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Theoretical and algorithmic results for a class of hierarchical fleet mix problems

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Abstract

In this paper a hierarchical workforce model is studied from both a theoretical and an algorithmic point of view. In the considered model workforce units can be substituted by higher qualified ones; external workforce can also be hired to cover low qualified jobs. A multilevel algorithm is proposed to solve the problems and its efficiency is analyzed by means of cut conditions and discrete convexity properties. Finally, the results of a computational test are provided.

Key words: Discrete programming, discrete convexity, optimal fleet mix.
AMS - 2000 Math. Subj. Class. 90C10, 90C90, 90C31.
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1 Introduction

The general fleet size problem can be summarized as follows: how many vehicles should a firm have in its transport fleet to meet a fluctuating work load? And which should be the optimal fleet mix?

The study of this kind of problems is an important topic of management science because of their practical applications (see for example [1, 3, 5, 7]). Emergency medical services involve decisions about the optimal fleet mix. In fact, ambulance services can be divided into categories according to the urgency of the requests, according to the need of a doctor on board, according to the equipment installed in the vehicle.

Two more examples are the need for a firm to decide how many internal and external technicians to employ, and the need for a transport company to determine the optimal number of trucks to own with respect to their storage capacity.

A critical market factor for each firm is also its capability to respond to customer requests not too late. In particular, service firms often have contractual constraints that force them to respond to the demand of service within a time window. In the case of uncertainty in the demand for services there is the need to cover the peaks at a reasonable cost.

These kinds of problems are said to be hierarchical when the fleet is heterogeneous and the relative units can be grouped in hierarchical sets according to different capabilities or peculiarities (see for all [1, 5]). An example of hierarchical fleet mix is the optimization of vehicles that can transport different kinds of goods, such as edible and non-edible goods. In facts, stainless steel vehicles are able to carry both types of goods, mild steel ones can only carry non-edible liquids.

In general, these are nonlinear integer problems, since the objective function is made by nonlinear penalty functions while the variables are integers since they represent fleet units.

In this paper a class of hierarchical fleet mix problems is studied from both a theoretical and an algorithmic point of view. In particular, we will point out that these problems can be efficiently solved without the use of any heuristics, so that the global optimal solution is guaranteed.

2 Definitions and preliminary results

In this paragraph a hierarchical programming model is proposed. Three different formulations are considered, according to the complexity of the fleet mix. The first one considers an homogeneous fleet, divided into internal and external employed units, the second and the third formulations, respectively, consider two and three kinds of internal employees, with different capabilities.

The aim is to prove that, using a parametric approach, the problems can be solved hierarchically by means of the ones having a smaller number of internal employees kinds.

2.1 Problem $P^{(1)}$

The first model $P^{(1)}$ determines the best fleet composition in order to guarantee a certain service level (that is the number of daily fulfilled requests) and a reasonable labour cost (related to both internal and external units). For this reason the model considers a variable x_1, that is the number of internal units, and a vector $z = [z_i]$ where z_i represents the number of external units employed at day $i \in \{1, \ldots, n\}$. We assume that each external employee can satisfy only one request in a day. On the contrary, every day each internal employee can respond to a number of requests equal to β_{x_1}.

A detailed structure of the model is given in the following definition ([1]).

Definition 2.1 Let $P^{(1)} \equiv P^{(1)}\left(M^{(1)}, \mu^{(1)}, k\right)$ be the following problem:

$$P^{(1)} : \begin{cases} \min f^{(1)}(x_1, z) \\ (x_1, z) \in S_1 \end{cases}$$

where $k \in \Re$, $\mu^{(1)}, M^{(1)} \in Z_+^n$, $b_1 \in Z_+$, $B_1 \in Z_+ \cup \{+\infty\}$, with $0 \leqq \mu^{(1)} \leqq M^{(1)}$, $0 \leq b_1 \leq B_1$ and

$$f^{(1)}(x_1, z) = k + nc_{x_1}x_1 + c_z \sum_{i=1}^{n} z_i + c_{w_1} \sum_{i=1}^{n} w_i^{(1)}(x_1, z_i)$$

$$w_i^{(1)}(x_1, z_i) = \max\left\{0; M_i^{(1)} - \beta_{x_1}x_1 - z_i\right\} \quad \forall i = 1, \ldots, n$$

$$S^{(1)} = \left\{ \begin{array}{c} (x_1, z) \in Z_+ \times Z_+^n \text{ such that} \\ \mu_i^{(1)} \leq \beta_{x_1}x_1 + z_i \ \forall i = 1, \ldots, n \\ b_1 \leq x_1 \leq B_1 \end{array} \right\}$$

with $\beta_{x_1} \in Z_{++}$ and $c_{x_1}, c_z, c_{w_1} \in \Re_{++}$.

The objective function is composed by three cost factors: the cost c_{x_1} for any single internal employee, the cost c_z for any single external unit, the penalty cost c_{w_1}. The constant value k can be interpreted as a fixed cost.

In particular, the cost c_{w_1} represents the penalty marginal cost, that is the cost of the shortage of workforce, while $\max\left\{0; M_i^{(1)} - \beta_{x_1}x_1 - z_i\right\}$ is the

[1] From now on we denote with Z the set of integers, with Z_+ the set of nonnegative integers and with Z_{++} the set of positive integers. Analogously, $\Re$ is the set of real numbers, $\Re_+$ is the set of nonnegative reals and $\Re_{++}$ is the set of positive reals.

number of not fulfilled requests in each day $i \in \{1, \ldots, n\}$. Notice that the parameter $M_i^{(1)}$ is the maximum number of requests at day $i \in \{1, \ldots, n\}$. Parameter $\mu_i^{(1)}$ is used to provide quality of service constraints. In particular, these constraints guarantee that the global number of employed units is able to cover, each day $i \in \{1, \ldots, n\}$, a number of requests greater that $\mu_i^{(1)}$. Finally, notice that b_1 represents the minimum number of internal units while B_1 is the maximum number of internal employees which can be hired.

The following theorem states an upper bound for variable x_1.

Theorem 2.1 *Let (x_1, z) be an optimal solution of problem $P^{(1)}$ and let*

$$u_1 = \min \left\{ \left\lceil \frac{\max\limits_{i=1,\ldots,n} \left\{ M_i^{(1)} \right\}}{\beta_{x_1}} \right\rceil, B_1 \right\}$$

Then, it is $b_1 \leq x_1 \leq u_1$.

