On the maximal domains of pseudoconvexity of some classes of generalized fractional functions

Alberto Cambini *- Laura Martein †

Abstract

The aim of this paper is to characterize the maximal domains of pseudoconvexity of two classes of generalized fractional functions: the sum of a linear and a linear fractional function and the sum of two linear fractional functions. Firstly, by using Charnes-Cooper's variable transformation, the sum of two linear ratios is tranformed into the sum of a linear and a linear fractional function. Successively, the maximal domains of pseudoconvexity of this last class of functions are studied. Taking into account that Charnes-Cooper's transformation preserves pseudoconvexity, the obtained results will allow us to reach our aim.

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1 Introduction

The optimization of a sum of linear ratios arises in various areas, such as multi-stage stochasting shipping [1], layered manifacturing [29, 30], cluster analysis [33], multiobjective bond portfolio [24], and combinatorial optimization [32] (for several others economic

*Department of Statistics and Applied Mathematics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, Italy
†Department of Statistics and Applied Mathematics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, Italy

applications see [21, 35]). This kind of problems has attracted the interest of researchers for a number of years and different approaches for solve them have been proposed. For instance, in [20] an approach in the so-called image space is suggested, in [18, 19, 26, 28] branch-and-bound procedures are given, and in [5] suitable transformations are used which reduce the problem in another one simpler to be handle.

For the particular case of the sum of two linear ratios, some others algorithms have been proposed, also in the framework of bicriteria problems, where a compromise solution is sought (see for all [10, 11]).

The different suggested approaches point out that the problem of minimizing (maximizing) the sum of ratios is difficult to be solved since, also in the case of two linear ratios, it does not have any generalized convexity properties and, in particular, it may have several local optima not global. Since the local-global property is verified by the class of pseudoconvex functions, starting from the pioneer works of Martos [31] and Schaible [34] related to quadratic functions, particular attention has been devoted, recently, to the characterization of the maximal domains of pseudoconvexity of some classes of generalized fractional functions [7, 12, 14, 15].

In this paper we shall characterize the maximal domains of pseudoconvexity of the sum of two linear fractional functions. With this aim in mind, firstly the sum of two linear ratios is tranformed, by using Charnes-Cooper's variable transformation, into the sum of a linear and a linear fractional function. Successively, the maximal domains of pseudoconvexity of this last class of functions are studied. Taking into account that Charnes-Cooper's transformation preserves pseudoconvexity, the obtained results will allow us to reach our aim.

2 Statement of the problem

Consider the function

$$h(x) = \frac{m^T x + m_0}{p^T x + p_0} + \frac{q^T x + q_0}{b^T x + b_0}$$

defined on $H = \{x \in \Re^n : p^T x + p_0 > 0,\ b^T x + b_0 > 0\}$, where $m, p, q, b \in \Re^n,\ p, b \neq 0$ and $m_0, p_0, q_0, b_0 \in \Re$, $p_0, b_0 \neq 0$.

By applying Charnes-Cooper's transformation

$$y = \frac{x}{p^T x + p_0}, \tag{2.1}$$

whose inverse is

$$x = \frac{p_0 y}{1 - p^T y}, \tag{2.2}$$

function h is transformed into

$$\psi(y) = \frac{(p_0 m - m_0 p)^T y}{p_0} + \frac{(p_0 q - q_0 p)^T y + q_0}{(p_0 b - b_0 p)^T y + b_0} + \frac{m_0}{p_0}$$

while domain H is transformed into

$$H^* = \left\{ y \in \Re^n : \frac{1 - p^T y}{p_0} > 0,\ \frac{(p_0 b - b_0 p)^T y + b_0}{p_0} > 0 \right\}.$$

Since Charnes-Cooper's transformation preserves pseudoconvexity [9], the study of the pseudoconvexity of the sum of two linear fractional functions can be performed by means of the characterization of pseudoconvexity of the sum of a linear and a linear fractional function.

More precisely, we have the following theorem.

Theorem 2.1 *The function $h(x)$ is pseudoconvex on a convex set $S \subseteq H$ if and only if the function $\psi(y)$ is pseudoconvex on $S^* \subseteq H^*$, where S^* is the image under Charnes-Cooper's transformation (2.2) of the set S.*

As we have already mentioned, the main purpose of this paper is to find necessary and sufficient conditions for the pseudoconvexity of the sum of two linear fractional functions. In order to achieve this result, firstly, we shall characterize in the next section the maximal domains of pseudoconvexity of the sum of a linear and a linear fractional function.

3 Pseudoconvexity of the sum of a linear and a linear fractional function

Consider the function

$$f(x) = a^T x + \frac{c^T x + c_0}{d^T x + d_0}$$

defined on $D = \{x \in \Re^n : d^T x + d_0 > 0\}$, where $a, c, d \in \Re^n$, $d \neq 0$, $c_0, d_0 \in \Re$.

In general, f is not pseudoconvex since it may have local minimum points which are not global as is shown in the following example.

Example 3.1 *Consider the function* $f(x_1, x_2) = -x_1 + x_2 + \dfrac{-2x_1 - 7x_2 - 6}{x_1 + x_2 + 1}$ *on the convex set* $S = \{(x_1, x_2) \in \Re^2 : \ x_1 \geq 0, \ 0 \leq x_2 \leq 4, \ x_1 - x_2 \leq 4\}$. *It is easy to verify that* $(0, 0)$ *is a local minimum point, while the global minimum is attained at* $(8, 4)$.

In this section we shall present a general approach which allows us to characterize the pseudoconvexity of f on an arbitrary convex set $S \subseteq D$. The obtained results will also allow us to study in the next Section the pseudoconvexity of the sum of two linear fractional functions.

