where $s > -1/2$. The Bernstein-type estimator (5) follows from the definition (1) of the Bernstein-type approximation $B_m^{(s)}(g;x)$, where $s > -1/2$, and $\bar{x} \in [0,1]$.

Let X be a $k$-variate random variable with values $\mathrm{x} \in [0,1]^k$, where $\mathrm{X} = (X_1,\ldots,X_k)^T$, with distribution function $F$, and finite $k$-variate mean $\mu = E[X]$, $\mu = (\mu_1,\ldots,\mu_k)^T$. We want to estimate $\theta = g(\mu)$, where $g:[0,1]^k \to \Re^1$. The natural estimator of $\theta$ is $\hat{\theta} = g(\overline{\mathrm{x}})$, where $\overline{\mathrm{x}} = (\bar{x}_1,\ldots,\bar{x}_k)^T$ is the $k$-variate sample mean, $\bar{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, $i = 1,\ldots,k$, of $n$ i.i.d. observations of $\mathrm{X}$. Following Pallini (2005), the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ is defined as

$$B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}}) = \sum_{\nu_1=0}^{m_1}\cdots\sum_{\nu_k=0}^{m_k} g\begin{pmatrix} m_1^{-s}\left(m_1^{-1}\nu_1 - \bar{x}_1\right) + \bar{x}_1 \\ \vdots \\ m_k^{-s}\left(m_k^{-1}\nu_k - \bar{x}_k\right) + \bar{x}_k \end{pmatrix} \cdot \binom{m_1}{\nu_1}\cdots\binom{m_k}{\nu_k}\bar{x}_1^{\nu_1}(1-\bar{x}_1)^{m_1-\nu_1}\cdots\bar{x}_k^{\nu_k}(1-\bar{x}_k)^{m_k-\nu_k}, \tag{6}$$

where $s > -1/2$. The multivariate Bernstein-type estimator (6) follows from the definition (2) of the Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, where $s > -1/2$, and $\overline{\mathrm{x}} \in [0,1]^k$.

*4.2. Orders of error of the Bernstein-type estimators*

For the Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$, given by (5), where $s > -1/2$, we have

$$B_m^{(s)}(g;\bar{x}) = g(\bar{x}) + O\left(m^{-2s-1}\right), \tag{7}$$

as $m \to \infty$, and

$$B_m^{(s)}(g;\bar{x}) = g(\mu) + O\left(m^{-2s-1}\right) + O_p(n^{-1/2}), \tag{8}$$

as $m \to \infty$ and $n \to \infty$. See Pallini (2005).

For the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, given by (6), where $s > -1/2$, we have

$$B_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}}) = g(\bar{\mathrm{x}}) + \sum_{i=1}^{k} O\left(m_i^{-2s-1}\right), \tag{9}$$

as $m_i \to \infty$, $i = 1,\ldots,k$, and

$$B_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}}) = g(\mu) + \sum_{i=1}^{k} O\left(m_i^{-2s-1}\right) + O_p\left(n^{-1/2}\right), \tag{10}$$

as $m_i \to \infty$, $i = 1,\ldots,k$, and $n \to \infty$. See Pallini (2005).

*4.3. Asymptotic variances of the Bernstein-type estimators*

We denote by $\sigma^2$ the asymptotic variance of $n^{1/2} g(\bar{x})$, as $n \to \infty$,

$$\sigma^2 = \{g'(\mu)\}^2 \nu^2, \tag{11}$$

where $g'(x) = (dx)^{-1} dg(x)$, $x \in [0,1]$; and $\nu^2 = E[(X-\mu)^2]$. For $m^{-2s-1} \le n^{-1/2}$, and $s > -1/2$, the distribution of the Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$, given by (5), is asymptotically normal,

$$n^{1/2}\left\{B_m^{(s)}(g;\bar{x}) - g(\mu)\right\} \xrightarrow{d} N(0,\sigma^2), \tag{12}$$

as $m \to \infty$ and $n \to \infty$, where $\sigma^2$ is given by (11). See Pallini (2005).

We denote by $\sigma^2$ the asymptotic variance of $n^{1/2} g(\bar{\mathrm{x}})$, $\bar{\mathrm{x}} = (\bar{x}_1,\ldots,\bar{x}_k)^T$, as $n \to \infty$,

$$\sigma^2 = \sum_{i_1=1}^{k}\sum_{i_2=1}^{k} \left(\partial x_{i_1}\right)^{-1} \partial g(x_1,\ldots,x_{i_1},\ldots,x_k)\Big|_{\mathrm{x}=\mu} \left(\partial x_{i_2}\right)^{-1} \partial g(x_1,\ldots,x_{i_2},\ldots,x_k)\Big|_{\mathrm{x}=\mu} \nu_{i_1 i_2}. \tag{13}$$

where $\nu_{i_1 i_2} = E[(X_{i_1} - \mu_{i_1})(X_{i_2} - \mu_{i_2})]$, $i_1, i_2 = 1,\ldots,k$. For $m_i^{-2s-1} \le n^{-1/2}$, and $s > -1/2$, $i = 1,\ldots,k$, the distribution of the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}})$, given by (6), is asymptotically normal,

$$n^{1/2}\left\{B_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}}) - g(\mu)\right\} \xrightarrow{d} N(0,\sigma^2), \tag{14}$$

as $m_i \to \infty$, $i = 1,\ldots,k$, and $n \to \infty$, where $\sigma^2$ is given by (13). See Pallini (2005).

## 5. SUPEREFFICIENT ESTIMATION

*5.1. Superefficient estimators*

There exists a bounded and real-valued function $h$, with real values $(h \circ g)(x)$, for $x \in [0,1]$.

Actually, we want to suppose that there exists an entire class of functions of $g(\mu)$, where $\mu = E[X]$, that can reasonably be viewed as population characteristics or parameters. Such characteristics or paramterers can be written as $\theta = h \circ g(\mu)$. The 'otimal' degree of approximtion will always work on the domain of the functions

$\theta = h \circ g(\mu)$ to estimate, being a function of $g(\bar{x})(1-g(\bar{x}))$. One can not require the form $h$, but one can obtain a superefficient estimator of general form by minimizing the nonlinear degree of approximation.