Proof First notice that from $0 \leq \mu^{(1)} \leq M^{(1)}$ it yields that for all $h \in Z_+$, for all $z \in Z_+^n$, and for all $i = 1, \ldots, n$, it is:

$$(u_1 + h, z) \in S^{(1)} \quad \text{and} \quad w_i^{(1)}(u_1 + h, z_i) = 0. \tag{1}$$

Being $x_1 \in Z_+$ we just have to prove that $x_1 \leq u_1$. With this aim, let us suppose, by contradiction, that the optimal solution $(\tilde{x}_1, \tilde{z}) \in S^{(1)}$ of problem $P^{(1)}$ is such that $\tilde{x}_1 = u_1 + h$, with $h > 0$. Taking into account of (1), we then get:

$$f^{(1)}(\tilde{x}_1, \tilde{z}) \geq f^{(1)}(\tilde{x}_1, 0) > f^{(1)}(u_1, 0)$$

and this is a contradiction since $P^{(1)}$ is a minimization problem and $(u_1, 0)$ is feasible. The result is then proved taking into account that $x_1 \leq B_1$. □

In order to find the exact optimal solution, a parametric approach is used. Setting variable x_1 fixed as a parameter, an explicit solution for the vector z, depending on the value of x_1, can be stated.

Theorem 2.2 *Let us consider problem $P^{(1)}$ and assume $x_1 \in [b_1, u_1]$ to be a fixed parameter. Then, the optimal value $\hat{z}_i(x_1)$ of variables z_i, $i = 1, \ldots, n$, is given by:*

$$\hat{z}_i(x_1) = \begin{cases} \max\left\{0; M_i^{(1)} - \beta_{x_1} x_1\right\} & \text{if } c_z \leq c_{w_1} \\ \max\left\{0; \mu_i^{(1)} - \beta_{x_1} x_1\right\} & \text{if } c_z > c_{w_1} \end{cases}$$

and the corresponding penalty value is

$$w_i^{(1)}(x_1, \hat{z}_i(x_1)) = \begin{cases} 0 & \text{if } c_z \leq c_{w_1} \\ \max\left\{0; M_i^{(1)} - \max\left\{\beta_{x_1} x_1; \mu_i^{(1)}\right\}\right\} & \text{if } c_z > c_{w_1} \end{cases}$$

Proof By means of the same lines of the proof of Theorem 2.1 it yields:

$$\max\{0; \mu_i^{(1)} - \beta_{x_1} x_1\} \leq \hat{z}_i(x_1) \leq \max\{0; M_i^{(1)} - \beta_{x_1} x_1\}.$$

From the definition of function $f^{(1)}(x_1, z)$ we get:

$$f^{(1)}(x_1, z) = k + nc_{x_1} x_1 + c_z \sum_{i=1}^{n} (z_i + w_i^{(1)}(x_1, z_i)) + (c_{w_1} - c_z) \sum_{i=1}^{n} w_i^{(1)}(x_1, z_i)$$

Hence, noticing that for all $i = 1, \ldots, n$:

$$z_i + w_i^{(1)}(x_1, z_i) = \max\{z_i; M_i^{(1)} - \beta_{x_1} x_1\}$$

and recalling that $\hat{z}_i(x_1) \leq \max\{0; M_i^{(1)} - \beta_{x_1} x_1\}$, it is:

$$f^{(1)}(x_1, z) = k + nc_{x_1} x_1 + c_z \sum_{i=1}^{n} \max\{0; M_i^{(1)} - \beta_{x_1} x_1\} + (c_{w_1} - c_z) \sum_{i=1}^{n} w_i^{(1)}(x_1, z_i)$$

The result then follows noticing that if $c_z < c_{w_1}$ then function $w_i^{(1)}(x_1, z_i)$ has to be minimized by choosing the biggest suitable value for z_i, if $c_z > c_{w_1}$ then function $w_i^{(1)}(x_1, z_i)$ has to be maximized by choosing the smallest suitable value for z_i. If $c_z = c_{w_1}$ then $f^{(1)}(x_1, z)$ is independent to z; in other words the cost of the penalty is equal to the cost of technicians z; for marketing purposes it is better to avoid penalties, so that the biggest suitable value for z_i can be chosen. The value of the penalty function follows trivially. □

The previous results allow to solve problem $P^{(1)}$ by means of the following single variable problem:

$$\bar{P}^{(1)} : \begin{cases} \min \ \varphi(x_1) = f(x_1, \hat{z}(x_1)) \\ b_1 \leq x_1 \leq u_1 \end{cases} \tag{2}$$

where $\hat{z}(x_1) = (\hat{z}_1(x_1), \ldots, \hat{z}_n(x_1))$ is given in the previous theorem. In other words, problem $P^{(1)}$ can be solved by simply comparing the values of $\varphi(x_1)$

for all $x_1 \in [b_1, u_1]$ and then by determining the optimal value $\hat{z}(x_1)$, as it is described in procedure "Min$P^{(1)}()$".

Procedure **Min**$P^{(1)}$(*inputs*: ... ; *outputs*: x_1^*, z^*, val^*)
  Compute u_1;
  # *optional*: update u_1 by means of UB;
  if $b_1 \leq u_1$ *then*
    Determine $\hat{z}(x_1)$ and $\phi(x_1)$;
    x_1^*:= ***MinDiscr***(ϕ, b_1,u_1);
    $z^* := \hat{z}(x_1^*)$ and $val^* := \phi(x_1^*)$;
  end if;
end proc.

Notice that a subprocedure "MinDiscr(ϕ, b_1,u_1)" is used in procedure "Min$P^{(1)}()$" in order to determine the minimum of function ϕ for the integer values in the interval $[b_1,u_1]$. The way this subprocedure can be implemented will be discussed later in order to improve the efficiency of the solution method. Notice also the use of the global variable UB in the optional step of procedure "Min$P^{(1)}()$"; this will allow to improve the performance of the method, as it will be discussed in Subsection 3.2.