In order to have a self-contained paper, we recall that a differentiable function φ defined on a convex set $S \subseteq \Re^n$, is called pseudoconvex if

$$x_1, \ x_2 \in S, \ \varphi(x_1) > \varphi(x_2) \Rightarrow \nabla\varphi(x_1)^T (x_2 - x_1) < 0. \tag{3.3}$$

A useful result for establishing the pseudoconvexity of a function which will be used through the paper is the following:

Let φ be a twice continuously differentiable function defined on an open convex set $S \subseteq \Re^n$. Then, φ is pseudoconvex on S if and only if the following conditions hold:

$$x \in S, \ u \in \Re^n, \ u^T \nabla\varphi(x_0) = 0 \ \Rightarrow u^T \nabla^2 \varphi(x_0) u \geq 0 \tag{3.4}$$

$$x_0 \in S, \ \nabla\varphi(x_0) = 0 \Rightarrow \ \ x_0 \text{ is a local minimum for } \varphi. \tag{3.5}$$

The gradient and the Hessian matrix of f are given, respectively, by

$$\nabla f(x) = a + \frac{1}{d^T x + d_0} \left(c - \frac{c^T x + c_0}{d^T x + d_0} d \right) \tag{3.6}$$

$$\nabla^2 f(x) = \frac{1}{(d^T x + d_0)^2}\left[-(cd^T + dc^T) + 2\frac{c^T x + c_0}{d^T x + d_0} dd^T\right] \tag{3.7}$$

By assuming the pseudoconvexity of f, the following theorem points out that $\nabla f(x)$ vanishes only for the particular case stated in the following theorem.

Theorem 3.1 *Assume that f is pseudoconvex on D. Then, f has a critical point if and only if (3.8) holds.*

$$\exists\ \alpha > 0,\ \exists\ \gamma \in \Re : a = \alpha d,\ \ c = \gamma d,\ \ c_0 - \gamma d_0 > 0. \tag{3.8}$$

Proof Assume that (3.8) holds.
Then, we have $f(x) = \alpha d^T x + \frac{c_0 - \gamma d_0}{d^T x + d_0} + \gamma$, so that $\nabla f(x) = \frac{\alpha (d^T x + d_0)^2 - (c_0 - \gamma d_0)}{(d^T x + d_0)^2} d$.
It follows that every point satisfying the equation $d^T x + d_0 = \sqrt{\frac{c_0 - \gamma d_0}{\alpha}}$ is a critical point.
Assume now that $x_0 \in D$ is a critical point.
From (3.4), we necessarily have, $v^T \nabla^2 f(x_0) v \geq 0$ for all $v \in \Re^n$, i.e.,

$$\frac{2}{(d^T x + d_0)^2}\left(-c^T v d^T v + \frac{c^T x_0 + c_0}{d^T x_0 + d_0}(d^T v)^2\right) \geq 0,\ \ \forall v \in \Re^n, \tag{3.9}$$

and, consequently, c and d are linearly dependent so that there exists $\gamma \in \Re$ such that $c = \gamma d$ and, in turn, $a = \frac{c_0 - \gamma d_0}{(d^T x_0 + d_0)^2} d$.
By replacing c in (3.9), we obtain $v^T \nabla^2 f(x_0) v = \frac{2(c_0 - \gamma d_0)}{(d^T x_0 + d_0)^3}(d^T v)^2 \geq 0, \forall v \in \Re^n$. This last inequality implies $c_0 - \gamma d_0 > 0$. By setting $\alpha = \frac{c_0 - \gamma d_0}{(d^T x_0 + d_0)^2}$, (3.8) is achieved. □

The following theorem shows that the linear independence of a, c, d implies that f is not pseudoconvex on S, whatever the open convex set S be.

Theorem 3.2 *Let f be pseudoconvex on an open convex set $S \subseteq D$. Then, the vectors a, c, d are linearly dependent.*

Proof The thesis clearly holds if $n = 2$. Let $n \geq 3$ and assume that $rank[a, c, d] = 3$.
We have $\nabla f(x) \neq 0,\ \forall x \in S$, and $rank[\nabla f(x), a, d] = 3,\ \forall x \in S$.

Let $A = \begin{bmatrix} \nabla^T f(x) \\ a^T \\ d^T \end{bmatrix}$; for every fixed $x \in S$ the linear map $A : \Re^n \to \Re^3$ is surjective so that there exist w_1, w_2 such that $\nabla f(x)^T w_1 = 0$, $a^T w_1 = 0$, $d^T w_1 < 0$ and $\nabla f(x)^T w_2 = 0$, $a^T w_2 > 0$, $d^T w_2 = 0$.

By setting $w = w_1 + w_2$ we have $\nabla f(x)^T w = 0$, $a^T w > 0$, $d^T w < 0$.

We shall prove that $\nabla f(x)^T w = 0$ implies $w^T \nabla^2 f(x) w < 0$, $\forall x \in S$.

The equality $\nabla f(x)^T w = 0$ implies

$$a^T w + \frac{c^T w}{d^T x + d_0} - \frac{c^T x + c_0}{(d^T x + d_0)^2} d^T w = 0. \tag{3.10}$$

If $x \in S$ is such that $c^T x + c_0 = 0$, from (3.10) $c^T w = -a^T w(d^T x + d_0) < 0$, so that $w^T \nabla^2 f(x) w < 0$.

Consider now the case $c^T x + c_0 \neq 0$.

If $c^T w = 0$, from (3.10) we have $\frac{c^T x + c_0}{d^T x + d_0} d^T w = a^T w(d^T x + d_0)$ so that

$$w^T \nabla^2 f(x) w = \frac{2}{(d^T x + d_0)^2} a^T w \; d^T w (d^T x + d_0) < 0.$$

If $c^T w \neq 0$, from (3.10) we have $c^T w = \frac{c^T x + c_0}{d^T x + d_0} d^T w - a^T w(d^T x + d_0)$ so that

$$w^T \nabla^2 f(x) w = \frac{2}{d^T x + d_0} a^T w \; d^T w < 0.$$

To sum up, condition $\nabla f(x)^T w = 0$ implies that $w^T \nabla^2 f(x) w < 0$ for all $x \in S$ and, consequently, f is not pseudoconvex on S (see (3.4)) and this contradicts the assumption. The linear dependence of a, c, d follows. □

The following theorem characterizes the maximal open domains of pseudoconvexity of function f.

