On the generalized convexity of quadratic functions

Alberto Cambini *- Laura Martein †

Abstract

In this paper we shallpresent a new approach in studying the quasiconvexity and the pseudoconvexity of a quadratic function. All classical results and new ones are obtained.

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1 Introduction

Generalized convexity of quadratic functions has been widely studied; the main historical references are Martos [11, 12, 13], Ferland [8], Cottle and Ferland [6], Schaible [14, 19, 18, 21].

In this paper we shall put together some results related to generalized convex quadratic functions. After noting that quasiconvexity can differ from convexity only on a proper subset S of $\Re^n$ and that quasiconvexity reduces to pseudoconvexity on an open set, we shall characterize the maximal domains of quasiconvexity and pseudoconvexity of a non-convex quadratic function. All the results that we are going to develop are obtained by means of an approach based on the second order characterization of pseudoconvexity. The

*Department of Statistics and Applied Mathematics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, Italy

†Department of Statistics and Applied Mathematics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, Italy

ghiven approach differs from the one suggested in [22]
These results will be specified in order to obtain the criteria established by Martos [11, 12, 13] related to generalized convexity over the nonnegative orthant $\Re^n_+$.
The suggested approach allows also to characterize the pseudoconvexity of a function which is the sum between a linear function and the product of affine functions.

2 Preliminary results

In this section we shall establish some properties of an $n \times n$ symmetric matrix Q. With this aim we introduce the following notations:

- $\lambda_1, \lambda_2, , ..., \lambda_n$ are the eigenvalues of the $n \times n$ symmetric matrix Q;
- $\{v^1, v^2, ..., v^n\}$ is an orthonormal basis of eigenvectors associated with $\lambda_1, \lambda_2, , ..., \lambda_n$. In order to define each of the eigenvectors uniquely, we shall assume that the first component of any eigenvector is positive (this can be obtained by multiplying it by (-1) if necessary).
- $kerQ$ is the kernel of Q, i.e., $kerQ = \{x \in \Re^n : Qx = 0\}$;
- $rankQ$ is the rank of Q, i.e., the maximum number of linearly independent columns (or rows) of Q;
- $\nu_-(Q)$ is the number of the negative eigenvalues of Q (according to their multiplicity).

Regarding the number of the negative eigenvalues of Q we have the following useful lemma.

Lemma 2.1 *Let Q be an $n \times n$ symmetric matrix and assume the existence of two vectors u, w such that*

$$u^T Qu < 0, \quad w^T Qw < 0, \quad u^T Qw = 0.$$

Then, Q has at least two negative eigenvalues.

Proof Let $u = \sum_{i=1}^{n} \alpha_i v^i$, $w = \sum_{i=1}^{n} \beta_i v^i$, $\alpha_i, \beta_i \in \Re, i = 1, ..., n$. We have

$$u^T Q u = \sum_{i=1}^{n} \alpha_i^2 \lambda_i, \; w^T Q w = \sum_{i=1}^{n} \beta_i^2 \lambda_i, \; u^T Q w = \sum_{i=1}^{n} \alpha_i \beta_i \lambda_i.$$

The assumptions imply that $\sum_{i=1}^{n} \alpha_i^2 \lambda_i < 0$ and $\sum_{i=1}^{n} \beta_i^2 \lambda_i < 0$, so that at least one eigenvalue is negative. Without loss of generality assume $\lambda_1 < 0$. If $\alpha_1 = 0$ or $\beta_1 = 0$, then obviously we have a second negative eigenvalue. If $\alpha_1 \beta_1 \neq 0$, we have

$$\sum_{i=1}^{n} (\beta_1 \alpha_i - \alpha_1 \beta_i)^2 \lambda_i = \beta_1^2 \sum_{i=1}^{n} \alpha_i^2 \lambda_i + \alpha_1^2 \sum_{i=1}^{n} \beta_i^2 \lambda_i - 2\alpha_1 \beta_1 \sum_{i=1}^{n} \alpha_i \beta_i \lambda_i =$$

$$= \beta_1^2 \sum_{i=1}^{n} \alpha_i^2 \lambda_i + \alpha_1^2 \sum_{i=1}^{n} \beta_i^2 \lambda_i.$$

Consequently, $\sum_{i=1}^{n} (\beta_1 \alpha_i - \alpha_1 \beta_i)^2 \lambda_i < 0$ so that a second negative eigenvalue exists and the thesis is achieved. □

Now we shall consider a symmetric matrix having one simple negative eigenvalue and we shall establish, for such a matrix, some fundamental properties which will be used in the next sections in characterizing the quasiconvexity and pseudoconvexity of quadratic functions.

From now on we shall assume $\lambda_1 < 0, \; \lambda_i > 0, i = 2, .., p$ and $\lambda_i = 0, i = p+1, .., n$.

The following lemma holds.

Lemma 2.2 *Let Q be an $n \times n$ symmetric matrix and assume $\nu_-(Q) = 1$. Then:*

i) if $u \in \Re^n$ is such that $u^T v^1 = 0$, then either $u \in kerQ$ or $u^T Q u > 0$;

ii) $u \in kerQ$ if and only if $u^T Q u = 0$ and $u^T v^1 = 0$.