2.2 Problem $P^{(2)}$

The second model $P^{(2)}$ considers two different kind of internal units: x_1 and x_2. This model belongs to hierarchical fleet mix problems, where the fleet can be divided into two hierarchical classes according to their characteristics. The units belonging to the most qualified class (denoted with 2) can answer to all the requests, the other ones (denoted with 1) can answer only to the requests asking for a lower qualification. As a consequence, the model structure involves an additional variable x_2 related to units that can fulfill both the requests of type 1 and type 2. Two different constraints are then needed to manage the different levels of requests.

Definition 2.2 Let $P^{(2)} \equiv P^{(2)}\left(M^{(1)}, M^{(2)}, \mu^{(1)}, \mu^{(2)}, k\right)$ be the following problem:

$$P^{(2)} : \begin{cases} \min f^{(2)}(x_1, x_2, z) \\ (x_1, x_2, z) \in S^{(2)} \end{cases}$$

where $k \in \Re$, $\mu^{(1)}, \mu^{(2)}, M^{(1)}, M^{(2)} \in Z_+^n$, $b_1, b_2 \in Z_+$, $B_1, B_2 \in Z_+ \cup \{+\infty\}$, with $0 \leqq \mu^{(1)} \leqq M^{(1)}$, $0 \leqq \mu^{(2)} \leqq M^{(2)}$, $0 \leq b_1 \leq B_1$, $0 \leq b_2 \leq B_2$, and

$$f^{(2)}(x_1, x_2, z) = k + n\left(c_{x_1}x_1 + c_{x_2}x_2\right) + c_z \sum_{i=1}^{n} z_i +$$

$$+ c_{w_1} \sum_{i=1}^{n} w_i^{(1)}(x_1, x_2, z_i) + c_{w_2} \sum_{i=1}^{n} w_i^{(2)}(x_2)$$

$$w_i^{(2)}(x_2) = \max\left\{0; M_i^{(2)} - \beta_{x_2}x_2\right\}$$

$$w_i^{(1)}(x_1, x_2, z_i) = \max\left\{0; M_i^{(1)} - \beta_{x_1}x_1 - z_i + \min\left\{0; M_i^{(2)} - \beta_{x_2}x_2\right\}\right\}$$

$$S^{(2)} = \left\{ \begin{array}{c} (x_1, x_2, z) \in Z_+ \times Z_+ \times Z_+^n \ \text{ such that} \\ \mu_i^{(2)} \leq \beta_{x_2}x_2 \ \forall i = 1, \ldots, n \\ \mu_i^{(1)} + \mu_i^{(2)} \leq \beta_{x_1}x_1 + \beta_{x_2}x_2 + z_i \ \forall i = 1, \ldots, n \\ b_1 \leq x_1 \leq B_1 \ , \ b_2 \leq x_2 \leq B_2 \end{array} \right\}$$

with $\beta_{x_1}, \beta_{x_2} \in Z_{++}$ and $c_{x_1}, c_z, c_{w_1} \in \Re_{++}$.

The differences between problem $P^{(1)}$ and $P^{(2)}$ can be summarized as follows. The objective function of $P^{(2)}$ has two additional cost components: the cost of x_2 units and the penalty costs for request of type 2. As far as the penalty costs is concerned, the type 1 penalty cost depends on all the kinds of fleet units, the type 2 penalty cost depends only on x_2.

An interval of variation for x_2 is determined in the following theorem which can be proved analogously to Theorem 2.1.

Theorem 2.3 *Let (x_1, x_2, z) be an optimal solution of problem $P^{(2)}$. Then, $x_2 \in [l_2, u_2]$ where:*

$$l_2 = \max\left\{ \left\lceil \frac{\max\limits_{i=1,\ldots,n} \left\{\mu_i^{(2)}\right\}}{\beta_{x_2}} \right\rceil, b_2 \right\}$$

$$u_2 = \min\left\{ \left\lceil \frac{\max\limits_{i=1,\ldots,n} \left\{M_i^{(1)} + M_i^{(2)}\right\}}{\beta_{x_2}} \right\rceil, B_2 \right\}$$

The following theorem states the equivalence between problems $P^{(1)}$ and $P^{(2)}$ when x_2 is fixed as a parameter.

Theorem 2.4 *Let us consider problems $P^{(1)}$ and $P^{(2)}$ and let $\bar{x}_2 \in [l_2, u_2]$. Then,*

$$P^{(2)}\left(M^{(1)}, M^{(2)}, \mu^{(1)}, \mu^{(2)}, k\right)\Bigg|_{x_2 = \bar{x}_2} \equiv P^{(1)}\left(\tilde{M}^{(1)}, \tilde{\mu}^{(1)}, \tilde{k}\right)$$

where for all $i = 1, \ldots, n$ it is

$$\begin{aligned}
\tilde{M}_i^{(1)} &= \max\left\{0; M_i^{(1)} + \min\left\{0; M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\}\right\} \\
\tilde{\mu}_i^{(1)} &= \max\left\{0; \mu_i^{(1)} + \mu_i^{(2)} - \beta_{x_2}\bar{x}_2\right\} \\
\tilde{k} &= k + nc_{x_2}\bar{x}_2 + c_{w_2}\sum_{i=1}^{n} \max\left\{0; M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\}
\end{aligned}$$

It results also that $0 \leq \tilde{\mu}_i^{(1)} \leq \tilde{M}_i^{(1)}$ $\forall i = 1, \ldots, n$.

Proof First notice that problems $P^{(1)}$ and $P^{(2)}$ given in the assumptions share the same feasible region; with this aim notice that, being $\beta_{x_1}x_1 + z_i \geq 0$ $\forall i = 1, \ldots, n$, in the case $\mu_i^{(1)} + \mu_i^{(2)} - \beta_{x_2}\bar{x}_2 < 0$ the region is not affected by the fact that $\tilde{\mu}_i^{(1)}$ is fixed at 0. Problems $P^{(1)}$ and $P^{(2)}$ share also the same objective function; with this aim notice that, being $-\beta_{x_1}x_1 - z_i \leq 0$ $\forall i = 1, \ldots, n$, in the case $M_i^{(1)} + \min\left\{0; M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\} < 0$ the penalty function $w_i^{(1)}$ is equal to zero and this value is maintained when $\tilde{M}_i^{(1)}$ is fixed at 0. Let us finally prove that $0 \leq \tilde{\mu}_i^{(1)} \leq \tilde{M}_i^{(1)}$ $\forall i = 1, \ldots, n$. With this aim, first notice that in the case $M_i^{(1)} + M_i^{(2)} \leq \beta_{x_2}\bar{x}_2$ it is $\tilde{M}_i^{(1)} = \tilde{\mu}_i^{(1)} = 0$ since $0 \leq \mu_i^{(1)} \leq M_i^{(1)}$ and $0 \leq \mu_i^{(2)} \leq M_i^{(2)}$. Assume now $M_i^{(1)} + M_i^{(2)} > \beta_{x_2}\bar{x}_2$; from $M_i^{(1)} \geq 0$ it yields that

$$\begin{aligned}
\tilde{M}_i^{(1)} &= \max\left\{0; M_i^{(1)} + \min\left\{0; M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\}\right\} \\
&= \max\left\{0; \min\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\}\right\} \\
&= \min\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\}
\end{aligned}$$