Theorem 3.3 *Consider the function f. The following conditions hold:*

i) if $a = \alpha d$, $\alpha \geq 0$, then f is pseudoconvex on D;

ii) if $c = \gamma d$, $c_0 - \gamma d_0 \geq 0$, then f is pseudoconvex on D;

iii) if $a = \alpha d$, $\alpha < 0$, and $c = \gamma d$, $c_0 - \gamma d_0 < 0$, then f is pseudoconvex on every convex set S such that:

$$S \subseteq \{x \in \Re^n : d^T x + d_0 > d_0^*\}$$

or

$$S \subseteq \{x \in \Re^n : 0 < d^T x + d_0 < d_0^*\}$$

where $d_0^* = \sqrt{\dfrac{c_0 - \gamma d_0}{\alpha}}$;

iv) if $c = \beta a + \gamma d$, $\beta > 0$ *and* $rank[a, d] = 2$, *then* f *is pseudoconvex on every convex set* S *such that:*

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ d^T x + d_0 > 0\};$$

v) if $c = \beta a + \gamma d$, $\beta < 0$ *and* $rank[a, d] = 2$, *then* f *is pseudoconvex on every convex set* S *such that:*

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ d^T x + d_0 + \beta > 0\}$$

or

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 < 0,\ 0 < d^T x + d_0 < -\beta\}.$$

In any other case f *is not pseudoconvex on* $S \subseteq D$ *whatever the open convex set* S *be.*

Proof According to Theorem 3.2, we must analyze the exhaustive cases $rank[a, c, d] = 1$, $rank[a, c, d] = 2$.

- $rank[a, c, d] = 1$.

Let $a = \alpha d$, $c = \gamma d$. We have

$$\nabla f(x) = \frac{1}{(d^T x + d_0)^2}[\alpha(d^T x + d_0)^2 - (c_0 - \gamma d_0)]d$$

$$\nabla^2 f(x) = \frac{2}{(d^T x + d_0)^3}(c_0 - \gamma d_0) d d^T.$$

If $c_0 - \gamma d_0 \geq 0$, then $\nabla^2 f(x)$ is positive semidefinite on D so that f is convex (in particular, pseudoconvex) on D.

Consider now the case $c_0 - \gamma d_0 < 0$.

If $\alpha \geq 0$, then $\nabla f(x) \neq 0$ for all $x \in D$ so that $\nabla f(x)^T v = 0$ implies that $v^T \nabla^2 f(x) v = 0$ and, consequently, f is pseudoconvex on D.

If $\alpha < 0$, then $\nabla f(x) = 0$, $\forall x \in D^* = \{x \in \Re^n : d^T x + d_0 = d_0^*\}$, with $d_0^* = \sqrt{\dfrac{c_0 - \gamma d_0}{\alpha}}$; choosing v such that $d^T v \neq 0$, we have $v^T \nabla^2 f(x) v < 0$ for every $x \in D^*$, so that f is not pseudoconvex on every open convex set S such that $S \cap D^* \neq \emptyset$, while it is pseudoconvex on every convex set $S \subset D$ such that $S \cap D^* = \emptyset$. Consequently, $iii)$ holds.

- $rank[a, c, d] = 2$.

The following two exhaustive cases occur:

a) $a = \alpha d$ and $rank[c, d] = 2$;

b) $c = \beta a + \gamma d$ and $rank[a, d] = 2$.

a) We have

$$\nabla f(x) = \frac{1}{d^T x + d_0}\left[c + \left(\alpha(d^T x + d_0) - \frac{c^T x + c_0}{d^T x + d_0}\right) d\right]$$

The linear independence of c and d implies that $\nabla f(x) \neq 0, \ \forall x \in D$. For every $v \in \Re^n$ such that $\nabla f(x)^T v = 0$, we have $v^T \nabla^2 f(x) v = \frac{2}{d^T x + d_0}\alpha(d^T v)^2$. Consequently, if $\alpha \geq 0$, then f is pseudoconvex on D, while, in the case $\alpha < 0$, f is not pseudoconvex on every open convex set $S \subseteq D$ since we can choose v such that $d^T v \neq 0$.

Note that if $a = \alpha d$, $\alpha \geq 0$, then f is pseudoconvex on D either when $rank[a, c, d] = 1$ or when $rank[a, c, d] = 2$, and thus $i)$ holds.

b) We have

$$\nabla f(x) = \frac{1}{d^T x + d_0}\left[(d^T x + d_0 + \beta)\, a - \frac{\beta a^T x + c_0 - \gamma d_0}{d^T x + d_0}\, d\right]$$

$$\nabla^2 f(x) = \frac{1}{(d^T x + d_0)^2}\left[-\beta(ad^T + da^T) + 2\frac{\beta a^T x + c_0 - \gamma d_0}{d^T x + d_0}\, dd^T\right].$$

It follows that $\nabla f(x) = 0$ if and only if $\beta < 0$ and $x \in \Gamma$ where

$$\Gamma = \{x \in \Re^n : d^T x + d_0 + \beta = 0, \ \beta a^T x + c_0 - \gamma d_0 = 0\}.$$

Since $rank[a, d] = 2$, for every $x \in \Gamma$ the Hessian matrix $\nabla^2 f(x)$ is indefinite so that f is not pseudoconvex on every open convex set S such that $S \cap \Gamma \neq \emptyset$.