Proof *i*) Let $u = \sum_{i=1}^{n} \alpha_i v^i$. We have $0 = u^T v^1 = \alpha_1$, so that $u = \sum_{i=2}^{n} \alpha_i v^i$. If $u \notin kerQ$, there exists $i \in \{2, .., p\}$ such that $\alpha_i \neq 0$. It follows that $u^T Q u = \sum_{i=2}^{p} (\alpha_i)^2 \lambda_i > 0$.

ii) If $u \in kerQ$, obviously we have $u^T Q u = 0$ and $u^T v^1 = 0$. The converse statement follows directly from *i*). □

Consider now the following opposite cones associated with the matrix Q:

$$T = \{x : x^T Q x \leq 0,\ x^T v^1 \geq 0\},\ -T = \{x : x^T Q x \leq 0,\ x^T v^1 \leq 0\}$$

We shall see in the next section that cones T and $-T$ will play a fundamental role in characterizing the maximal domains of the quasiconvexity and pseudoconvexity of a quadratic function.

The following theorems hold, where ∂T denotes the boundary of T. Note that since T and $-T$ are opposite cones, the properties of $-T$ can be easily derived from the ones which will be established for T.

Theorem 2.1 *Let Q be an $n \times n$ symmetric matrix and assume $\nu_-(Q) = 1$. Then:*
i) $kerQ = T \cap (-T)$;
ii) T is a pointed cone if and only if $kerQ = \{0\}$.

Proof $i)$ From ii) of Lemma 2.2 we have $kerQ \subseteq T \cap (-T)$. If $x \in T \cap (-T)$ we necessarily have $x^T Q x \leq 0,\ x^T v^1 = 0$; consequently, i) of Lemma 2.2 implies that $x \in kerQ$.
$ii)$ Since T is pointed if and only if $T \cap (-T) = \{0\}$, the thesis follows from $i)$. □

Theorem 2.2 *Let Q be an $n \times n$ symmetric matrix and assume $\nu_-(Q) = 1$. Then:*
i) $x_0 \in intT$ if and only if $x_0^T Q x_0 < 0$ and $x_0^T v^1 > 0$;
ii) $x_0 \in \partial T \setminus kerQ$ if and only if $x_0^T Q x_0 = 0$ and $x_0^T v^1 > 0$;
iii) $intT \cap int(-T) = \emptyset$;
iv) $T \cup (-T) = \{x \in \Re^n : x^T Q x \leq 0\}$;
v) $int(T \cup (-T)) = intT \cup int(-T)$.

Proof $i)$ This is obvious.
$ii)$ This follows by noting that $x_0^T Q x_0 = 0$ if and only if $x_0 \in \partial T \cup \partial(-T)$ and that $x_0 \notin kerQ$ if and only if $x_0^T v^1 \neq 0$.
$iii)$ It follows from i) and from its analogous result for cone $-T$.

iv) This follows directly from the definitions of T and $-T$.
v) Since $int(T \cup (-T)) = \{x \in \Re^n : x^T Qx < 0\} \supseteq intT \cup int(-T)$, we must prove that $intT \cup int(-T) \supseteq \{x \in \Re^n : x^T Qx < 0\}$. Let x such that $x^T Qx < 0$. From Lemma 2.2 we necessarily have $x^T v^1 \neq 0$ and the thesis follows. □

The following theorem points out the convexity of cones T and $-T$.

Theorem 2.3 *Let Q be an $n \times n$ symmetric matrix. If $\nu_-(Q) = 1$, then T is a closed convex cone.*

Proof Let P be the orthonormal matrix which has the eigenvectors $v^1, ..., v^n$ as columns, and let H be the diagonal matrix with the first p diagonal entries given by $(-\lambda_1)^{-\frac{1}{2}}$, $(\lambda_2)^{-\frac{1}{2}}$, ..., $(\lambda_p)^{-\frac{1}{2}}$, and all the others equal to 1. It is well known that the linear transformation $x = PHy$ reduces the quadratic form $x^T Qx$ to the canonical form $\sum_{i=2}^{p} y_i^2 - y_1^2 = \| \bar{y} \|^2 - y_1^2$, where $\bar{y} = (y_2, .., y_p)^T$.
Consider set $C = \{(y_1, \bar{y}) : \| \bar{y} \|^2 - y_1^2 \leq 0,\ y_1 \geq 0\} = \{(y_1, \bar{y}) : \| \bar{y} \| \leq y_1,\ y_1 \geq 0\}$. It is easy to verify that C is a closed cone; we shall prove that C is convex. Let $z = (z_1, \bar{z}) \in C, w = (w_1, \bar{w}) \in C$. Since $\| \bar{z} \| \leq z_1$, $\| \bar{w} \| \leq w_1$, we have $\| t\bar{z} + (1-t)\bar{w} \| \leq t \| \bar{z} \| + (1-t) \| \bar{w} \| \leq tz_1 + (1-t)w_1$ for all $t \in [0,1]$. Consequently, $tz + (1-t)w \in C$ for all $t \in [0,1]$ so that C is convex.
Taking into account that $x^T v^1 = y^T H^T P^T v^1$ and that $v^1 = Pe^1$, where e^1 is the unit vector $e^1 = (1, 0, .., 0)^T$, we have $x^T v^1 = y^T He^1 = (-\lambda_1)^{-\frac{1}{2}} y^T e^1 = (-\lambda_1)^{-\frac{1}{2}} y_1$. Consequently, $y_1 \geq 0$ if and only if $x^T v^1 \geq 0$ and this implies $PH(C) = T$. The thesis follows from the linearity of the transformation PH. □