As a consequence, it results

$$\begin{aligned}\tilde{M}_i^{(1)} - \tilde{\mu}_i^{(1)} &= \min\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\} - \max\left\{0; \mu_i^{(1)} + \mu_i^{(2)} - \beta_{x_2}\bar{x}_2\right\} \\ &= \min\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} - \beta_{x_2}\bar{x}_2\right\} + \min\left\{0; -\mu_i^{(1)} - \mu_i^{(2)} + \beta_{x_2}\bar{x}_2\right\} \\ &= \min\left\{\begin{array}{c} M_i^{(1)}; M_i^{(1)} - \mu_i^{(1)} - \mu_i^{(2)} + \beta_{x_2}\bar{x}_2; \\ M_i^{(1)} + M_i^{(2)} - \beta_{x_2}\bar{x}_2; M_i^{(2)} + M_i^{(1)} - \mu_i^{(1)} - \mu_i^{(2)} \end{array}\right\} \geq 0\end{aligned}$$

being $\mu_i^{(2)} \leq \beta_{x_2}\bar{x}_2$, $M_i^{(1)} \geq 0$, $M_i^{(1)} \geq \mu_i^{(1)}$ and $M_i^{(2)} \geq \mu_i^{(2)}$. □

The previous result suggests to approach problem $P^{(2)}$ by iteratively solving the equivalent problems $P^{(1)}$ for x_2 from l_2 to u_2, as it is described in procedure "Min$P^{(2)}$()".

Procedure Min$P^{(2)}$(*inputs*: ... ; *outputs*: x_1^*, x_2^*, z^*, val^*)
  Compute u_2 and l_2;
  # *optional*: update u_2 by means of UB;
  if $l_2 \leq u_2$ *then*
    set $\bar{x}_2 := l_2$ and $val^* = +\infty$;
    while $\bar{x}_2 \leq u_2$
      compute $\tilde{M}^{(1)}$, $\tilde{\mu}^{(1)}$, $\tilde{k}$;
      [x_1', z', val']:= ***MinP***$^{(1)}$($\tilde{M}^{(1)}$, $\tilde{\mu}^{(1)}$, $\tilde{k}$);
      if $val' < val^*$ *then*
        $x_1^* := x_1'$; $x_2^* := \bar{x}_2$; $z^* := z'$; $val^* := val'$;
        # *optional*: $UB := val^*$; update u_2 by means of UB;
      end if;
      $\bar{x}_2 := \bar{x}_2 + 1$;
    end while;
  end if;
end proc.

The use of the global variable UB in the optional steps of procedure "Min$P^{(2)}$()", aimed to improve the performance of the method, will be discussed in Subsection 3.2.

2.3 Problem $P^{(3)}$

The third model $P^{(3)}$ considers three different kinds of internal units: x_1, x_2 and x_3 according to three different levels of requests. Type 3 fleet units can fulfill all the three kind of requests, type 2 units can answer to requests of the first and second type, type 1 and external units can satisfy only type 1 requests. The model is formulated as follows.

Definition 2.3 Let $P^{(3)} \equiv P^{(3)}\left(M^{(1)}, M^{(2)}, M^{(3)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}, k\right)$ be the following problem:

$$P^{(3)} : \begin{cases} \min f^{(3)}(x_1, x_2, x_3, z) \\ (x_1, x_2, x_3, z) \in S^{(3)} \end{cases}$$

where $k \in \Re$, $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, M^{(1)}, M^{(2)}, M^{(3)} \in Z_+^n$, $b_1, b_2, b_3 \in Z_+$, $B_1, B_2, B_3 \in Z_+ \cup \{+\infty\}$, with $0 \leqq \mu^{(1)} \leqq M^{(1)}$, $0 \leqq \mu^{(2)} \leqq M^{(2)}$, $0 \leqq \mu^{(3)} \leqq M^{(3)}$, $0 \leq b_1 \leq B_1$, $0 \leq b_2 \leq B_2$, $0 \leq b_3 \leq B_3$, and

$$f^{(3)}(x_1, x_2, x_3, z) = n(c_{x_1}x_1 + c_{x_2}x_2 + c_{x_3}x_3) + c_z \sum_{i=1}^{n} z_i + c_{w_1} \sum_{i=1}^{n} w_i^{(1)}(x_1, x_2, x_3, z_i) +$$
$$+ c_{w_2} \sum_{i=1}^{n} w_i^{(2)}(x_2, x_3) + c_{w_3} \sum_{i=1}^{n} w_i^{(3)}(x_3) + k$$
$$w_i^{(3)}(x_3) = \max\left\{0; M_i^{(3)} - \beta_{x_3}x_3\right\}$$
$$w_i^{(2)}(x_2, x_3) = \max\left\{0; M_i^{(2)} - \beta_{x_2}x_2 + \min\left\{0; M_i^{(3)} - \beta_{x_3}x_3\right\}\right\}$$
$$w_i^{(1)}(x_1, x_2, x_3, z_i) = \max\left\{\begin{array}{l} 0; M_i^{(1)} - \beta_{x_1}x_1 - z_i \\ + \min\left\{0; M_i^{(2)} - \beta_{x_2}x_2 + \min\left\{0; M_i^{(3)} - \beta_{x_3}x_3\right\}\right\} \end{array}\right\}$$

$$S^{(3)} = \left\{\begin{array}{c} (x_1, x_2, x_3, z) \in Z_+ \times Z_+ \times Z_+ \times Z_+^n \text{ such that} \\ \mu_i^{(3)} \leq \beta_{x_3}x_3 \;\; \forall i = 1, \ldots, n \\ \mu_i^{(2)} + \mu_i^{(3)} \leq \beta_{x_2}x_2 + \beta_{x_3}x_3 \;\; \forall i = 1, \ldots, n \\ \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} \leq \beta_{x_1}x_1 + \beta_{x_2}x_2 + \beta_{x_3}x_3 + z_i \;\; \forall i = 1, \ldots, n \\ b_1 \leq x_1 \leq B_1 \;,\;\; b_2 \leq x_2 \leq B_2 \;,\;\; b_3 \leq x_3 \leq B_3 \end{array}\right\}$$

The following theorem, which is analogous to Theorems 2.1 and 2.3, states a lower and an upper bound for variable x_3.