If $\nabla f(x) \neq 0$ and $d^T x + d_0 + \beta = 0$, then $\nabla f(x)^T v = 0$ implies that $d^T v = 0$, so that $v^T \nabla^2 f(x) v = 0$. If $\nabla f(x) \neq 0$ and $d^T x + d_0 + \beta \neq 0$, then $\nabla f(x)^T v = 0$ implies that $a^T v = \frac{\beta a^T x + c_0 - \gamma d_0}{(d^T x + d_0 + \beta)(d^T x + d_0)} d^T v$, so that

$$v^T \nabla^2 f(x) v = \frac{2}{(d^T x + d_0)^2}\frac{\beta a^T x + c_0 - \gamma d_0}{d^T x + d_0 + \beta}(d^T v)^2.$$

Consequently, if $\beta > 0$, then $\nabla f(x) \neq 0$ for all $x \in D$ and f is pseudoconvex on every convex set S such that $S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0\} \cap D$ and $iv)$ holds.

If $\beta < 0$, then f is pseudoconvex on S if

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ d^T x + d_0 + \beta > 0\}$$

or

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 < 0,\ d^T x + d_0 + \beta < 0\} \cap D$$

and $v)$ holds.
If $\beta = 0$ and $c_0 - \gamma d_0 \geq 0$, then f is convex on D. Consequently, if $c = \gamma d$ and $c_0 - \gamma d_0 \geq 0$, f is convex (in particular, pseudoconvex) on D either when $rank[a, c, d] = 1$ or when $rank[a, c, d] = 2$ so that $ii)$ holds.
The proof is complete. □

As a particular case of Theorem 3.3, we obtain the following result established in [7].

Corollary 3.1 *The function f is pseudoconvex on D if and only if it assumes one of the following forms:*
i) $f(x) = \alpha d^T x + \dfrac{c^T x + c_0}{d^T x + d_0}$, $\alpha \geq 0$;
ii) $f(x) = a^T x + \dfrac{c_0 - \gamma d_0}{d^T x + d_0} + \gamma$, $c_0 - \gamma d_0 \geq 0$.

Remark 3.1 *Note that in case ii) of the previous Corollary the function f is convex on D.*

The following theorem states the results given in Theorem 3.3 in terms of pseudoconcavity.

Theorem 3.4 *Consider the function f. The following conditions hold.*
i) If $a = \alpha d$, $\alpha \leq 0$, then f is pseudoconcave on D.
ii) If $c = \gamma d$, $c_0 - \gamma d_0 \leq 0$, then f is pseudoconcave on D.
iii) If $a = \alpha d$, $\alpha > 0$, and $c = \gamma d$, $c_0 - \gamma d_0 > 0$, then f is pseudoconcave on every convex set S such that:

$$S \subseteq \{x \in \Re^n : d^T x + d_0 > d_0^*\}$$

or

$$S \subseteq \{x \in \Re^n : 0 < d^T x + d_0 < d_0^*\}$$

where $d_0^* = \sqrt{\dfrac{c_0 - \gamma d_0}{\alpha}}$.

iv) If $c = \beta a + \gamma d$, $\beta > 0$ *and* $rank[a, d] = 2$, *then* f *is pseudoconcave on every convex set* S *such that:*

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 < 0,\ d^T x + d_0 > 0\}.$$

v) If $c = \beta a + \gamma d$, $\beta < 0$ *and* $rank[a, d] = 2$, *then* f *is pseudoconcave on every convex set* S *such that:*

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 < 0,\ d^T x + d_0 + \beta > 0\}$$

or

$$S \subseteq \{x \in \Re^n : \beta a^T x + c_0 - \gamma d_0 > 0,\ 0 < d^T x + d_0 < -\beta\}.$$

In any other case f *is not pseudoconcave on* $S \subseteq D$, *whatever the open convex set* S *be.*

By combining Theorem 3.3 and Theorem 3.4, the characterization of the pseudolinearity of f is achieved.

Theorem 3.5 *Consider the function* f*.The following conditions hold:*
i) if $a = 0$, *then* f *is pseudolinear on* D;
ii) if $c = \gamma d$, *with* $c_0 - \gamma d_0 = 0$, *then* f *is pseudolinear on* D;
iii) if $a = \alpha d$, $c = \gamma d$, *with* $\alpha(c_0 - \gamma d_0) < 0$, *then* f *is pseudolinear on* D;
iv) if $a = \alpha d$, $c = \gamma d$, *with* $\alpha(c_0 - \gamma d_0) > 0$, *then* f *is pseudolinear on every convex set* $S \subseteq D$ *such that*

$$S \subseteq \{x \in \Re^n : d^T x + d_0 > d_0^*\}$$

or

$$S \subseteq \{x \in \Re^n : 0 < d^T x + d_0 < d_0^*\}$$

where $d_0^* = \sqrt{\dfrac{c_0 - \gamma d_0}{\alpha}}$.

In any other case f *is not pseudolinear on* $S \subseteq D$ *whatever the open convex set* S *be.*

We have studied the pseudoconvexity of f by assuming $d^T x + d_0 > 0$; the obtained results can be easily adapted to the case $d^T x + d_0 < 0$, re-writing the function f as follows:

$f(x) = a^T x + \dfrac{-c^T x - c_0}{-d^T x - d_0}$.

In the following example we characterize the maximal open convex domains of the pseudoconvexity of function f when the denominator is either positive or negative.