Remark 2.1 *Given a convex set C and a linear map A, one has $A(riC) = ri(AC)$*[1]*, but, in general, the image of a closed convex set is not closed. When C is a closed convex cone*

[1]Let S be a convex set and let W be the smallest linear manifold containing S. Then, the relative interior of S, denoted by riS, is the set of all interior points of S with respect to the topology induced by $\Re^n$ on W

such that $C \cap (-C) = kerA$, then $A(clC) = cl(AC)$. Consequently, from i) of Theorem 2.1 and from Theorem 2.3, we have the following corollary.

Corollary 2.1 *Let Q be an $n \times n$ symmetric matrix and assume $\nu_-(Q) = 1$. Then:*
i) $Q(intT) = ri(Q(T))$, $Q(int(-T)) = ri(Q(-T))$;
ii) $Q(T)$ and $Q(-T)$ are closed convex cones.

Consider now the set

$$Z = \{z \in \Re^n \setminus \{0\} : \exists w \in intT \text{ such that } z^T w = 0\}$$

and denote with T^+ and T^- the positive polar and the negative polar of T, respectively. The following theorem characterizes Z in terms of the two polar cones.

Theorem 2.4 *Let Q be an $n \times n$ symmetric matrix. If $\nu_-(Q) = 1$, then $Z = (T^+ \cup T^-)^c$.*

Proof Since $w \in intT$ if and only if either $z^T w > 0$ for all $z \in T^+$ or $z^T w < 0$ for all $z \in T^-$, we necessarily have $Z \cap T^+ = \emptyset$ and $Z \cap T^- = \emptyset$. Consequently, $Z \subseteq (T^+ \cup T^-)^c$. Consider now an element $z \in (T^+ \cup T^-)^c$ and assume by contradiction that $z^T w \neq 0 \ \forall \ w \in intT$. The convexity of $intT$ implies $z^T w > 0 \ \forall \ w \in intT$ or $z^T w < 0 \ \forall \ w \in intT$, i.e., $z \in (T^+ \cup T^-)$, and this is a contradiction. It follows that $(T^+ \cup T^-)^c \subseteq Z$ and the thesis is achieved. □

The following lemma characterizes the image of the cones T and $-T$ under the linear transformation $z = Qx$. The obtained results will play a fundamental role in characterizing the maximal domains of quasiconvexity of a quadratic function.

Lemma 2.3 *Let Q be an $n \times n$ symmetric matrix and assume $\nu_-(Q) = 1$. Then: $Q(intT) = riT^-$, $Q(T) = T^-$, $Q(int(-T)) = riT^+$, $Q(-T) = T^+$, $Q((T \cup (-T))^c) = Z \cap (kerQ)^\perp$.*

Proof First of all we shall prove that $Q(intT) \subseteq riT^-$, $Q(int(-T)) \subseteq riT^+$, $Q((T \cup (-T))^c) \subseteq (riT^- \cup riT^+)^c$.
Let $x_0 \in intT$. Since $x_0^T Q x_0 < 0$, $Qx_0 \notin T^+$ and, taking into account Lemma 2.1,

$Qx_0 \notin Z$. Consequently, $Qx_0 \in T^-$and, from Corollary 2.1, $Q(intT) \subseteq riT^-$. Similarly we have $Q(int(-T)) \subseteq riT^+$.

Now we shall prove that $Q((T \cup (-T))^c) \cap riT^- = \emptyset$.

Let $z_0 \in (T \cup (-T))^c$, i.e. $z_0^T Q z_0 > 0$, and let $x_0 \in intT$, so that $Qx_0 \in riT^-$. If $Qz_0 \in riT^-$, then $Q([z_0, x_0]) \subseteq riT^-$ because of the convexity of riT^-. On the other hand, $x_0 \in intT$, $z_o \notin T$ imply the existence of $\bar{x} \in [z_0, x_0] \cap \partial T$ for which $Q\bar{x} \in \partial T^-$. Since $Qx_0 \in riT^-$, and this is a contradiction.

In a similar way it can be proven that $Q((T\cup(-T))^c)\cap riT^+ = \emptyset$, so that $Q((T\cup(-T))^c) \subseteq (riT^- \cup riT^+)^c$.

Since $kerQ = T \cup (-T)$ implies $T^+ \cap T^- \subseteq (kerQ)^\perp = ImQ = \{w = Qx,\ x \in \Re^n\}$, from Corollary 2.1 the thesis is achieved. □

3 Quadratic functions

In this section we shall characterize quadratic functions which are generalized convex.