Theorem 2.5 *Let (x_1, x_2, x_3, z) be an optimal solution of problem $P^{(3)}$. Then, $x_3 \in [l_3, u_3]$ where:*

$$l_3 = \max\left\{\left\lceil \frac{\max\limits_{i=1,\dots,n}\left\{\mu_i^{(3)}\right\}}{\beta_{x_3}} \right\rceil, b_3\right\}$$
$$u_3 = \min\left\{\left\lceil \frac{\max\limits_{i=1,\dots,n}\left\{M_i^{(1)} + M_i^{(2)} + M_i^{(3)}\right\}}{\beta_{x_3}} \right\rceil, B_3\right\}$$

As mentioned in the introduction, one of the aims of this paper is to establish the equivalence between the three presented formulations. Since the equivalence between $P^{(2)}$ and $P^{(1)}$ has been proved in the previous section, it is sufficient, here, to prove the equivalence between $P^{(2)}$ and $P^{(3)}$.

Theorem 2.6 *Consider problems $P^{(2)}$ and $P^{(3)}$ and let $\bar{x}_3 \in [l_3, u_3]$. Then,*

$$P^{(3)}\left(M^{(1)}, M^{(2)}, M^{(3)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}, k\right)\Bigg|_{x_3 = \bar{x}_3} \equiv P^{(2)}\left(\tilde{M}^{(1)}, \tilde{M}^{(2)}, \tilde{\mu}^{(1)}, \tilde{\mu}^{(2)}, \tilde{k}\right)$$

where for all $i = 1, \dots, n$ it is

$$\tilde{M}_i^{(2)} = \max\left\{0; M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}$$
$$\tilde{M}_i^{(1)} = \max\left\{0; M_i^{(1)} + \min\left\{0; M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}\right\}$$

$$\tilde{\mu}_i^{(2)} = \max\left\{0; \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}$$
$$\tilde{\mu}_i^{(1)} = \max\left\{0; \mu_i^{(1)} + \min\left\{0; \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}$$
$$\tilde{k} = k + nc_{x_3}\bar{x}_3 + c_{w_3}\sum_{i=1}^{n}\max\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}$$

It results also that $\tilde{M}_i^{(2)} \geq \tilde{\mu}_i^{(2)} \geq 0$ and $\tilde{M}_i^{(1)} \geq \tilde{\mu}_i^{(1)} \geq 0$ $\forall i = 1, \dots, n$.

Proof First notice that problems $P^{(3)}$ and $P^{(2)}$ given in the assumptions share the same feasible region; with this aim notice that, being $\beta_{x_2}x_2 \geq 0$ $\forall i = 1, \ldots, n$ in the case $\mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3 < 0$ the region is not affected by the fact that $\tilde{\mu}_i^{(2)}$ is fixed at 0. The same observation holds for $\tilde{\mu}_i^{(1)}$.

Problems $P^{(3)}$ and $P^{(2)}$ share also the same objective function; with this aim let us consider two different cases:

1) if $M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} \geq 0$ the equivalence can be stated by setting $\tilde{M}_i^{(1)} = M_i^{(1)}$ and $\tilde{M}_i^{(2)} = M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}$

2) if $M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} < 0$ the equivalence can be stated by setting $\tilde{M}_i^{(1)} = M_i^{(1)} + M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}$ and $\tilde{M}_i^{(2)} = 0$

Let us now prove that $\tilde{M}_i^{(2)} \geq \tilde{\mu}_i^{(2)} \geq 0$ $\forall i = 1, \ldots, n$. With this aim, first notice that in the case $M_i^{(2)} + M_i^{(3)} \leq \beta_{x_3}\bar{x}_3$ it is $\tilde{M}_i^{(2)} = \tilde{\mu}_i^{(2)} = 0$ since $M_i^{(2)} \geq \mu_i^{(2)} \geq 0$ and $M_i^{(3)} \geq \mu_i^{(3)} \geq 0$. Assume now $M_i^{(2)} + M_i^{(3)} > \beta_{x_3}\bar{x}_3$; from $M_i^{(2)} \geq 0$ it yields that

$$\begin{aligned}
\tilde{M}_i^{(2)} &= \max\left\{0; M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\} \\
&= \max\left\{0; \min\left\{M_i^{(2)}; M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\} \\
&= \min\left\{M_i^{(2)}; M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}
\end{aligned}$$

As a consequence, it results

$$\begin{aligned}
\tilde{M}_i^{(2)} - \tilde{\mu}_i^{(2)} &= \min\left\{M_i^{(2)}; M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} - \max\left\{0; \mu_i^{(3)} + \mu_i^{(2)} - \beta_{x_3}\bar{x}_3\right\} \\
&= \min\left\{M_i^{(2)}; M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} + \min\left\{0; -\mu_i^{(3)} - \mu_i^{(2)} + \beta_{x_3}\bar{x}_3\right\} \\
&= \min\left\{\begin{array}{cc} M_i^{(2)} \; ; & M_i^{(2)} - \mu_i^{(3)} - \mu_i^{(2)} + \beta_{x_3}\bar{x}_3 \\ M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3 \; ; & M_i^{(2)} + M_i^{(3)} - \mu_i^{(3)} - \mu_i^{(2)} \end{array}\right\} \geq 0
\end{aligned}$$

being $\mu_i^{(3)} \leq \beta_{x_3}\bar{x}_3$, $M_i^{(2)} \geq \mu_i^{(2)}$ and $M_i^{(3)} \geq \mu_i^{(3)}$.