We suppose that $g(\bar{x})$ has values $g \in [0,1]$. Following (1) and (3), for any composite function $(h \circ g)(\bar{x}) = h(g(\bar{x}))$, we have

$$\left| B_m^{(s)}(h; g(\bar{x})) - (h \circ g)(\bar{x}) \right| \leq \frac{1}{2} Q m^{-2s-1} g(\bar{x})(1-g(\bar{x})), \tag{15}$$

where $s > -1/2$. Given the degree of approximation $\varphi_m^{(s)}(\bar{x}) = m^{-2s-1} g(\bar{x})(1-g(\bar{x}))$, obtained as (15), where $s > -1/2$, we also have

$$\varphi_m^{(s)}(g(\bar{x})) = 0,$$

for $(h \circ g)(\bar{x}) = a + bg(\bar{x})$, where $a \in R^1$, $b \in R^1$, and $s > -1/2$. Furthermore, we have

$$\varphi_m^{(s)}(g(\bar{x})) \leq \varphi_m^{(s)}(g(\bar{x}) - g),$$

for every value $g \in [0,1]$, where

$$\varphi_m^{(s)}(g(\bar{x}) - g) = m^{-2s-1} g(\bar{x})(1-g(\bar{x})) + (g(\bar{x}) - g)^2, \tag{16}$$

and $s > -1/2$. See Appendix 8.3. The degree of approximation $\varphi_m^{(s)}(\bar{x}) = m^{-2s-1} g(\bar{x})(1-g(\bar{x}))$ in (15) has a minimizer $\hat{g}$, that is defined as

$$\hat{g} = (1 - m^{-2s-1}) g(\bar{x}) + \frac{1}{2} m^{-2s-1}, \tag{17}$$

where $s > -1/2$. The minimizer $\hat{g}$ given by (17) takes values on $(0,1)$, $\hat{g} \in (0,1)$, for every $s > -1/2$. See Appendix 8.3.

The degree of approximation $\varphi_m^{(s)}(\bar{x}) = m^{-2s-1} g(\bar{x})(1-g(\bar{x}))$ in (15) is a measure of dispersion that can be minimized when taken around a specific value for $g(\bar{x})$. From (1), it follows that the minimizer $\hat{g}$ given by (15) is valid for the set of all binomial r.v.'s $bi(m, g(\bar{x}))$, where $g(\bar{x}) = (1 - m^{-2s-1})^{-1} \left( \hat{g} - \frac{1}{2} m^{-2s-1} \right)$, and $s > -1/2$. We can call this set a base of the binomial r.v.'s $bi(m, g(\bar{x}))$, in the sense that the elements in the base admit the same minimizer $\hat{g}$ given by (17).

If $g = g(\mu)$, where $\mu = E[X]$, then we have

$$\begin{aligned} VAR[\hat{g}] &= (1 - m^{-2s-1})^2 VAR[g(\bar{x})] \\ &= (1 - m^{-2s-1})^2 n^{-1} \sigma^2, \end{aligned} \tag{18}$$

where the variance $\sigma^2$ is defined in (11), and $s > -1/2$. Of course, $(1 - m^{-2s-1})^2 < 1$, for $s > -1/2$. Most importantly, for $s > -1/2$, the minimizer $\hat{g}$ is a superefficient estimator of $g(\mu)$, because

$$VAR[\hat{g}] \le VAR[g(\bar{x})]. \tag{19}$$

We have the case $VAR[\hat{g}] = VAR[g(\bar{x})]$ in (19), by letting $m \to \infty$. See Appendix 8.3.

We suppose that $g(\overline{\mathrm{x}})$, where $\overline{\mathrm{x}} = (\bar{x}_1, \ldots, \bar{x}_k)^T$, has values $g \in [0,1]$. Following (2) and (4), for any bounded and continuous composite function $(h \circ g)(\overline{\mathrm{x}}) = h(g(\overline{\mathrm{x}}))$, we have

$$\left| B_{\mathrm{m}}^{(s)}(h; g(\overline{\mathrm{x}})) - (h \circ g)(\overline{\mathrm{x}}) \right| \le \frac{1}{2} Q m^{-2s-1} g(\overline{\mathrm{x}})\left(1 - g(\overline{\mathrm{x}})\right), \tag{20}$$

where $s > -1/2$. Given the degree of approximation $\varphi_m^{(s)}(\overline{\mathrm{x}}) = m^{-2s-1} g(\overline{\mathrm{x}})(1 - g(\overline{\mathrm{x}}))$, obtained as (20), where $s > -1/2$, we also have

$$\varphi_m^{(s)}(g(\overline{\mathrm{x}}) - g) = m^{-2s-1} g(\overline{\mathrm{x}})\left(1 - g(\overline{\mathrm{x}})\right) + (g(\overline{\mathrm{x}}) - g)^2, \tag{21}$$

where the value $g \in [0,1]$, and $s > -1/2$. See Appendix 8.3. The degree of approximation $\varphi_m^{(s)}(\overline{\mathrm{x}}) = m^{-2s-1} g(\overline{\mathrm{x}})(1 - g(\overline{\mathrm{x}}))$ in (20) has the minimizer $\hat{g}$, that is defined as

$$\hat{g} = \left(1 - m^{-2s-1}\right) g(\overline{\mathrm{x}}) + \frac{1}{2} m^{-2s-1}, \tag{22}$$

where $s > -1/2$. The minimizer $\hat{g}$ given by (22) takes values on $(0,1)$, $\hat{g} \in (0,1)$, for every $s > -1/2$. See Appendix 8.3.

We can define the base of all binomial r.v.'s $bi(m, g(\bar{x}))$ that yields the same minimizer $\hat{g}$ given by (22).

If $g = g(\mu)$, where $\mu = (\mu_1, \ldots, \mu_k)^T = E[\mathrm{X}]$, then the minimizer $\hat{g}$ is a superefficient estimator of $g(\mu)$. In particular, we have

$$\begin{aligned} VAR[\hat{g}] &= \left(1 - m^{-2s-1}\right)^2 VAR[g(\overline{\mathrm{x}})] \\ &= \left(1 - m^{-2s-1}\right)^2 n^{-1} \sigma^2, \end{aligned} \tag{23}$$

where the variance $\sigma^2$ is defined in (13), and $s > -1/2$. Of course, $\left(1 - m^{-2s-1}\right)^2 < 1$, for $s > -1/2$. Finally, we have

$$VAR[\hat{g}] \le VAR[g(\overline{\mathrm{x}})]. \tag{24}$$

We have the case $VAR[\hat{g}] = VAR[g(\overline{\mathrm{x}})]$ in (24), by letting $m \to \infty$ in (22). See Appendix 8.3.