Example 3.2 *Consider the function*

$$f(x_1, x_2) = 2x_1 + 32x_2 + \frac{-2x_1 + 3x_2 + 2}{3x_1 + 13x_2 + 1}$$

The function is not pseudoconvex on $D = \{(x_1, x_2) \in \Re^2 : 3x_1 + 13x_2 + 1 > 0\}$ *since* $f(10, -2) < f(10.33, -2.03)$, $\nabla f(10.33, -2.03)^T(-0.33, 0.03) > 0$*; nevertheless, it is easy to verify that iv) of Theorem 3.3 holds with* $\beta = \frac{1}{2}$, $\gamma = -1$, *so that* f *is pseudoconvex on every convex set* $S \subseteq D_1^+ = \{(x_1, x_2) \in \Re^2 : x_1 + 16x_2 + 3 > 0,\ 3x_1 + 13x_2 + 1 > 0\}$.

In order to study the pseudoconvexity on the half-plane $\{(x_1, x_2) \in \Re^2 : 3x_1 + 13x_2 + 1 < 0\}$, *we can re-write the function as follows:* $f(x_1, x_2) = 2x_1 + 32x_2 + \dfrac{2x_1 - 3x_2 - 2}{-3x_1 - 13x_2 - 1}$.

By re-applying again Theorem 3.3, case v) occurs so that f *is pseudoconvex on every convex set contained in* $D_1^- = \{(x_1, x_2) \in \Re^2 : x_1 + 16x_2 + 3 < 0,\ 3x_1 + 13x_2 + \frac{3}{2} < 0\}$ *or in* $D_2^- = \{(x_1, x_2) \in \Re^2 : x_1 + 16x_2 + 3 > 0,\ -\frac{1}{2} < 3x_1 + 13x_2 + 1 < 0\}$.

The obtained maximal domains of pseudoconvexity are drawn in Figure 1.

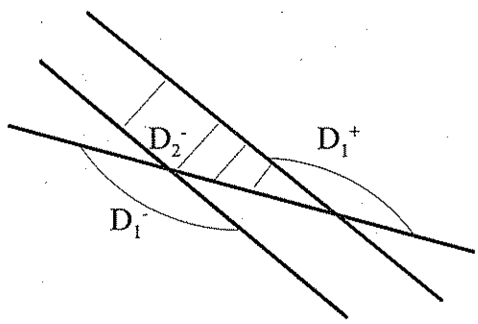

Figure 1: Maximal domains of pseudoconvexity

4 Sum of two linear fractional functions

In this section we shall characterize the maximal domains of pseudoconvexity of the sum of two linear fractional functions, by means of the results stated in the previous section.

Consider again

$$h(x) = \frac{m^T x + m_0}{p^T x + p_0} + \frac{q^T x + q_0}{b^T x + b_0}$$

defined on $H = \{x \in \Re^n : p^T x + p_0 > 0,\ b^T x + b_0 > 0\}$.

The following theorem characterizes the maximal open domains of pseudoconvexity of function h.

Theorem 4.1 *Consider the function h on H. The following conditions hold:*

i) If there exists $\alpha \geq 0$ such that

$$p_0 m - m_0 p = \alpha(p_0 b - b_0 p), \tag{4.11}$$

then h is pseudoconvex on H;

ii) If there exists $\gamma \in \Re$ such that

$$p_0 q - q_0 p = \gamma(p_0 b - b_0 p),\ \frac{q_0 - \gamma b_0}{p_0} \geq 0, \tag{4.12}$$

then h is pseudoconvex on H;

iii) If there exist $\alpha < 0$, $\gamma \in \Re$ such that $p_0 m - m_0 p = \alpha(p_0 b - b_0 p)$, $p_0 q - q_0 p = \gamma(p_0 b - b_0 p)$, $\frac{q_0 - \gamma b_0}{p_0} < 0$, then h is pseudoconvex on every convex set S such that:

$$S \subseteq \{x \in \Re^n : p^T x + p_0 > 0,\ (b - h_0^* p)^T x + b_0 - h_0^* p_0 > 0\}$$

or

$$S \subseteq \{x \in \Re^n :\ (b - h_0^* p)^T x + b_0 - h_0^* p_0 < 0\} \cap H,$$

where $h_0^* = \sqrt{\frac{q_0 - \gamma b_0}{\alpha p_0}}$;

iv) If $rank[p_0 m - m_0 p, p_0 b - b_0 p] = 2$, and there exist $\beta > 0$, $\delta \in \Re$ such that

$$p_0 q - q_0 p = \beta(p_0 m - m_0 p) + \delta(p_0 b - b_0 p), \tag{4.13}$$

then h is pseudoconvex on every convex set S such that

$$S \subseteq \{x \in \Re^n : v^T x + v_0 > 0\} \cap H$$

where $v = \frac{\beta(p_0 m - m_0 p) + (q_0 - \delta b_0)p}{p_0}$, $v_0 = q_0 - \delta b_0$.

Furthermore, h is pseudoconvex on H if and only if there exist $\beta > 0$, $\delta \in \Re$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ such that (4.19), (4.20),and (4.21):

$$\beta(p_0 m - m_0 p) = \lambda_1(-p) + \lambda_2(p_0 b - b_0 p) \tag{4.14}$$

$$\frac{q_0 - \delta b_0 - (\lambda_1 + \lambda_2 b_0)}{p_0} \geq 0. \tag{4.15}$$

v) If $rank[p_0 m - m_0 p, p_0 b - b_0 p] = 2$, *and there exist* $\beta < 0$, $\delta \in \Re$, *such that*

$$p_0 q - q_0 p = \beta(p_0 m - m_0 p) + \delta(p_0 b - b_0 p), \tag{4.16}$$

then h *is pseudoconvex on every convex set* S *such that:*

$$S \subseteq \{x \in \Re^n : p^T x + p_0 > 0,\ v^T x + v_0 > 0,\ v^{*T} x + v_0^* > 0\}$$

or

$$S \subseteq \{x \in \Re^n :\ v^T x + v_0 < 0,\ v^{*T} x + v_0^* < 0\} \cap H,$$

where $v^* = b + \beta p$, $v_0^* = b_0 + \beta p_0$.
In any other case h *is not pseudoconvex on* $S \subseteq H$ *whatever the open convex set* S *be.*