Consider the following quadratic function

$$Q(x) = \frac{1}{2}\, x^T Q x + q^T x \tag{3.1}$$

where Q is an $n \times n$ symmetric matrix, $q \in \Re^n$.

We shall refer to

$$Q_0(x) = \frac{1}{2}\, x^T Q x \tag{3.2}$$

as the quadratic form associated with (3.1).

In order to have a self-contained paper, we recall the definition of a quasiconvex function and of a pseudoconvex function.

Definition 3.1 *Let f be a differentiable function defined on an open set containing the convex set $S \subseteq \Re^n$.*

i) f is quasiconvex on S if

$$x_1, x_2 \in S,\ f(x_1) \geq f(x_2) \Rightarrow (x_2 - x_1)^T \nabla f(x_1) \leq 0 \tag{3.3}$$

ii) f is pseudoconvex on S if

$$x_1, x_2 \in S,\ f(x_1) > f(x_2) \Rightarrow (x_2 - x_1)^T \nabla f(x_1) < 0 \tag{3.4}$$

Remark 3.1 *In general, a pseudoconvex function is quasiconvex too; nevertheless, quasiconvexity is equivalent to pseudoconvexity when the function does not have critical points.*

A useful second order characterization of pseudoconvexity of $Q(x)$ is the following one: $Q(x)$ is pseudoconvex on an open convex set S if and only if (3.5) holds

$$x \in S,\ w \in \Re^n, w^T(Qx + q) = 0 \Rightarrow w^T Q w \geq 0. \tag{3.5}$$

The following theorem shows that quasiconvexity reduces to convexity if the domain is the whole space $\Re^n$.

Theorem 3.1 *The quadratic function $Q(x)$ is quasiconvex on $\Re^n$ if and only if $Q(x)$ is convex on $\Re^n$.*

Proof Since convexity implies quasiconvexity, the converse statement remains to be proven. By contradiction, assume that $Q(x)$ is not convex. Then, Q has at least one negative eigenvalue λ_1. Let w be a normalized eigenvector associated with λ_1 and let $\varphi(t) = Q(tw) = \frac{1}{2}\lambda_1 t^2 + t\, q^T w, t \in \Re$. The restriction $\varphi(t)$ has a strict local maximum point at $\bar{t} = -\frac{q^T w}{\lambda_1}$ and consequently, $\varphi(t)$, and in turns $Q(x)$, is not quasiconvex and this contradicts the assumption. □

Theorem 3.1 implies that quasiconvexity can differ from convexity only on a proper subset S of $\Re^n$. From now on, following Martos [13], we will insert the word "merely" to distinguish quadratic quasiconvex (pseudoconvex, etc.) functions that are not convex.

Remark 3.2 *Recalling that the Hessian matrix of a twice quasiconvex function evaluated at a point of its domain has at most one negative eigenvalue, a necessary condition for a quadratic function to be merely quasiconvex is that the matrix Q has one simple negative eigenvalue, i.e., $\nu_-(Q) = 1$.*

The following theorem shows that quasiconvexity reduces to pseudoconvexity on every open convex set of $\Re^n$.

Theorem 3.2 *The quadratic function $Q(x)$ is quasiconvex on an open convex set $S \subseteq \Re^n$ if and only if it is pseudoconvex on S.*

Proof Since pseudoconvexity implies quasiconvexity, the converse statement remains to be proven. The thesis follows from Theorem 3.1 if $Q(x)$ is convex, otherwise Q has one simple negative eigenvalue λ_1. Let w be a normalized eigenvector associated with λ_1. Taking into account Remark 3.1, it is sufficient to prove that the gradient of $Q(x)$ cannot vanish. By contradiction, assume the existence of $x_0 \in S$ such that $\nabla Q(x_0) = Qx_0 + q = 0$, and consider the restriction $\varphi(t) = Q(x_0 + tw)$. Since $\varphi(t) = \frac{1}{2}\lambda_1 t^2 + Q(x_0)$, this restriction has a strict local maximum point at $t = 0$, so that $\varphi(t)$ and in turns $Q(x)$, is not quasiconvex, which contradicts the assumption. □

Note that Theorem 3.2 implies:

- a quadratic function which is merely pseudoconvex on an open convex set S has no critical points;
- a quadratic function which is merely quasiconvex on a convex set S is merely pseudoconvex at least on $intS$;
- a quadratic function which is merely pseudoconvex on an open convex set S is merely quasiconvex (not necessarily pseudoconvex) on the closure of S.

Remark 3.3 *It is important to point out that any characterization of pseudoconvexity of a quadratic function $Q(x)$ on an open convex set S allows us to simultaneously obtain criteria for the quasiconvexity of $Q(x)$ on the closure of S. This fact simplifies the analysis in the sense that, in order to characterize the quasiconvexity of $Q(x)$ on S, it is sufficient to study the pseudoconvexity on the interior of S.*

Now we are able to find the maximal domains of quasiconvexity (pseudoconvexity) of a quadratic form and of a quadratic function.