### *5.2. Orders of error of superefficient estimators*

Letting $n$ and $\bar{x}$ fixed, we have that

$$\hat{g} = g(\bar{x}) + O\left(m^{-2s-1}\right),$$

where $s > -1/2$, as $m \to \infty$. Furthermore, we have that

$$\hat{g} = g(\mu) + O(m^{-2s-1}) + O_p(n^{-1/2}) + O_p(m^{-2s-1}n^{-1/2}),$$

where $s > -1/2$, as $m \to \infty$ and $n \to \infty$.

Similarly, with $n$ and $\bar{x}$ fixed, we have that

$$\hat{g} = g(\bar{\mathrm{x}}) + O(m^{-2s-1}),$$

where $\bar{\mathrm{x}} = (\bar{x}_1, \ldots, \bar{x}_k)^T$, and $s > -1/2$, as $m \to \infty$. Furthermore, we have that

$$\hat{g} = g(\mu) + O(m^{-2s-1}) + O_p(n^{-1/2}) + O_p(m^{-2s-1}n^{-1/2}),$$

where $\mu = (\mu_1, \ldots, \mu_k)^T$, and $s > -1/2$, as $m \to \infty$ and $n \to \infty$.

*5.3. Asymptotic variances of the Bernstein-type estimators*

Recalling (23), we have

$$n^{1/2}\{\hat{g} - g(\mu)\} \xrightarrow{d} N(0, \sigma^2), \tag{25}$$

where $\sigma^2$ is given by (11), as $m \to \infty$ and $n \to \infty$. See Appendix 8.3. Similarly, we have

$$n^{1/2}\{\hat{g} - g(\mu)\} \xrightarrow{d} N(0, \sigma^2), \tag{26}$$

where $\sigma^2$ is given by (13), as $m_i \to \infty$, $i = 1, \ldots, k$, and $n \to \infty$. See Appendix 8.3.

*5.4. Superefficient estimation of multivariate means*

Let $\mathrm{X}$ be a $k$-variate random variable with values $\mathrm{x} \in [0,1]^k$, where $\mathrm{X} = (X_1, \ldots, X_k)^T$, $\mathrm{x} = (x_1, \ldots, x_k)^T$, with distribution function $F$, and finite $k$-variate mean $\mu = E[X]$, $\mu = (\mu_1, \ldots, \mu_k)^T$. We want to estimate the $k$-variate mean $\mu$. The natural estimator of $\mu$ is the $k$-variate sample mean $\bar{\mathrm{x}} = (\bar{x}_1, \ldots, \bar{x}_k)^T$, where $\bar{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, for every $i = 1, \ldots, k$. From subsection 5.1, it follows the definition of the $k$-variate superefficient estimator $\hat{\mu}$ of $\mu$,

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_k \end{pmatrix} = \begin{pmatrix} \left(1 - m_1^{-2s-1}\right)\bar{x}_1 + \frac{1}{2}m_1^{-2s-1} \\ \vdots \\ \left(1 - m_k^{-2s-1}\right)\bar{x}_k + \frac{1}{2}m_k^{-2s-1} \end{pmatrix}. \tag{27}$$

In fact, for the covariance matrix $VAR[\hat{\mu}]$, the inequality $VAR[\hat{\mu}] \le VAR[\bar{x}]$ shows that the estimator

$\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_k)^T$ is superefficient. Following subsections 5.1 to 5.3, we also have that

$$n^{1/2}\{\hat{\mu} - \mu\} \xrightarrow{d} N(0, \xi^2),$$

where the asymptotic variance $\xi^2 = (1 - m^{-2s-1})^2 v^2$, and $v^2$ can be deduced from (11), where $s > -1/2$, as $n \to \infty$. Following subsections 5.1 to 5.3, we also have that

$$\{n_1^{1/2}(\hat{\mu}_1 - \mu_1), \ldots, n_k^{1/2}(\hat{\mu}_k - \mu_k)\} \xrightarrow{d} N(0, \xi^2),$$

where the asymptotic covariance matrix $\xi^2 = (\xi_{i_1 i_2})_{i_1=1,\ldots,k,\, i_2=1,\ldots,k} = ((1 - m_{i_1}^{-2s-1})(1 - m_{i_2}^{-2s-1}) v_{i_1 i_2})_{i_1=1,\ldots,k,\, i_2=1,\ldots,k}$, and $v^2 = \sum_{i_1=1}^{k} \sum_{i_1=2}^{k} v_{i_1 i_2}$ can be deduced from (13), where $s > -1/2$, as $n_i \to \infty$, $i = 1, \ldots, k$.

# 6. SIMULATION STUDY

For all the Monte Carlo simulations we performed, we always used the statistical measures of dispersion $VAR_B[g(\overline{\mathrm{X}})]$ and $VAR_B[\hat{g}]$, that are defined as

$$VAR_B[g(\overline{\mathrm{X}})] = (B-1)^{-1} \sum_{b=1}^{B} (g(\overline{x}_b) - g(\mu))^2$$

and

$$VAR_B[\hat{g}] = (B-1)^{-1} \sum_{b=1}^{B} (\hat{g}_b - g(\mu))^2,$$

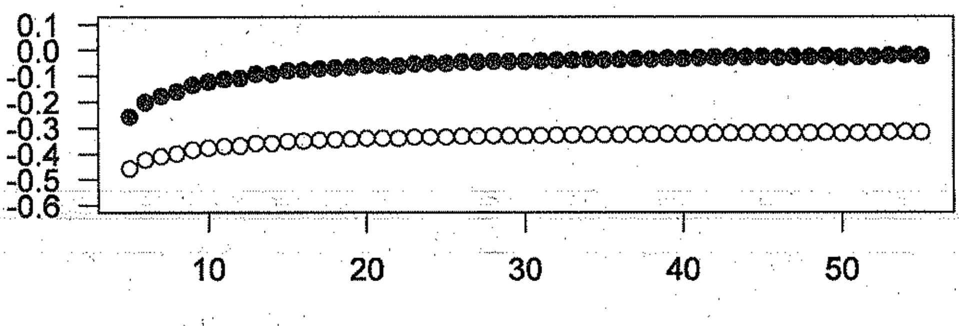
where $g(\mu)$ is the population value. One can observe that $g(\overline{x}_b)$ is the smooth function of the mean $\overline{x}_b$, and $g(\overline{x}_b)$ and $\hat{g}_b = (1 - m^{-2s-1}) g(\overline{x}_b) + \frac{1}{2} m^{-2s-1}$, where $s > -1/2$, are the values from the $b$-th simulation of size $n$, for $s > -1/2$, where $b = 1, \ldots, B$. The same notation is chosen for $g(\overline{\mathrm{X}}_b)$, $\overline{\mathrm{X}}_b$, and $\hat{g}_b = (1 - m^{-2s-1}) g(\overline{\mathrm{X}}_b) + \frac{1}{2} m^{-2s-1}$, where $s > -1/2$.