Proof From Theorem 2.1, the maximal domains of pseudoconvexity of $h(x)$ are the image under Charnes-Cooper's transformation $y = \dfrac{x}{p^T x + p_0}$ of the intersection of H^* with the maximal domains of pseudoconvexity of ψ which can be expressed as follows:

$$\psi(y) = \frac{(p_0 m - m_0 p)^T y}{p_0} + \frac{\left(\frac{p_0 q - q_0 p}{p_0}\right)^T y + \frac{q_0}{p_0}}{\left(\frac{p_0 b - b_0 p}{p_0}\right)^T y + \frac{b_0}{p_0}} + \frac{m_0}{p_0}.$$

i) and *ii)* follow from i) and ii) of Theorem 3.3, respectively, since H^* is contained in the maximal domain of pseudoconvexity of ψ which is $\{y \in \Re^n : \dfrac{(p_0 b - b_0 p)^T y + b_0}{p_0} > 0\}$.
iii) The assumptions imply iii) of Theorem 3.3, so that the maximal domains of pseudoconvexity of ψ are

$$D_1 = \{y \in \Re^n : \frac{(p_0 b - b_0 p)^T y + b_0}{p_0} > h_0^*\}$$

$$D_2 = \{y \in \Re^n :\ 0 < \frac{(p_0 b - b_0 p)^T y + b_0}{p_0} < h_0^*\}$$

where $h_0^* = \sqrt{\dfrac{q_0 - \gamma b_0}{\alpha p_0}}$.
By considering the image under Charnes-Cooper's transformation of $D_1 \cap H^*$ and of $D_2 \cap H^*$, we get the thesis.

iv) From iv) of Theorem 3.3, the maximal domain of pseudoconvexity of $h(x)$ is the image under Charnes-Cooper's transformation of

$$\left\{y \in \Re^n : \frac{\beta(p_0 m - m_0 p)^T y + q_0 - \delta b_0}{p_0} > 0\right\} \cap H^*,$$

i.e.,

$$\left\{x \in \Re^n : \frac{\beta p_0 m - (\beta m_0 - q_0 + \delta b_0)p)^T x}{p_0} + q_0 - \delta b_0 > 0\right\} \cap H.$$

Now we prove that $H^* \subset D = \left\{y \in \Re^n : \frac{\beta(p_0 m - m_0 p)^T y + q_0 - \delta b_0}{p_0} > 0\right\}$ if and only if (4.20), (4.21) hold. In fact, the half-space $\left\{y \in \Re^n : \frac{\beta(p_0 m - m_0 p)^T y}{p_0} \geq 0\right\}$ supports the cone $C = \left\{y \in \Re^n : -\frac{p^T y}{p_0} \geq 0, \ \frac{(p_0 b - b_0 p)^T y}{p_0} \geq 0\right\}$ at the origin if and only if (4.20), together with the non-negativity of $\lambda_1, \ \lambda_2$, holds. Furthermore, let $\bar{y}$ such that $1 - p^T \bar{y} = 0, \ \beta(p_0 b - b_0 p)^T \bar{y} + b_0 = 0$; then i $\bar{y} + C \subset D$ if and only if (4.21) holds.

v) By applying v) of Theorem 3.3 and Charnes-Cooper's transformation, we get the thesis.

The proof is complete. □

As a direct consequence of the previous theorem, we have the following corollary.

Corollary 4.1 *Consider the function h. If* $rank[m, p, q, b] = 4$, *then h is not pseudoconvex on* $S \subseteq H$ *whatever the open convex set S be.*

Remark 4.1 *Referring to iv) of Theorem 4.1, we have* $\{x \in \Re^n : v^T x + v_0 > 0\} \cap H = \emptyset$ *if and only if (4.20) holds with* $\lambda_1 \leq 0, \ \lambda_2 \leq 0$.

Remark 4.2 *It is important to note that regarding characterization of pseudoconvexity given in Theorem 4.1, there is no difference between using* $y = \frac{x}{p^T x + p_0}$ *and* $y = \frac{x}{b^T x + b_0}$.

Remark 4.3 *Let us note that when* $rank[p, b] = 2$, *the domain H of the function* $h(x)$ *is non-empty. When* $rank[p, b] = 1$, *we have* $H = \emptyset$ *if and only if* $k < 0$ *and* $p_0 k - b_0 > 0$.

In terms of pseudoconcavity we have the following results.

Theorem 4.2 *Consider the function h on H. The following conditions hold:*

i) If there exists $\alpha \leq 0$ such that

$$p_0 m - m_0 p = \alpha(p_0 b - b_0 p), \tag{4.17}$$

then h is pseudoconcave on H;

ii) If there exists $\gamma \in \Re$ such that

$$p_0 q - q_0 p = \gamma(p_0 b - b_0 p), \ \frac{q_0 - \gamma b_0}{p_0} \leq 0, \tag{4.18}$$

then h is pseudoconcave on H;

iii) If there exist $\alpha > 0$, $\gamma \in \Re$ such that $p_0 m - m_0 p = \alpha(p_0 b - b_0 p)$, $p_0 q - q_0 p = \gamma(p_0 b - b_0 p)$, $\frac{q_0 - \gamma b_0}{p_0} > 0$, then h is pseudoconcave on every convex set S such that:

$$S \subseteq \{x \in \Re^n : p^T x + p_0 > 0, \ (b - h_0^* p)^T x + b_0 - h_0^* p_0 > 0\}$$

or

$$S \subseteq \{x \in \Re^n : \ (b - h_0^* p)^T x + b_0 - h_0^* p_0 < 0\} \cap H,$$

where $h_0^* = \sqrt{\frac{q_0 - \gamma b_0}{\alpha p_0}}$;

iv) If $rank[p_0 m - m_0 p, p_0 b - b_0 p] = 2$, and there exist $\beta > 0$, $\delta \in \Re$ such that

$$p_0 q - q_0 p = \beta(p_0 m - m_0 p) + \delta(p_0 b - b_0 p), \tag{4.19}$$

then h is pseudoconcave on every convex set S such that

$$S \subseteq \{x \in \Re^n : v^T x + v_0 < 0\} \cap H$$

where $v = \frac{\beta(p_0 m - m_0 p) + (q_0 - \delta b_0)p}{p_0}$, $v_0 = q_0 - \delta b_0$.