Let us now prove that $\tilde{M}_i^{(1)} \geq \tilde{\mu}_i^{(1)} \geq 0$ $\forall i = 1, \ldots, n$. With this aim, first notice that in the case $M_i^{(1)} + M_i^{(2)} + M_i^{(3)} \leq \beta_{x_3}\bar{x}_3$ it is $\tilde{M}_i^{(2)} = \tilde{\mu}_i^{(2)} = 0$ since $M_i^{(1)} \geq \mu_i^{(1)} \geq 0$, $M_i^{(2)} \geq \mu_i^{(2)} \geq 0$ and $M_i^{(3)} \geq \mu_i^{(3)} \geq 0$.

Assume now $M_i^{(1)} + M_i^{(2)} + M_i^{(3)} > \beta_{x_3}\bar{x}_3$; from $M_i^{(1)} \geq 0$ it yields that

$$\begin{aligned}
\tilde{M}_i^{(1)} &= \max\left\{0; M_i^{(1)} + \min\left\{0; M_i^{(2)} + \min\left\{0; M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}\right\} \\
&= \max\left\{0; M_i^{(1)} + \min\left\{0; \min\left\{M_i^{(2)}; M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}\right\} \\
&= \max\left\{0; \min\left\{M_i^{(1)}; \min\left\{M_i^{(1)} + M_i^{(2)}; M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}\right\} \\
&= \min\left\{M_i^{(1)}; \min\left\{M_i^{(1)} + M_i^{(2)}; M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\} \\
&= \min\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mu}_i^{(1)} &= \max\left\{0; \mu_i^{(1)} + \min\left\{0; \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\} \\
&= \max\left\{0; \min\left\{\mu_i^{(1)}; \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}
\end{aligned}$$

As a consequence, it results

$$\begin{aligned}
\tilde{M}_i^{(2)} - \tilde{\mu}_i^{(2)} = \min&\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} + \\
&- \max\left\{0; \min\left\{\mu_i^{(1)}; \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\} = \\
= \min&\left\{M_i^{(1)}; M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3\right\} + \\
&+ \min\left\{0; -\min\left\{\mu_i^{(1)}; \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}\right\}
\end{aligned}$$

and hence $\tilde{M}_i^{(2)} - \tilde{\mu}_i^{(2)}$ is equal to:

$$\min\left\{\begin{array}{l}
M_i^{(1)}; \\
M_i^{(1)} - \min\left\{\mu_i^{(1)}; \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}; \\
M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3; \\
M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3 - \min\left\{\mu_i^{(1)}; \mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)} - \beta_{x_3}\bar{x}_3\right\}
\end{array}\right\} \geq 0$$

noticing that $M_i^{(1)} \geq \mu_i^{(1)}$, $M_i^{(2)} \geq \mu_i^{(2)}$, $M_i^{(3)} \geq \mu_i^{(3)}$ and $M_i^{(1)} + M_i^{(2)} + M_i^{(3)} - \beta_{x_3}\bar{x}_3 > 0$ by assumption. □

The previous result suggests to approach problem $P^{(3)}$ by iteratively solving the equivalent problems $P^{(2)}$ for x_3 from l_3 to u_3, and then using the equivalence between $P^{(2)}$ and $P^{(1)}$, as it is described in procedure "Min$P^{(3)}$()".

Procedure Min$P^{(3)}$(*inputs*: ... ; *outputs*: x_1^*, x_2^*, x_3^*, z^*, val^*)
  Compute u_3 and l_3;
  # *optional*: update u_3 by means of UB;
  if $l_3 \leq u_3$ *then*
    set $\bar{x}_3 := l_3$ and $val^* = +\infty$;
    while $\bar{x}_3 \leq u_3$
      compute $\tilde{M}^{(1)}$, $\tilde{\mu}^{(1)}$, $\tilde{M}^{(2)}$, $\tilde{\mu}^{(2)}$, $\tilde{k}$;
      $[x_1', x_2', z', val']$:= ***MinP***$^{(2)}(\tilde{M}^{(1)}, \tilde{\mu}^{(1)}, \tilde{M}^{(2)}, \tilde{\mu}^{(2)}, \tilde{k})$;
      if $val' < val^*$ *then*
        $x_1^* := x_1'$; $x_2^* := x_2'$; $x_3^* := \bar{x}_3$; $z^* := z'$; $val^* := val'$;
        # *optional*: $UB := val^*$; update u_3 by means of UB;
      end if;
      $\bar{x}_3 := \bar{x}_3 + 1$;
    end while;
  end if;
end proc.

The use of the global variable UB in the optional steps of procedure "Min$P^{(3)}$()" will be discussed in Subsection 3.2.

3 Algorithm improvements

The aim of this section is to study some results aimed to improve the performance of the solution method presented in the previous section. This can be done both studying theoretical properties of function $\phi(x_1)$ and stating better bounds for variables x_1, x_2 and x_3.

3.1 Discrete convexity of $\phi(x_1)$

In procedure "Min$P^{(1)}$()" a subroutine named "MinDiscr()" has to be used in order to minimize function $\phi(x_1)$ in the interval $[b_1, u_1]$. This can be done, for example, by evaluating function $\phi(x_1)$ for all the $u_1 - b_1 + 1$ integer values of the interval $[b_1, u_1]$. A much more efficient technique can be used in the case function $\phi(x_1)$ verifies some generalized convexity property.

With this aim, let us recall the following concept of discrete convexity introduced and studied by Cambini-Riccardi-Yuceer in [2] (see also [4, 6, 8]).

Definition 3.1 A set $X \subseteq Z$ is said to be a *discrete reticulum* if

$$\{z \in Z : \min\{x, y\} \leq z \leq \max\{x, y\}\} \subseteq X \quad \forall x, y \in X.$$

Definition 3.2 Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. Function f is said to be a *discrete convex function* if for all $x \in X$ such that $x + 1 \in X$ and $x - 1 \in X$, it is:

$$f(x+1) + f(x-1) \geq 2f(x) \tag{3}$$

The two following fundamental results follow directly from the previous definitions.

Theorem 3.1 *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. If f is a convex function over the convex hull of X, then it is also discrete convex over X.*

Theorem 3.2 *Let $f, g : X \to \Re$, where X is a discrete reticulum, be two discrete convex functions and let $\alpha \in \Re$, $\alpha > 0$. Then, $(f + g)(x)$ and $\alpha f(x)$ are discrete convex functions.*

For the sake of convenience, it is worth providing the following characterization of the discrete convexity concept.