We did always choose a number $B = 100000$ of independent Monte Carlo replications.

## 6.1. *The univariate variance*

For the univariate variance example $g(\mu)$ we have a bivariate r.v. $X = (X_1, X_1^2)^T$, with finite mean $\mu = (\mu_1, \mu_2)^T = (E[X_1], E[X_1^2])^T$. In particular, we define the univariate variance $g(\mu)$ as

$$g(\mu) = \mu_2 - (\mu_1)^2.$$



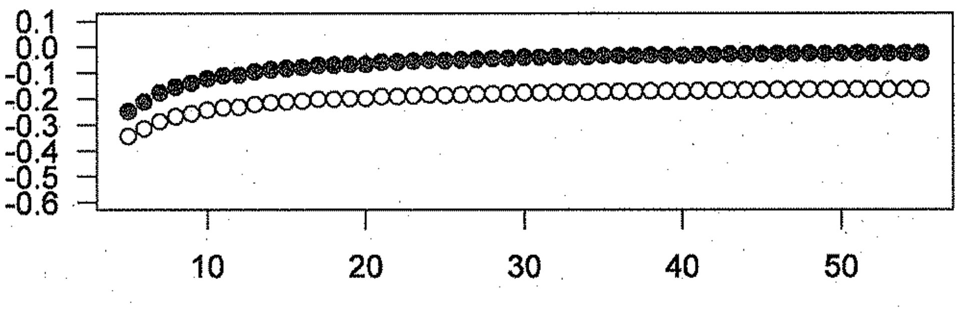
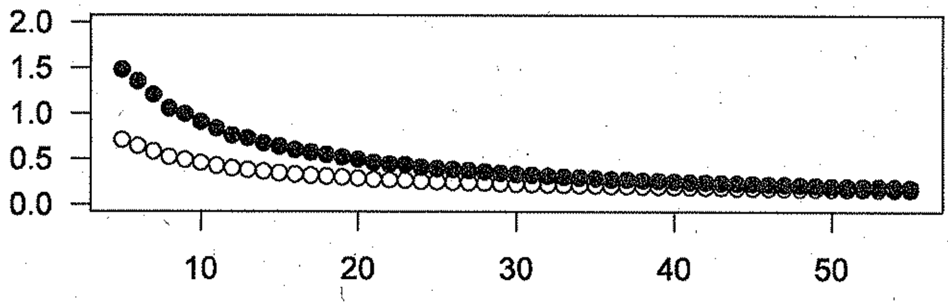
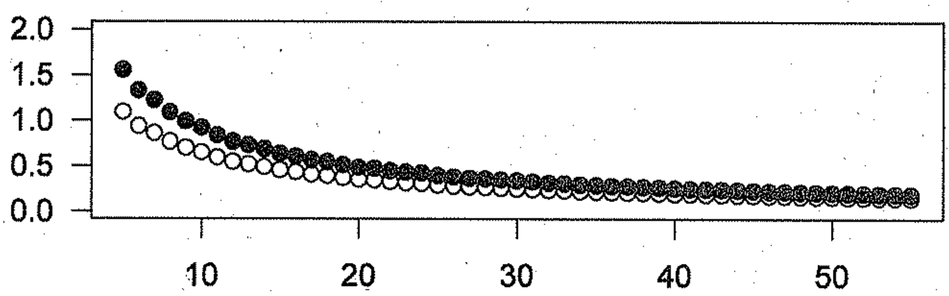




*Figure 1* – The univariate variance example. The Monte Carlo bias, symbol (●), and the Monte Carlo superefficient bias, symbol (○). Random samples of sizes $n = 5, 6, \ldots, 55$ from the Gamma distribution, with parameter $\alpha_1 = 1.25$. Monte Carlo biases for $m_1 = m_2 = 2$, and $s = 0.15$ (panel (a)), and for $m_1 = m_2 = 3$, and $s = 0.25$ (panel (b)).




(a)




(a)

*Figure 2* – The univariate variance example. The Monte Carlo variance, symbol ( ● ), and the Monte Carlo superefficient variance, symbol ( ○ ). Random samples of sizes $n = 5, 6, \ldots, 55$ from the Gamma distribution, with parameter $\alpha_1 = 1.25$. Monte Carlo variances for $m_1 = m_2 = 2$, and $s = 0.15$ (panel (a)), and for $m_1 = m_2 = 3$, and $s = 0.25$ (panel (b)).