Furthermore, h is pseudoconcave on H if and only if there exist $\beta > 0$, $\delta \in \Re$, $\lambda_1 \leq 0$, $\lambda_2 \leq 0$ such that (4.19), (4.20),and (4.21):

$$\beta(p_0 m - m_0 p) = \lambda_1(-p) + \lambda_2(p_0 b - b_0 p) \tag{4.20}$$

$$\frac{q_0 - \delta b_0 - (\lambda_1 + \lambda_2 b_0)}{p_0} \leq 0. \tag{4.21}$$

v) If $rank[p_0m - m_0p, p_0b - b_0p] = 2$, *and there exist* $\beta < 0,\ \delta \in \Re$, *such that*

$$p_0q - q_0p = \beta(p_0m - m_0p) + \delta(p_0b - b_0p), \tag{4.22}$$

then h is pseudoconcave on every convex set S such that:

$$S \subseteq \{x \in \Re^n : p^Tx + p_0 > 0,\ v^Tx + v_0 < 0,\ v^{*T}x + v_0^* > 0\}$$

or

$$S \subseteq \{x \in \Re^n :\ v^Tx + v_0 > 0,\ (v^*)^Tx + v_0^* < 0\} \cap H,$$

where $v^* = b + \beta p$, $v_0^* = b_0 + \beta p_0$.
In any other case h is not pseudoconcave on $S \subseteq H$ *whatever the open convex set S be.*

By combining the characterization of pseudoconvexity and pseudoconcavity of the function of h, stated in Theorem 4.1 and in Theorem 4.2, respectively, the following result on pseudolinearity is obtained.

Theorem 4.3 *Consider the function h on H. The following conditions hold:*
i) If $m = \frac{m_0}{p_0}p$, *then h is pseudolinear on H;*
ii) If $q = \frac{q_0}{b_0}b$, *then h is pseudolinear on H;*
iii) If there exist $\alpha,\ \gamma \in \Re$, *such that*

$$p_0m - m_0p = \alpha(p_0b - b_0p),\ p_0q - q_0p = \gamma(p_0b - b_0p),\ \alpha\frac{q_0 - \gamma b_0}{p_0} < 0,$$

then h is pseudolinear on H;
iv) If there exist $\alpha,\ \gamma \in \Re$ *such that*

$$p_0m - m_0p = \alpha(p_0b - b_0p),\ p_0q - q_0p = \gamma(p_0b - b_0p),\ \alpha\frac{q_0 - \gamma b_0}{p_0} > 0,$$

then h is pseudolinear on every convex set S such that:

$$S \subseteq \{x \in \Re^n : p^Tx + p_0 > 0,\ (b - h_0^*p)^Tx + b_0 - h_0^*p_0 > 0\}$$

or

$$S \subseteq \{x \in \Re^n :\ (b - h_0^*p)^Tx + b_0 - h_0^*p_0 < 0\} \cap H,$$

where $h_0^* = \sqrt{\dfrac{q_0 - \gamma b_0}{\alpha p_0}}$.
In any other case h is not pseudolinear on $S \subseteq H$ *whatever the open convex set S be.*

Remark 4.4 *Consider case iv) of Theorem 4.3. If $\alpha > 0$ ($\alpha < 0$), from i) of Theorem 4.1 (Theorem 4.2), h is pseudoconvex (pseudoconcave) on H and, consequently, is not pseudoconcave (pseudoconvex) on H. Nevertheless, h is pseudoconcave (pseudoconvex) on $\{x \in \Re^n : (b - h_0^* p)^T x + b_0 - h_0^* p_0 > 0\} \cap H$, and on $\{x \in \Re^n : (b - h_0^* p)^T x + b_0 - h_0^* p_0 < 0\} \cap H$. This means that the points of the set $\{x \in \Re^n : (b - h_0^* p)^T x + b_0 - h_0^* p_0 = 0\} \cap H$ must have particular properties as it is shown in the following theorem.*

Theorem 4.4 *Consider case iv) of Theorem 4.3. If $\alpha > 0$ ($\alpha < 0$), then every point $\bar{x}$ of the set $\{x \in \Re^n : (b - h_0^* p)^T x + b_0 - h_0^* p_0 = 0\} \cap H$ is a global minimum (maximum) point for h on H.*

Proof Taking into account that $p_0 m - m_0 p = \alpha(p_0 b - b_0 p)$, $p_0 q - q_0 p = \gamma(p_0 b - b_0 p)$, the function h can be expressed as follows:

$$h(x) = \frac{m_0}{p_0} - \alpha \frac{b_0}{p_0} + \gamma + \frac{\alpha p_0 (b^T x + b_0)^2 + (q_0 - \gamma b_0)(p^T x + p_0)^2}{p_0 (b^T x + b_0)(p^T x + p_0)}.$$