Theorem 3.3 *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. Function f is a discrete convex function if and only if the following property holds:*

$$\frac{f(x+h) - f(x)}{h} \geq \frac{f(x) - f(x-k)}{k} \quad \forall h \geq 1, \ \forall k \geq 1 \tag{4}$$

Proof Noticing that (3) follows from (4) by setting $k = h = 1$, we just have to prove that if f is discrete convex then (4) holds. Let us first prove, as a preliminary result, that the discrete convexity of f implies:

$$f(x+h) - f(x+h-1) \geq f(x+1) - f(x) \quad \forall h \geq 1 \tag{5}$$

This property is trivial in the case $h = 1$, while for $h > 1$ condition (3) implies:

$$\begin{aligned}[f(x+h) &- f(x+h-1)] - [f(x+1) - f(x)] = \\ &= \sum_{j=1}^{h-1} \{[f(x+j+1) - f(x+j)] - [f(x+j) - f(x+j-1)]\} \\ &= \sum_{j=1}^{h-1} [f(x+j+1) + f(x+j-1) - 2f(x+j)] \geq 0\end{aligned}$$

Notice also that from (5) it yields:

$$f(x) - f(x-1) \geq f(x-k+1) - f(x-k) \quad \forall k \geq 1 \tag{6}$$

Conditions (5) and (6) allow us to prove that:

$$f(x+h) - f(x) = \sum_{j=1}^{h} (f(x+j) - f(x+j-1)) \geq h(f(x+1) - f(x))$$

$$f(x) - f(x-k) = \sum_{j=1}^{k} (f(x-j+1) - f(x-j)) \leq k(f(x) - f(x-1))$$

As a conclusion, the discrete convexity of f implies:

$$\frac{f(x+h) - f(x)}{h} \geq f(x+1) - f(x) \geq f(x) - f(x-1) \geq \frac{f(x) - f(x-k)}{k}$$

so that the result is proved. □

The previous characterization allows us to state the following properties.

Theorem 3.4 *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum, be a discrete convex function and let $x_0 \in X$ such that $x_0 + 1 \in X$. The following properties hold:*

i) if $f(x_0) < f(x_0+1)$ then $f(x) \geq f(x_0)$ $\forall x \in X$, $x \geq x_0$;

ii) if $f(x_0) > f(x_0+1)$ then $f(x) \geq f(x_0+1)$ $\forall x \in X$, $x \leq x_0 + 1$;

iii) if $f(x_0) = f(x_0+1)$ then $f(x) \geq f(x_0)$ $\forall x \in X$.

Proof *i)* Assume by contradiction that there exists $y \in X$, $y \geq x_0$, such that $f(y) < f(x_0) < f(x_0+1)$; hence $y > x_0 + 1$, so that (4) implies:

$$\frac{f(y) - f(x_0+1)}{y - x_0 - 1} \geq \frac{f(x_0+1) - f(x_0)}{1}$$

Since $f(y) - f(x_0+1) < 0$ and $y - x_0 - 1 > 0$, it follows $f(x_0+1) - f(x_0) < 0$ which is a contradiction.

The proofs for *ii)* and *iii)* are analogous. □

Notice that these results imply the global optimality of local optima.

Corollary 3.1 *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum, be a discrete convex function and let $x_0 \in X$. If the following condition holds:*

$$f(x_0) \leq f(x) \quad \forall x \in \{x_0 - 1, x_0 + 1\} \cap X$$

then, x_0 is an global minimum for f over X.

The following result related to the discrete convexity of function $\varphi(x_1)$ in problem $P^{(1)}$ can now be stated.

Theorem 3.5 *Consider problem $P^{(1)}$ and function $\varphi(x_1)$ as defined in (2). Then, function $\varphi(x_1)$ is discrete convex.*

Proof First notice that

$$\varphi(x_1) = k + nc_{x_1}x_1 + \sum_{i=1}^{n} \psi_i(x_1) \tag{7}$$

where

$$\psi_i(x_1) = c_z \hat{z}_i(x_1) + c_{w_1} w_i^{(1)}(x_1, \hat{z}(x_1)) \tag{8}$$

Taking into account of Theorems 3.1 and 3.2 and noticing that $k + nc_{x_1}x_1$ is trivially a discrete convex function, by means of (7) it is sufficient to prove that for all $i = 1, \ldots, n$ functions $\psi_i(x_1)$ defined in (8) are convex, and hence also discrete convex. Two exhaustive cases have to be considered.

(Case $c_z \leq c_w$) Let $i \in \{1, \ldots, n\}$, by means of simple calculations we get:

$$\psi_i(x_1) = \begin{cases} c_z(M_i^{(1)} - \beta_{x_1}x_1) & \text{if } \beta_{x_1}x_1 < M_i^{(1)} \\ 0 & \text{if } \beta_{x_1}x_1 \geq M_i^{(1)} \end{cases}$$

which results to be convex and continuous over $\Re$ being $c_z > 0$ and $\beta_{x_1} > 0$.
(Case $c_z > c_{w_1}$) Let $i \in \{1, \ldots, n\}$, by means of simple calculations we get:

$$\psi_i(x_1) = \begin{cases} c_z(\mu_i^{(1)} - \beta_{x_1}x_1) + c_w(M_i^{(1)} - \mu_i^{(1)}) & \text{if } \beta_{x_1}x_1 \leq \mu_i^{(1)} \\ c_w(M_i^{(1)} - \beta_{x_1}x_1) & \text{if } \mu_i^{(1)} < \beta_{x_1}x_1 < M_i^{(1)} \\ 0 & \text{if } \beta_{x_1}x_1 \geq M_i^{(1)} \end{cases}$$

which results to be convex and continuous in $\Re$ being $c_z\beta_{x_1} > c_w\beta_{x_1} > 0$. □

By means of Theorems 3.1 and 3.5, the minimizer of $\varphi(x_1)$ in the interval $[b_1, u_1]$ can be obtained in $O(u_1)$ steps just by computing the values $\varphi(x_1)$ for x_1 from b_1 to u_1 until a local minimum is reached. Actually, since the interval to be scanned is bounded, a faster logarithmic procedure based on a sort of bisection method is proposed in the following procedure "MinDiscrConv()".