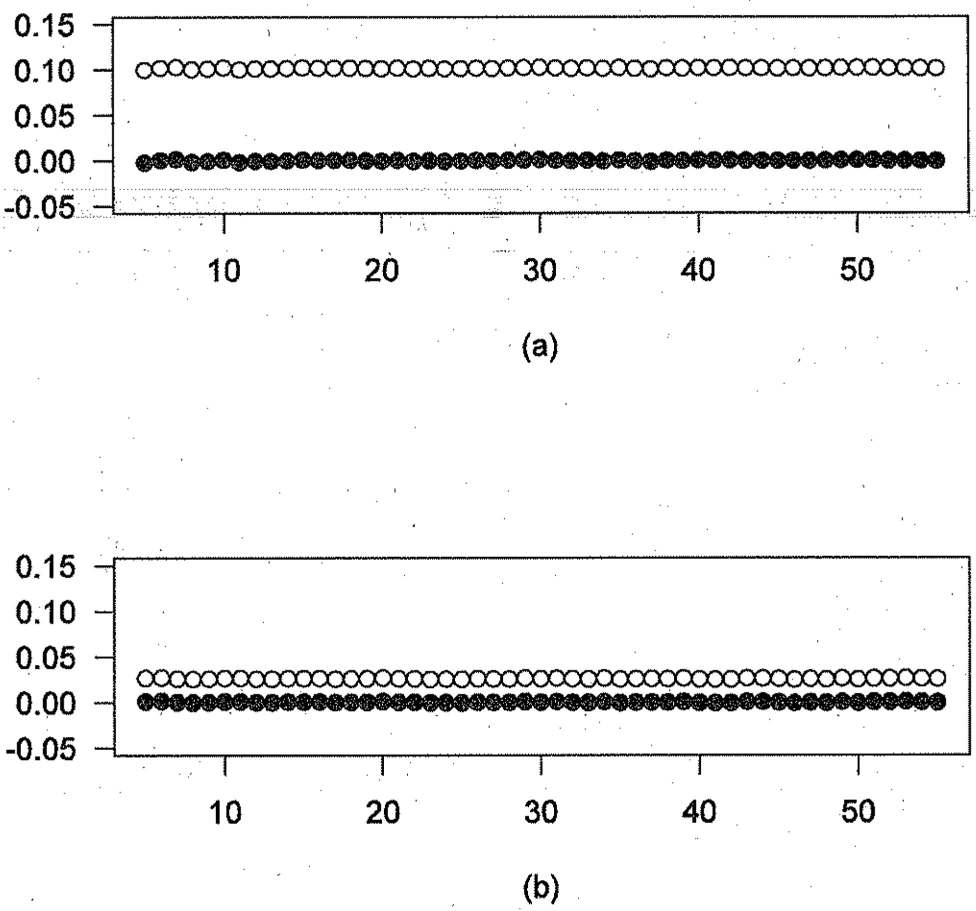
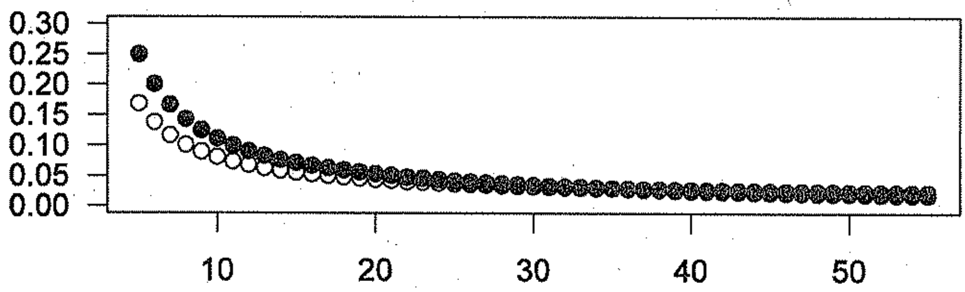
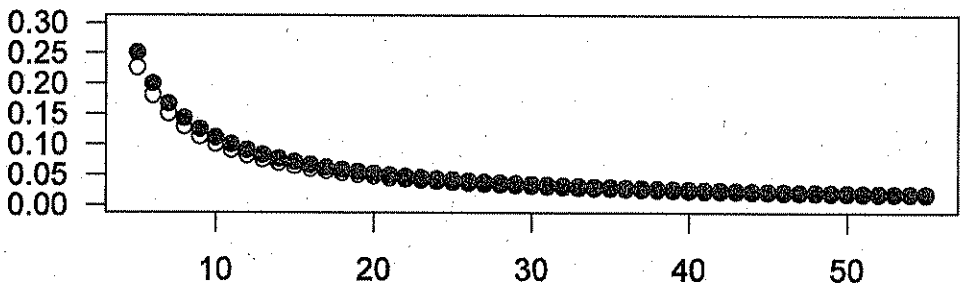



*Figure 3* – The correlation coefficient example. The Monte Carlo bias, symbol (●), and the Monte Carlo superefficient bias, symbol (○). Random samples of sizes $n = 5,6,\ldots,55$ from the bivariate normal distribution, with independent marginals $|N(0,1)|$. Monte Carlo biases for $m_1 = m_2 = m_3 = m_4 = m_5 = 2$, and $s = 0.65$ (panel (a)), and for $m_1 = m_2 = m_3 = m_4 = m_5 = 3$, and $s = 0.85$ (panel (b)).




(a)




(b)

*Figure 4* – The correlation coefficient example. The Monte Carlo variance, symbol (●), and the Monte Carlo superefficient variance, symbol (○). Random samples of sizes $n = 5,6,\ldots,55$ from the bivariate normal distribution, with independent marginals $|N(0,1)|$. Monte Carlo variances for $m_1 = m_2 = m_3 = m_4 = m_5 = 2$, and $s = 0.65$ (panel (a)), and for $m_1 = m_2 = m_3 = m_4 = m_5 = 3$, and $s = 0.85$ (panel (b)).

Following the definition (22), the superefficient estimator $\hat{g}$ can be obtained as $\hat{g}=\left(1-m^{-2s-1}\right)\left(\bar{x}_2-(\bar{x}_1)^2\right)+\frac{1}{2}m^{-2s-1}$, where $s>-1/2$.

Figure 1 compares the Monte Carlo bias of the standard estimator $g(\overline{\mathrm{X}})$ with the bias of the superefficient estimator $\hat{g}$. Figure 2 compares the Monte Carlo variance of the standard estimator $g(\overline{\mathrm{X}})$ with the Monte Carlo variance of the superefficient estimator $\hat{g}$; the variance of $\hat{g}$ always outperform the variance of $g(\overline{\mathrm{X}})$.

*6.2. The correlation coeffficient*

For the correlation coeffcient example $g(\mu)$ we have a 5-variate r.v. $X=(X_1,X_2,X_1^2,X_2^2,X_1X_2)^T$, with finite mean $\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)^T=(E[X_1],E[X_2],E[X_1^2],E[X_2^2],E[X_1X_2])^T$. In particular, we define the corrrelation coefficient $g(\mu)$ as

$$g(\mu)=\frac{(\mu_5-\mu_1\mu_2)}{\left\{\mu_3-(\mu_1)^2\right\}^{1/2}\left\{\mu_4-(\mu_2)^2\right\}^{1/2}}.$$

Following the definition (22), we have the minimizer $\hat{g}=\left(1-m^{-2s-1}\right)g(\overline{\mathrm{x}})+\frac{1}{2}m^{-2s-1}$, where $g(\overline{\mathrm{x}})=\left\{\bar{x}_3-(\bar{x}_1)^2\right\}^{-1/2}\left\{\bar{x}_4-(\bar{x}_2)^2\right\}^{-1/2}(\bar{x}_5-\bar{x}_1\bar{x}_2)$, and $s>-1/2$.