Theorem 3.3 *Let Q be an $n \times n$ symmetric matrix. If $\nu_-(Q) = 1$, then the quadratic form $Q_0(x) = \frac{1}{2}\, x^T Q x$ is merely quasiconvex on the closed convex cones $T, -T$. Furthermore, T and $-T$ are the maximal domains of quasiconvexity of $Q_0(x)$.*

Proof Taking into account Remark 3.3, we shall prove that $Q_0(x)$ is pseudoconvex on $intT$. If not, from Remark 3.1 , there exist $x_0 \in intT$, $w \in \Re^n$ such that $w^T Q x_0 = 0$ and $w^T Q w < 0$. Since $x_0^T Q x_0 < 0$, from Lemma 2.1 Q has at least two negative eigenvalues, which contradicts the assumptions. Similarly, we obtain that $Q_0(x)$ is pseudoconvex on $int(-T)$. Consequently, $Q_0(x)$ is quasiconvex on T and on $-T$.
The maximality of the domains remains to be proven. To see this, assume that $Q_0(x)$ is pseudoconvex on an open set S such that $S \cap (T \cup (-T)) \neq \emptyset$ and let $y \in S$, $y \notin T \cup (-T)$. Then $y^T Q y > 0$ and from Lemma 2.3, $Qy \in Z$. Consequently, there exists $x_0 \in intT$ such that $x_0^T Q y = 0$, $x_0^T Q x_0 < 0$, which contradicts (3.5). The thesis is achieved. □

Taking into account Remark 3.2, Theorem 3.3 may be re-stated as follows.

Theorem 3.4. *A quadratic form $Q_0(x)$ is merely quasiconvex on a convex set $S \subset \Re^n$, with $intS \neq \emptyset$, if and only if*
i) $\nu_-(Q) = 1$;
ii) $S \subseteq T$, or $S \subseteq -T$.

We shall prove that the maximal domains of quasiconvexity of a quadratic function are obtained by the ones ($\pm T$) associated with the quadratic form by means of a suitable translation. To this end, firstly we shall state the following theorem which gives a necessary condition for a quadratic function to be quasiconvex and which points out that, unlike the convex case, the sum of a quasiconvex function with a linear function is not, in general, quasiconvex.

Theorem 3.5 *Assume that the quadratic function $Q(x) = \frac{1}{2}\, x^T Q x + q^T x$ is merely quasiconvex on an open convex set $S \subset \Re^n$. Then, $rankQ = rank[Q, q]$.*

Proof The thesis is trivial if $q = 0$. Consider $ImQ = \{Qx,\ x \in \Re^n\}$ and assume by contradiction that $q \notin ImQ$. Then, for every fixed $x \in \Re^n$, $Qx + q \notin ImQ$ and, in particular, $Qx + q \notin T^+ \cup T^- \subseteq ImQ$. From Lemma 2.3, we have $Qx + q \in Z$ so that, from Theorem 2.4, there exists $w \in intT$ such that $w^T(Qx + q) = 0$. Let $x_0 \in S$ and consider the restriction $\varphi(t) = Q(x_0 + tw)$. By means of simple calculations we have $\varphi'(0) = w^T(Qx_0 + q) = 0$, $\varphi''(0) = w^TQw < 0$ so that $t = 0$ is a strict local maximum for $\varphi(t)$ and this implies that $Q(x)$ is not quasiconvex on S, which contradicts the assumption. It follows that $q \in ImQ$ or, equivalently, $rankQ = rank[Q, q]$. □

Remark 3.4 *Note that $w \in ImQ$ if and only if $w \in (kerQ)^{\perp}$*[2]*. In particular, $rankQ = rank[Q, q]$ implies that $q \in (kerQ)^{\perp}$.*

Theorem 3.6 *Consider the quadratic function $Q(x) = \frac{1}{2}\, x^TQx + q^Tx$.*
If there exists $s \in \Re^n$ such that $Qs + q = 0$, then:
i) $Q(x)$ is merely quasiconvex on the closed convex cones $s + T, s - T$ if and only if $Q_0(x) = \frac{1}{2}\, x^TQx$ is merely quasiconvex on T, $-T$, respectively.
ii) If $\nu_-(Q) = 1$, then $s+T$ and $s-T$ are the maximal domains of quasiconvexity of $Q(x)$ and we have

$$s + T = \{x \in \Re^n : (x - s)^TQ(x - s) \leq 0,\ (v^1)^T(x - s) \geq 0\} \tag{3.6}$$

$$s - T = \{x \in \Re^n : (x - s)^TQ(x - s) \leq 0,\ (v^1)^T(x - s) \leq 0\} \tag{3.7}$$

Proof i) $Q(x)$ is pseudoconvex on $s \pm intT$ if and only if (3.5) holds with $S = s \pm intT$. Since $x \in s \pm intT$ if and only if $x - s = u \in \pm intT$, we have $Qx + q = Q(x - s) = Qu$ so that $Q(x)$ is merely pseudoconvex on $s \pm intT$ if and only if $Q_0(x)$ is merely pseudoconvex on $\pm intT$ and the thesis follows.
ii) Theorem 3.3 implies that $\pm T$ are the maximal domains of quasiconvexity of $Q_0(x)$ so that, taking into account *i)*, $s \pm T$ are the maximal domains of quasiconvexity of $Q(x)$. Finally, $x \in s \pm T$ if and only if $x - s \in \pm T$, so that (3.6), and (3.7) hold.
The proof is complete. □