Procedure **MinDiscrConv**(*inputs*: f, m, M ; *output*: ris)
  Let $a := m$ and $b := M$;
  while $a < b$ *do*
    let $c := \lfloor \frac{a+b}{2} \rfloor$;
    if $f(c+1) < f(c)$ *then* $a := c + 1$
      elseif $f(c+1) > f(c)$ *then* $b := c$
      else $a := c$ and $b := c$
    end if;
  end while;
  $ris := a$;
end proc.

The correctness of procedure "MinDiscrConv()" follows straightforward from Theorem 3.4. The logarithmic complexity yields noticing that in every iteration the current interval is divided into two equally long subintervals. It is worth also to point out that in every iteration of the while cycle two points are evaluated, that are c and $c + 1$.

3.2 Tighter upper bounds

As it has been described in Section 2, a key role in the solution procedures "Min$P^{(1)}$()", "Min$P^{(2)}$()" and "Min$P^{(3)}$()", is played by the values of the bounds $[b_1, u_1]$, $[l_2, u_2]$ and $[l_3, u_3]$, respectively. This happens since the procedures have to iteratively analyze all the values of variables x_1, x_2 and x_3 within those intervals. In this light, the performance of the solution methods can be improved by stating tighter bounds for x_1, x_2 and x_3. This can be done by using a global variable UB corresponding to the value of the incumbent solution; in facts, such a global variable allows to improve the upper bounds as described below:

$$\begin{aligned} u_1 &:= \min\left\{u_1; \left\lfloor \frac{UB-k}{nc_{x_1}} \right\rfloor\right\} \\ u_2 &:= \min\left\{u_2; \left\lfloor \frac{UB-k}{nc_{x_2}} \right\rfloor\right\} \\ u_3 &:= \min\left\{u_3; \left\lfloor \frac{UB-k}{nc_{x_3}} \right\rfloor\right\} \end{aligned}$$

These formulas are based on the fact that there is no need in procedure "Min$P^{(1)}$()" to continue the visit of x_1 in the case $f^{(1)}(x_1, z) \geq k + nc_{x_1}x_1 \geq UB$, and similarly in "Min$P^{(2)}$()" and "Min$P^{(3)}$()" whenever $f^{(2)}(x_1, x_2, z) \geq k + nc_{x_2}x_2 \geq UB$ and $f^{(3)}(x_1, x_2, x_3, z) \geq k + nc_{x_3}x_3 \geq UB$, respectively.

These updating assignment commands can be used in the optional steps of procedures "Min$P^{(1)}$()", "Min$P^{(2)}$()" and "Min$P^{(3)}$()", which have been already described in Section 2. Clearly, in the case the updated value of the upper bound gets smaller than the value of the lower bound, the procedure stops since the feasible region does not allow to find a better incumbent solution.

4 Computational results

In this paragraph the simulation results are presented. The procedures have been implemented in MatLab 7.4 rev.2007a and the test has been done with a MacOSX 10.4 computer with a dual core Core 2 Duo processor at 2.16 GHz. For each category 1000 instances of problems $P^{(3)}$ have been generated by using the "rand()" MatLab function. For the sake of concreteness, the cost parameters have been generated according to a hierarchical cost structure, that is $c_{x_3} \geq c_{x_2} \geq c_{x_1}$ and $c_{w_3} \geq c_{w_2} \geq c_{w_1}$. Every generated problem has been solved by using no improvements ("None"), only one improvement ("Conv" and "Bounds"), both improvements ("Conv & Bounds").

In the first computational test we fixed the variation intervals of $M_i^{(j)}$ and $\mu_i^{(j)}$ and we investigated the algorithm efficiency varying the number of working days n. Notice that this parameter implicitly determines the number of variables in the model: in facts, for each working day $i \in \{1, \dots, n\}$ a variable z_i is defined, together with the three variables x_1, x_2 and x_3.

We provide as a computational result the mean number of points evaluated in order to determine the minimum. The values obtained in each category are summarized in the following table.

n	None	Conv	Bounds	Conv&Bounds
20	14468 (100%)	9463 (65,4%)	1116 (7,7%)	692 (4,8%)
30	14970 (100%)	9708 (64,8%)	1058 (7,1%)	660 (4,4%)
50	15578 (100%)	10144 (65,1%)	953 (6,1%)	602 (3,9%)
100	16128 (100%)	10580 (65,6%)	804 (5,0%)	516 (3,2%)
150	16547 (100%)	10885 (65,7%)	747 (4,5%)	484 (2,9%)
200	17922 (100%)	11861 (66,2%)	477 (2,7%)	324 (1,8%)

Table 1: Algorithm efficiency with respect to n

In each category we fix as a 100% base the mean number of points evaluated when no improvement is used. In this light, the algorithm efficiency can be pointed out by observing that the combined use of the two proposed improvements significantly reduces the number of evaluated points.

In particular, the use of discrete convexity properties reduces the iterations to a 65% – 66% independently to the value of n. The improved upper bounds result to be as more efficient as bigger is the value of n. In any case, the use of both the two improvements provides the better performance.

In the second computational test the parameter n is fixed to 250 and we considered different ranges of intervals for $M_i^{(j)}$ and $\mu_i^{(j)}$. In this light, we aimed to test the performance of the algorithm when the wideness of the feasible region is given by $M_i^{(j)}$ and $\mu_i^{(j)}$. The results are presented in the following table.

range	None	Conv	Bounds	Conv&Bounds
small	16777 (100%)	11114 (66,24%)	658 (3,9%)	431 (2,5%)
wide	67781 (100%)	34256 (50,5%)	3069 (3,1%)	1123 (1,6%)

Table 2: Algorithm efficiency with respect to the range of $M_i^{(j)}$ and $\mu_i^{(j)}$

Clearly, the number of iterations increases with the wideness of the feasible region. Notice that in this case the use of discrete convexity properties results to be as more efficient as wider is the feasible region. Again, the better performance is obtained by using both the two improvements.

5 Conclusions

A class of hierarchical fleet mix problems, with various concrete applications, has been fully studied from both a theoretical and an algorithmic point of view. The improvements to the solution algorithm proposed in this paper resulted to be extremely efficient, so that problems with a wide feasible region can be solved in a reasonable time.

Further developments could be represented by the study of different ways to manage the stochastic demand of services.

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