Figure 3 compares the Monte Carlo bias of the standard estimator $g(\overline{\mathrm{X}})$ with the bias of the superefficient estimator $\hat{g}$. The bias of $g(\overline{\mathrm{X}})$ can turn out to be preferable. Figure 4 compares the Monte Carlo variance of the standard estimator $g(\overline{\mathrm{X}})$ with the Monte Carlo variance of the superefficient estimator $\hat{g}$; the variance of $\hat{g}$ always outperform the variance of $g(\overline{\mathrm{X}})$.

*6.3. The multivariate mean*

Following the univariate variance example and the corrrelation coefficient example in subsections 6.1 and 6.2, above, we only compared the variances $VAR^*[X_i]=VAR[\hat{\mu}_i]$ with the variances $VAR[X_i]$, where the superefficicent estimator $\hat{\mu}_i$ of the mean $\mu_i$ is defined by (27) as $\hat{\mu}_i=\left(1-m^{-2s-1}\right)\bar{x}_i+\frac{1}{2}m^{-2s-1}$, for every $i=1,\ldots,k$.

For the univariate variance example in subsection 6.1, we had the differences $VAR^*[X_1]-VAR[X_1]=-0.214$, $VAR^*[X_1^2]-VAR[X_1^2]=-1.716$, $VAR^*[X_1^3]-VAR[X_1^3]=-20.886$. For the correlation coefficient example in subsection 6.2, we had $VAR^*[X_1]-VAR[X_1]=-9.716$, $VAR^*[X_1^2]-VAR[X_1^2]=-14.793$, $VAR^*[X_1^3]-VAR[X_1^3]=-10.394$, $VAR^*[X_1^4]-VAR[X_1^4]=-15.782$, $VAR^*[X_1^5]-VAR[X_1^5]=-10.985$.

## 7. CONCLUDING REMARK

1). For the saturation class of the Bernstein polynomials, the proofs in subsections 8.1 and 8.2, of Appendix 7 below, can be substitutued with the proofs that are studied in Lorentz (1986b), chapter 7.

2). The supefficient estimator $\hat{g}$ given by (17) and (22) can be written as

$$\hat{g} = \left(1 - m^{-2s-1}\right)\tilde{g} + \frac{1}{2} m^{-2s-1}, \tag{28}$$

where $s > -1/2$, and where $\tilde{g}$ is any sample estimate of the popoulation quantity $g(\mu)$, such that $\tilde{g} \xrightarrow{p} g(\mu)$, as $n \to \infty$. The quantity $\tilde{g}$ can be a U-Statistic, a Von Mises Differrentiable Statistical Function, a M-Estimate, a L-esitimate, or a R-Estimate. See Serfling (1980), chapters 5 to 9, and Sen and Singer (1993). Any sample point $\tilde{g}$, that converges to $\mu$, in probability, as $n \to \infty$, can define a superefficient estimator $\hat{g}$ by (28).

3). The set of parameteres $\Theta_0 \subseteq \Theta$ for which the Information inequality does not hold is known to have Lebesgue measure zero. See LeCam (1953), Bahadur (1964) and Le Cam (1999). The supefficient estimator $\hat{g}$ given by (17) and (22), do not solve such situations, because $VAR[\hat{g}] \leq VAR[g(\bar{x})]$ and $VAR[\hat{g}] \leq VAR[g(\bar{\mathrm{x}})]$, also for the set $\Theta_0 \subseteq \Theta$ of parameters.

4). Beyond the domains $[0,1]$ and $[0,1]^k$, more general domains can be obtained following the theory for the Bernstein polynomials in Lorentz (1986a), chapter 2.

8. APPENDIX

*8.1. Lipschitz conditions*

A function $g$ defined on $[0,1]$, $g:[0,1] \to \Re^1$, satisfies a Lipschitz condition with constant $Q$ and exponent $\alpha$, or belongs to the class $Lip_Q\alpha$, where $Q > 0$, $\alpha > 0$, if

$$| g(x_0) - g(x) | \leq Q | x_0 - x |^{\alpha},$$

for every $x_0, x \in [0,1]$. The class $Lip_Q\alpha$ is a linear space. If $g \in Lip_Q\alpha$ on $[0,1]$, then $g$ is continuous, and uniformly continuous on $[0,1]$. If $g \in Lip_Q\alpha$, with $\alpha > 1$, then $g$ is a constant. If $g \in Lip_Q\alpha$, and if $g$ has a derivative satisfying $| g'(x) | \leq Q$, then $g \in Lip_Q 1$. If $\alpha < \beta$, then $Lip_Q\alpha \subset Lip_Q\beta$. See Davis (1963), chapter 1, and Lorentz (1986b), chapter 3.

We denote by $|\mathrm{x}|$, $\mathrm{x} \in [0,1]^k$, the Euclidean norm, $|\mathrm{x}| = \left(\sum_{i=1}^{k} (x_{0i} - x_i)^2\right)^{1/2}$. A function $g$, $g:[0,1]^k \to \Re^1$, satisfies a Lipschitz condition with constant $Q$ or belongs to the class $Lip_Q\alpha$, where $Q > 0$, and $\alpha > 0$, if

$$| g(\mathrm{x}_0) - g(\mathrm{x}) | \leq Q | \mathrm{x}_0 - \mathrm{x} |^{\alpha},$$

for every $\mathrm{x}_0, \mathrm{x} \in [0,1]^k$. See Kelley (1999), chapters 1 and 2, and Cheney and Light (2000), chapter 21.

*8.2. Saturation classes*

For the Bernstein-type approximation $B_m^{(s)}(g;x)$, given by (1), $s > -1/2$, $x \in [0,1]$, the Lipschitz condition $g' \in Lip_Q 1$ implies the degree of approximation $\varphi_m^{(s)}(x) = m^{-2s-1} x(1-x)$, $s > -1/2$, $x \in [0,1]$, in (3). We have

$$
\begin{aligned}
B_m^{(s)}(g;x)-g(x) &= \sum_{\nu=0}^{m}\left\{g\left(m^{-s}(m^{-1}\nu-x)+x\right)-g(x)\right\}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\\
&= \sum_{\nu=0}^{m}\left\{\int_x^{m^{-s}(m^{-1}\nu-x)+x} g'(t)\,dt\right\}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\\
&= \sum_{\nu=0}^{m}\left\{m^{-s}(m^{-1}\nu-x)\,g'(m^{-s}(m^{-1}\nu-x)+x)\right\}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\\
&\quad -\sum_{\nu=0}^{m}\left\{\int_x^{m^{-s}(m^{-1}\nu-x)+x}(t-x)\,g''(t)\,dt\right\}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu},
\end{aligned}
$$