[2]Given a linear subspace $W \subseteq \Re^n$, $W^{\perp} = \{z \in \Re^n :\ z^Tw = 0,\ \forall w \in W\}$

Remark 3.5 *If Q is a singular matrix, then a stationary point of $Q(x)$ is not unique. However, the characterization of the maximal domains of quasiconvexity is independent of the particular stationary point used. To see this, let s_1, s_2 be two distinct stationary points, i.e., $Qs_1 + q = Qs_2 + q = 0$. We have $s_1 = s_2 + u$, $u \in kerQ \subset T \cup (-T)$. It follows that $s_1 \pm T = s_2 + u \pm T = s_2 \pm T$.*

The previous results allow us to characterize the merely quasiconvexity of a quadratic function.

Theorem 3.7 *The quadratic function $Q(x)$ is merely quasiconvex on a convex set S with nonempty interior if and only if*
i) $\nu_-(Q) = 1$;
ii) there exists $s \in \Re^n$ such that $Qs + q = 0$;
iii) $S \subseteq s \pm T$.

Proof Assume that $Q(x)$ is merely quasiconvex on S. We necessarily have $\nu_-(Q) = 1$ and from Theorem 3.5, *ii)* follows; *iii)* is a direct consequence of Theorem 3.6.
Conversely, the thesis follows from Theorem 3.3 and from Theorem 3.6. □

Corollary 3.1 *If $Q(x)$ is merely quasiconvex on a convex set S with nonempty interior, then $Q_0(x)$ is merely quasiconvex on $S - s$, where s is such that $Qs + q = 0$.*

Proof The thesis follows from Theorem 3.7, taking into account Theorem 3.3. □

The following examples clarify the results given in Theorem 3.4 and in Theorem 3.7.

Example 3.1 *Consider the quadratic form $Q_0(x) = 2x_1^2 - x_2^2 - x_1x_2$.*
We have $Q = \begin{bmatrix} 4 & -1 \\ -1 & -2 \end{bmatrix}$, $\lambda_1 = 1 - \sqrt{10} < 0$, $\lambda_2 = 1 + \sqrt{10} > 0$, $v^1 = \frac{v}{\|v\|}$ with $v = (1, 3 + \sqrt{10})^T$.
Theorem 3.4 implies that $Q_0(x)$ is quasiconvex on the maximal domains T, $-T$. It is easy to verify that $T = \{x = \alpha(1,1)^T + \beta(-1,2)^T,\ \alpha,\ \beta \geq 0\}$, so that the positive and negative

polars of T are respectively,

$T^+ = \{x = \alpha_1(-1,1)^T + \beta_1(2,1)^T,\ \alpha_1,\ \beta_1 \geq 0\}$,

$T^- = \{x = \alpha_1(-1,1)^T + \beta_1(2,1)^T,\ \alpha_1,\ \beta_1 \leq 0\}$.

Now we shall verify that the image of T under the linear transformation Q is T^-.

In fact, $Q(T) = \{y = \alpha Q(1,1)^T + \beta Q(-1,2)^T,\ \alpha,\ \beta \geq 0\} = \{y = -3\alpha(-1,1)^T - 3\beta(2,1)^T,\ \alpha,\ \beta \geq 0\} = \{y = \alpha_1(-1,1)^T + \beta_1(2,1)^T,\ \alpha_1,\ \beta_1 \leq 0\} = T^-$.

By means of similar calculations we can verify that $Q(-T) = T^+$.

Let us note that the nonsingularity of Q implies that the quadratic function $Q(x) = Q_0(x) + q^Tx$ is quasiconvex for every $q \in \Re^n$ on the maximal domains $s+T$ and $s-T$, where $s = -Q^{-1}q$.

Example 3.2 *Consider the quadratic function* $Q(x) = -x_1^2 - x_2^2 - 2x_1x_2 + 2x_1 + 2x_2$. *We have* $Q = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$, $q = (2,2)^T$, $\lambda_1 = -4 < 0$, $\lambda_2 = 0$, $v^1 = \frac{v}{\|v\|}$ *with* $v = (1,1)^T$.
Note that $Q(x)$ is a concave function; nevertheless, since rank Q = rank $[Q,q]$, $Q(x)$ is also merely quasiconvex on $s+T$ and on $s-T$, where s is any vector of the kind $s = (s_1, -s_1+1)^T$, $s_1 \in \Re$. *Taking into account that* $x^TQx \leq 0\ \forall x \in \Re^2$, *we have* $s+T = \{x \in \Re^2 : (v^1)^T(x-s) \geq 0\} = \{x \in \Re^2 : x_1 + x_2 - 1 \geq 0\}$.

At last, we shall characterize the merely quasiconvexity of a quadratic function over a half-space.