Let $\bar{x}$ be such that $(b - h_0^* p)^T \bar{x} + b_0 - h_0^* p_0 = 0$ or, equivalently, $b^T \bar{x} + b_0 = h_0^* (p^T \bar{x} + p_0)$. We have

$$h(\bar{x}) = \frac{m_0}{p_0} - \alpha \frac{b_0}{p_0} + \gamma + 2\alpha h_0^*$$

i.e., the restriction of h on $\{x \in \Re^n : (b - h_0^* p)^T x + b_0 - h_0^* p_0 = 0\} \cap H$ is constant. Furthermore,

$$h(x) - h(\bar{x}) = \alpha \frac{(b^T x + b_0)^2 + (h_0^*)^2 (p^T x + p_0)^2}{(b^T x + b_0)(p^T x + p_0)} - 2\alpha h_0^* = \alpha \frac{[(b^T x + b_0) - h_0^* (p^T x + p_0)]^2}{(b^T x + b_0)(p^T x + p_0)}.$$

It follows that $\bar{x}$ is a global minimum (maximum) point of h if $\alpha > 0$ ($\alpha < 0$). □

5 Examples

In this section we shall give some examples in order to point out the results given in Section 4. Furthermore, we shall see how the obtained results may be utilized for costructing a pseudoconvex function which is the sum of two linear ratios starting from the sum between a linear and a linear fractional function which is pseudoconvex according to Theorem 3.3.

Example 5.1 *Consider the function*

$$h(x_1, x_2) = \frac{x_1 - x_2}{x_1 + 2x_2 + 1} + \frac{3x_1 + 3x_2 + 2}{2x_1 + x_2 + 1}$$

on $H = \{(x_1, x_2) \in \Re^2 : x_1 + 2x_2 + 1 > 0,\ 2x_1 + x_2 + 1 > 0\}$.
We have $p = (1,2)^T$, $b = (2,1)^T$, $m = (1,-1)^T$, $q = (3,3)^T$, $p_0 = b_0 = 1$, $m_0 = 0$, $q_0 = 2$; *consequentely,* $p_0 m - m_0 p = p_0 q - q_0 p = p_0 b - b_0 p = (1,-1)^T$.
Since i) of Theorem 4.1 holds with $\alpha = 1$, *the function is pseudoconvex on* H.
Furthermore, iii) of Theorem 4.3 is verified with $\gamma = 1$, *so that, taking into account that* $h_0^* = 1$, h *is pseudolinear on* $H_1 = \{(x_1, x_2) \in \Re^2 : x_1 + 2x_2 + 1 > 0,\ x_1 - x_2 > 0\}$ *or on* $H_2 = \{(x_1, x_2) \in \Re^2 : x_1 - x_2 < 0\} \cap H$.
Let us note that every point of the line $x_2 = x_1$ *is a global minimum point.*

Example 5.2 *Consider the function*

$$h(x_1, x_2) = \frac{x_1 + 2x_2}{x_1 - 1} + \frac{4x_1 - x_2 - 3}{3x_1 + x_2 - 1}$$

on $H = \{(x_1, x_2) \in \Re^2 : x_1 - 1 > 0,\ 3x_1 + x_2 - 1 > 0\}$.
We have $p = (1,0)^T$, $b = (3,1)^T$, $m = (1,2)^T$, $q = (4,-1)^T$, $p_0 = b_0 = -1$, $m_0 = 0$, $q_0 = -3$; *consequentely,* $p_0 m - m_0 p = (-1,-2)^T$, $p_0 q - q_0 p = (-1,1)^T$, $p_0 b - b_0 p = (-2,-1)^T$.
Case v) of Theorem 4.1 is verified with $\beta = -1$, $\delta = 1$. *Since* $v = (1,-2)^T$, $v_0 = -2$, $v^* = (2,1)^T$, $v_0^* = 0$, h *is pseudoconvex on the maximal domain:*

$$\{(x_1, x_2) \in \Re^2 : x_1 - 1 > 0,\ x_1 - 2x_2 - 2 > 0,\ 2x_1 + x_2 > 0\}.$$

At last, consider the function $f(x) = a^T x + \frac{c^T x + c_0}{d^T x + d_0}$ and assume that it is pseudoconvex on the maximal domain D_{max}. In order to construct a sum of two linear ratios which is pseudoconvex on some maximal domains, it is sufficient to choose a Charnes-Cooper's transformation $y = \frac{x}{p^T x + p_0}$ such that

$$\{x \in \Re^n : p^T x + p_0 > 0\} \cap D_{max} \neq \emptyset. \tag{5.23}$$

Example 5.3 *Consider the function given in Example 3.2 whose maximal domain of pseudoconvexity is* $D_{max} = \{(x_1, x_2) \in \Re^2 : x_1 + 16x_2 + 3 > 0,\ 3x_1 + 13x_2 + 1 > 0\}$.

Condition (5.23) is satisfied if we choose, for instance, $p = \alpha_1(3,13) + \alpha_2(1,16)$ *with* $\alpha_1\alpha_2 < 0$. *Setting* $p = (-1,19)^T$, $p_0 = 5$ *and applying Charnes-Cooper's transformation* $x = \frac{p_0 z}{1 - p^T z}$, *we get the function* $h(z_1,z_2) = \frac{10z_1 + 160z_2}{z_1 - 19z_2 + 1} + \frac{-8z_1 - 23z_2 + 2}{16z_1 + 46z_2 + 1}$ *which is pseudoconvex according to iii) of Theorem 4.1.*

Setting $p = (-2,3)^T$, $p_0 = 4$ *and applying Charnes-Cooper's transformation* $x = \frac{p_0 z}{1 - p^T z}$, *we get the function* $h(z_1,z_2) = \frac{8z_1 + 128z_2}{2z_1 - 3z_2 + 1} + \frac{-4z_1 + 6z_2 + 2}{14z_1 + 49z_2 + 1}$ *which is pseudoconvex on the maximal domain* $\{(z_1,z_2) \in \Re^2 : 10z_1 + 55z_2 + 3 > 0\} \cap H$ *according to iv) of Theorem 4.1.*

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e-mail of the authors:

acambini@ec.unipi.it; lmartein@ec.unipi.it