where $g''(x)=(dx)^{-2}d^2g(x)$, $x\in[0,1]$. We also have $|g'(t+h)-g'(t)|\le Q|h|$; then $|g''(x)|\le Q$, and

$$
\begin{aligned}
\left|B_m^{(s)}(g;x)-g(x)\right| &\le Q\sum_{\nu=0}^{m}\left\{\int_x^{m^{-s}(m^{-1}\nu-x)+x}(t-x)\,dt\right\}\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\\
&= \frac{1}{2}Q\sum_{\nu=0}^{m}m^{-2s}\left(m^{-1}\nu-x\right)^2\binom{m}{\nu}x^{\nu}(1-x)^{m-\nu}\\
&= \frac{1}{2}Q\,m^{-2s-1}\,x(1-x)\,.
\end{aligned}
$$

The converse is valid. The conclusion (3) implies that the Lipschitz condition $g'\in Lip_Q 1$. We suppose that $|g'(x)|\le Q$, where $Q>0$ is a finite constant. We have

$$
\frac{1}{2}Q|h|^{\alpha}\le\frac{1}{2}Q\,m^{-2s-1}\,x(1-x),
$$

where $\alpha\ge 1$, for some $h\in[0,1]$, and then

$$
|g'(x+h)-g'(x)|\le Q|h|\left(\frac{1}{2}\,|h|^{\alpha-1}\right). \tag{29}
$$

Since (29) holds, we have

$$
|g'(x+h)-g'(x)|\le Q|h|,
$$

and thus the Lipschitz condition $g'\in Lip_Q 1$.

Finally, from (29), it follows that a better degree of approximation, $\varphi_m^{(s)}(x)=0$, $s>-1/2$, $x\in[0,1]$, can be achieved for the linear functions $g(x)=a$, with derivative $g'(x)=0$, $g(x)=bx$, with $g'(x)=b$ and $g(x)=a+bx$, with $g'(x)=b$, where $x\in[0,1]$.

For the Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (2), $s>-1/2$, $\mathrm{x}\in[0,1]^{\mathrm{k}}$, the Lipschitz condition $grad\,g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\in Lip_Q 1$ implies the degree of approximation (4). We have

$$\left| B_{\mathrm{m}}^{(s)}(g;\mathrm{x})-\mathrm{g}(\mathrm{x})\right| \leq Q\sum_{i=1}^{k}\sum_{v_1=0}^{m_1}\cdots\sum_{v_k=0}^{m_k}\left\{\int_{x_i}^{m_i^{-s}(m_i^{-1}v_i-x_i)+x_i}(t_i-x_i)\,dt_i\right\}$$
$$\cdot\binom{m_1}{v_1}\cdots\binom{m_k}{v_k}x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_k^{v_k}(1-x_k)^{m_k-v_k}$$
$$=\frac{1}{2}Q\sum_{i=1}^{k}m_i^{-2s-1}x_i(1-x_i).$$

The converse is valid. The degree of approximation (4) implies that the Lipschitz condition $grad\, g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\in Lip_Q 1$. We suppose that $\left|grad\, g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\right|\leq Q$, where $Q>0$ is a finite constant. We have

$$\frac{1}{2}Q|\mathrm{h}|^{\alpha}\leq\frac{1}{2}Q\sum_{i=1}^{k}m_i^{-2s-1}x_i(1-x_i),$$

where $\alpha\geq 1$, for some $\mathrm{h}\in[0,1]^k$, $|\mathrm{h}|=\left(\sum_{i=1}^{k}h_i^2\right)^{1/2}$, and then

$$\left|\sum_{i=1}^{k}\{grad\, g(x_1+h_1,\ldots,x_i+h_i,\ldots,x_k+h_k)\vec{u}-grad\, g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\}\right|$$
$$\leq Q|\mathrm{h}|\left(\frac{1}{2}|\mathrm{h}|^{\alpha-1}\right). \tag{30}$$

Since (30) holds, we have

$$\left|\sum_{i=1}^{k}\{grad\, g(x_1+h_1,\ldots,x_i+h_i,\ldots,x_k+h_k)\vec{u}-grad\, g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\}\right|\leq Q|\mathrm{h}|,$$

and thus the Lipschitz condition $grad\, g(x_1,\ldots,x_i,\ldots,x_k)\vec{u}\in Lip_Q 1$, $\mathrm{x}\in[0,1]^k$.

Finally, from (30), it follows that a better degree of approximation, $\varphi_m^{(s)}(\mathrm{x})=0$, $s>-1/2$, $\mathrm{x}\in[0,1]^k$, can be achieved for all the linear functions $g$ of $\mathrm{x}\in[0,1]^{\mathrm{k}}$.

*8.3. Superefficient estimation*

Following (3), for the Bernstein-type estimator $B_m^{(s)}(g;x)$, given by (1), $s>-1/2$, $x\in[0,1]$, we can write $\varphi_m^{(s)}(g(\bar{x}))\leq\varphi_m^{(s)}(g(\bar{x})-g(\mu))$, $s>-1/2$, where the degree of approximation $\varphi_m^{(s)}(g(\bar{x})-g(\mu))$ is obtained as

$$\varphi_m^{(s)}(g(\bar{x})-g(\mu))=\sum_{v=0}^{m}\left\{m^{-s}\left(m^{-1}v-g(\bar{x})\right)+g(\bar{x})-g(\mu)\right\}^2\binom{m}{v}g(\bar{x})^v(1-g(\bar{x}))^{m-v}$$
$$=m^{-2s-1}g(\bar{x})(1-g(\bar{x}))+(g(\bar{x})-g(\mu))^2,$$

$s>-1/2$. Since

$$\left(dg(\overline{x})\right)^{-1} d\varphi_m^{(s)}(g(\overline{x})) = m^{-2s-1}(1-2g(\overline{x})) + 2(g(\overline{x}) - g(\mu)),$$

$$\left(dg(\overline{x})\right)^{-2} d^2\varphi_m^{(s)}(g(\overline{x})) = 2\left(1-m^{-2s-1}\right) > 0,$$