Theorem 3.8 *$Q(x)$ is merely quasiconvex on the half-space* $H = \{x \in \Re^n : h^Tx + h_0 \geq 0\}$ *if and only if (3.8) holds:*

$$\nu_-(Q) = 1,\ kerQ = h^\perp,\ \exists\, \beta \in \Re:\ q = \beta h,\ h_0 \leq \beta\frac{\| h \|^4}{h^TQh}. \tag{3.8}$$

Proof Assume that $Q(x)$ is merely quasiconvex on H. From Theorem 3.7, we have $H \subseteq s+T \subseteq \Gamma = \{x \in \Re^n : (v^1)^T(x-s) \geq 0\}$, or $H \subseteq s-T \subseteq \Gamma_1 = \{x \in \Re^n : (v^1)^T(x-s) \leq 0\}$. Since $H \subseteq \Gamma$ or $H \subseteq \Gamma_1$, ∂H and $\partial\Gamma$ are necessarily parallel hyperplanes so that $h = kv^1$, $k \neq 0$, i.e., h is an eigenvector associated with the negative eigenvalue λ_1. Obviously, $k > 0$ implies $H \subseteq \Gamma$, while $k < 0$ implies $H \subseteq \Gamma_1$. We shall limit ourselves to considering the case $k > 0$ since the other one is perfectly analogous.

Note that $H \subseteq \Gamma$ if and only if $h_0 \leq -h^T s$; when $h_0 = -h^T s$ we have $H = s + T = \Gamma$ and $T = \{x \in \Re^n : x^T Q x \leq 0, h^T x \geq 0\}$.
Since the quadratic form $\frac{1}{2}x^T Q x$ is merely quasiconvex on T and on $-T$, by means of continuity, we have $\frac{1}{2}x^T Q x \leq 0$, $\forall\ x \in \Re^n$ and, because $\nu_-(Q) = 1$, this implies that $kerQ = h^\perp$ and $ImQ = \{kh,\ k \in \Re\}$. Since $rankQ = rank(Q, q)$, there exists $\beta \in \Re$ such that $q = \beta h$. If $\beta = 0$, we have $s \in kerQ = h^\perp$ so that $h_0 \leq -h^T s = 0$ and (3.8) holds. If $\beta \neq 0$, we can choose any $s \in s_0 + h^\perp$, where $Qs_0 = -q$. In particular, $s = (s_0 + h^\perp) \cap ImQ$ is an eigenvector of Q and thus $Qs = \lambda_1 s$. It follows that $h_0 \leq -h^T s = -\frac{h^T Q s}{\lambda_1} = \frac{h^T \beta h}{\lambda_1} = \beta \frac{\|h\|^2}{\lambda_1} = \beta \frac{\|h\|^4}{h^T Q h}$.
Conversely, by choosing $s = -\frac{\beta}{\lambda_1}h$, it is easy to verify that (3.8) implies i), ii), and iii) of Theorem 3.7. □

Corollary 3.2 *$Q(x)$ is merely quasiconvex on the half-space $H = \{x \in \Re^n : h^T x + h_0 \geq 0\}$ if and only if $Q = \mu h h^T$, $q = \beta h$, with $\mu < 0$ and $h_0 \leq \frac{\beta}{\mu}$.*

4 Quadratic functions of nonnegative variables

By specifying the results given in the previous section, it is possible to establish criteria for generalized convex quadratic functions on $\Re^n_+$. These results were obtained for the first time by Martos in [11, 12], by introducing of the concept of positive subdefinite matrices. Now we shall characterize the quasiconvexity of a quadratic form on the nonnegative orthant. The first result points out the relationships between the nonnegative orthant and maximal cone T.

Theorem 4.1 *The quadratic form $Q_0(x)$ is merely quasiconvex on $\Re^n_+$ if and only if*
i) $\nu_-(Q) = 1$;
ii) $\Re^n_+ \subseteq T$.

Proof From Theorem 3.4, $Q_0(x)$ is merely quasiconvex on $\Re^n_+$ if and only if *i)* holds and either $\Re^n_+ \subseteq T$ or $\Re^n_+ \subseteq -T$. This last inclusion cannot hold; in fact $(v^1)^T x \leq 0$, $\forall x \in \Re^n_+$ implies that $v^1 \in \Re^n_-$ and this is a contradiction since the first nonzero component of v^1

is positive. The thesis follows. □

The following theorem characterizes a quadratic form on the nonnegative orthant in terms of the sign of the elements of matrix Q.

Theorem 4.2 *The quadratic form $Q_0(x)$ is merely quasiconvex on $\Re^n_+$ if and only if*
i) $\nu_-(Q) = 1$;
ii) $Q \leq 0$ [3].