$s > -1/2$, $x \in [0,1]$, the degree of approximation $\varphi_m^{(s)}(g(\overline{x}) - g(\mu))$ has a minimizer for

$$\hat{g} = \left(1-m^{-2s-1}\right) g(\overline{x}) + \frac{1}{2} m^{-2s-1}, \tag{31}$$

$s > -1/2$. Following (11), the variance of the estimator $\hat{g}$ given by (31) is

$$\begin{aligned} VAR[\hat{g}] &= \left(1-m^{-2s-1}\right)^2 VAR[g(\overline{x})] \\ &= \left(1-m^{-2s-1}\right)^2 n^{-1}\sigma^2, \end{aligned} \tag{32}$$

where $s > -1/2$, and $\sigma^2$ is given by (11). Thus, for $s > -1/2$, we have

$$\hat{g} = g(\mu) + O\left(\left(1-m^{-2s-1}\right)n^{-1/2}\right),$$

as $m \to \infty$ and $n \to \infty$.

$$\hat{g} = \left(1-m^{-2s-1}\right) g(\overline{x}) + \frac{1}{2} m^{-2s-1}.$$

It follows that

$$\hat{g} = g(\overline{x}) + O\left(m^{-2s-1}\right),$$

as $m \to \infty$. Finally, it also follows that

$$\begin{aligned} \left(g - g(\mu)\right) &= \left(g(\overline{x}) - g(\mu)\right) - m^{-2s-1} g(\overline{x}) + \frac{1}{2} m^{-2s-1} \\ &= \left(g(\overline{x}) - g(\mu)\right) - m^{-2s-1}\left(g(\mu) + O_p\left(n^{-1/2}\right)\right) + \frac{1}{2} m^{-2s-1} \\ &= O\left(m^{-2s-1}\right) + O_p\left(n^{-1/2}\right) + O_p\left(m^{-2s-1} n^{-1/2}\right), \end{aligned}$$

as $m \to \infty$ and $n \to \infty$.

*8.4. Superefficient estimation of multivariate means*

Following (4), for the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (2), $s > -1/2$, $\mathrm{x} \in [0,1]^{\mathrm{k}}$, we can write $\varphi_{\mathrm{m}}^{(s)}(\overline{\mathrm{x}}) \le \varphi_{\mathrm{m}}^{(s)}(\overline{\mathrm{x}} - \mu)$, $s > -1/2$, where the degree of approximation $\varphi_{\mathrm{m}}^{(s)}(\overline{\mathrm{x}} - \mu)$ is obtained as

$$\varphi_{\mathrm{m}}^{(s)}(\overline{\mathrm{x}} - \mu) = \sum_{i=1}^{k} \left\{ m_i^{-2s-1}\, \overline{x}_i (1-\overline{x}_i) + (\overline{x}_i - \mu_i)^2 \right\},$$

$s > -1/2$. The degree of approximation $\varphi_{\mathrm{m}}^{(s)}(\overline{\mathrm{X}} - \mu)$ has a minimizer for

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_k \end{pmatrix} = \begin{pmatrix} \left(1 - m_1^{-2s-1}\right)\overline{x}_1 + \frac{1}{2} m_1^{-2s-1} \\ \vdots \\ \left(1 - m_k^{-2s-1}\right)\overline{x}_k + \frac{1}{2} m_k^{-2s-1} \end{pmatrix}, \tag{33}$$

$s > -1/2$. Following (13), the variance of the estimator $\hat{\mu}$ given by (33) is

$$VAR[\hat{\mu}] = \begin{pmatrix} \left(1 - m_1^{-2s-1}\right)^2 VAR[\overline{x}_1] & \cdots & \left(1 - m_1^{-2s-1}\right)\left(1 - m_k^{-2s-1}\right) COV[\overline{x}_k, \overline{x}_1] \\ \cdots & \cdots & \cdots \\ \left(1 - m_k^{-2s-1}\right)\left(1 - m_1^{-2s-1}\right) COV[\overline{x}_1, \overline{x}_k] & \cdots & \left(1 - m_k^{-2s-1}\right)^2 VAR[\overline{x}_k] \end{pmatrix}$$
$$= n^{-1} \begin{pmatrix} \left(1 - m_1^{-2s-1}\right)^2 v_1^2 & \cdots & \left(1 - m_1^{-2s-1}\right)\left(1 - m_k^{-2s-1}\right) v_{1k} \\ \cdots & \cdots & \cdots \\ \left(1 - m_k^{-2s-1}\right)\left(1 - m_1^{-2s-1}\right) v_{k1} & \cdots & \left(1 - m_k^{-2s-1}\right)^2 v_k^2 \end{pmatrix}, \tag{34}$$

$s > -1/2$. Thus, for $s > -1/2$, we have

$$\hat{\mu}_i = \mu_i + O\left(\left(1 - m_i^{-2s-1}\right) n^{-1/2}\right),$$

as $m_i \to \infty$, $i = 1, \ldots, k$, and $n \to \infty$.

Following (13), the asymptotic variance of $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_k)^T$ is $\xi^2 = \left(\xi_{i_1 i_2}\right)_{i_1 = 1,\ldots,k;\, i_2 = 1,\ldots,k} = \left(\left(1 - m_{i_1}^{-2s-1}\right)\left(1 - m_{i_2}^{-2s-1}\right) v_{i_1 i_2}\right)_{i_1 = 1,\ldots,k;\, i_2 = 1,\ldots,k}$, where $v^2 = \sum_{i_1=1}^{k} \sum_{i_1=2}^{k} v_{i_1 i_2}$ can be deduced from (13), and $s > -1/2$, as $m_{i_1} \to \infty$, $m_{i_2} \to \infty$, $i_1, i_2 = 1, \ldots, k$, and $n \to \infty$.

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## RIASSUNTO

*Saturazione e superefficienza per alcune approsssimazioni del tipo di Bernstein*

Studiamo il grado di approssimazione 'ottimo' e le classi di saturazione per le approssimazioni del tipo di Bernstein che sono proposte in Pallini (2005). Vengono anche studiate situazioni di superefficienza, più precisamente, vengono anche studiati stimatori superefficienti per funzioni regolari di medie nella popolazione. Un esperimento di simulazione completa il contributo.

## SUMMARY

*Saturation and superefficiency for some approximations of the Bernstein type*

We study the 'optimal' degree of approximation and the saturation classes for the Bernstein-type approximations that are proposed in Pallini (2005). Situations of superefficiency, more precisely, superefficient estimators for smooth functions of population means are also studied. A simulation experiment completes the contribution.

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