Proof Assume that $i)$ and $ii)$ hold and let Γ be the subspace spanned by the normalized eigenvectors associated with the nonnegative eigenvalues of Q; we have $\Gamma = \{x \in \Re^n : (v^1)^T x = 0\}$. Since $x^T Q x \geq 0$ for all $x \in \Gamma$, and $x^T Q x \leq 0$ for all $x \in \Re^n_+$, Γ is a supporting hyperplane to $\Re^n_+$ at the origin, so that $v^1 \in \Re^n_+$. Consequently, the elements of $\Re^n_+$ satisfy the inequalities $x^T Q x \leq 0$, $(v^1)^T x \geq 0$ so that $\Re^n_+ \subseteq T$ and the thesis follows from Theorem 4.1.
Assume now that $Q_0(x)$ is merely quasiconvex on $\Re^n_+$.
From Theorem 4.1, $i)$ holds and furthermore $\Re^n_+ \subseteq T$ so that $x^T Q x \leq 0$ for all $x \in \Re^n_+$; in particular $(e^i)^T Q e^i = q_{ii} \leq 0$, $i = 1, ..., n$. Consider now the submatrix of Q, $Q_{ij} = \begin{bmatrix} q_{ii} & q_{ij} \\ q_{ij} & q_{jj} \end{bmatrix}$ and the restriction $\varphi(x_i, x_j) = \frac{1}{2}(q_{ii}x_i^2 + 2q_{ij}x_i x_j + q_{jj}x_j^2)$.
Since $\varphi(x_i, x_j) \leq 0$, $\forall (x_i, x_j) \in \Re^2_+$, we have $q_{ij} \leq 0$ when $q_{ii}q_{jj} = 0$. Consider the case $q_{ii} < 0$, $q_{jj} < 0$. The quasiconvexity of φ implies that Q_{ij} has at most one negative eigenvalue, so that $q_{ii}q_{jj} - q_{ij}^2 \leq 0$. If $q_{ii}q_{jj} - q_{ij}^2 < 0$, the equation $q_{ii}x_i^2 + 2q_{ij}x_i x_j + q_{jj}x_j^2 = 0$ has, for every fixed x_j (or x_i), two roots which cannot be positive since $\varphi(x_i, x_j) \leq 0$, $\forall (x_i, x_j) \in \Re^2_+$; consequently, $q_{ij} \leq 0$.
If $q_{ii}q_{jj} - q_{ij}^2 = 0$, we have $\varphi(x_i, x_j) = \frac{1}{2q_{ii}}(q_{ii}x_i + q_{ij}x_j)^2$. This function has a line r of critical points which are global maximum points; since the quasiconvexity of φ implies that $r \cap int\Re^2_+ = \emptyset$, we necessarily have $q_{ij} \leq 0$.
It follows that $q_{ij} \leq 0$, $\forall i, j = 1, .., n$, i.e., $Q \leq 0$ and the proof is complete. □

[3] $Q \leq 0$ means $q_{ij} \leq 0$, $\forall i, j$

Theorem 4.3 *Let $Q(x)$ be merely quasiconvex on $\Re^n_+$. Then, $Q_0(x)$ is merely quasiconvex on $\Re^n_+$.*

Proof From ii) of Theorem 3.6, either $\Re^n_+ \subseteq s+T$ or $\Re^n_+ \subseteq s-T$. Let $v^1_j > 0$ be the first nonzero component of v^1; since $te^j \in \Re^n_+$, $\forall t > 0$, we have $(v^1)^T(te^j - s) = tv^1_j - (v^1)^T s > 0$, for a large enough t. It follows that $\Re^n_+ \subseteq s+T$. Consequently, we must prove that $\Re^n_+ \subseteq T$, i.e., $x^TQx \leq 0$, $(v^1)^T x \geq 0$, $\forall x \in \Re^n_+$. Assume the existence of $\bar{x}$ such that $\bar{x}^TQ\bar{x} > 0$ ($(v^1)^T\bar{x} < 0$). Since $t\bar{x} \in \Re^n_+$, $\forall t > 0$, for a large enough t we have $(t\bar{x}-s)^TQ(t\bar{x}-s) > 0$ ($(v^1)^T(t\bar{x}-s) < 0$) and this is a contradiction.
Consequently, $\Re^n_+ \subseteq T$, so that Q_0 is merely quasiconvex on $\Re^n_+$. □

The following example shows that the converse statement of Theorem 4.3 does not hold; we need some additional assumptions on the vector q which will be given in Theorem 4.4.

Example 4.1 *Consider the quadratic function $Q(x_1, x_2) = -x_1x_2 + x_1 - x_2$.*
We have $Q = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
$Q_0(x_1, x_2) = -x_1x_2$ *is merely quasiconvex on $\Re^2_+$ according to Theorem 4.2. On the other hand, $Q(x_1, x_2)$ is not quasiconvex on $\Re^2_+$, since its restriction on line $x_2 = x_1 + 2$ has a strict local maximum point at $(1, 3)$.*

Theorem 4.4 *A quadratic function $Q(x) = \frac{1}{2}\, x^TQx + q^Tx$ is merely quasiconvex on $\Re^n_+$ if and only if*
i) $\nu_-(Q) = 1$;
ii) $Q \leq 0$;
iii) there exists $s \in \Re^n$ such that $Qs + q = 0$, $q^Ts \geq 0$;
iv) $q \leq 0$.

Proof Assume that *i)-iv)* hold. From Theorem 3.6 we must prove that $\Re^n_+ \subseteq s+T = \{x \in \Re^n : (x-s)^TQ(x-s) \leq 0,\ (v^1)^T(x-s) \geq 0\}$.
We have $(x-s)^TQ(x-s) = x^TQx + 2q^Tx - q^Ts$, so that *ii)*, *iii)* and *iv)* imply that $(x-s)^TQ(x-s) \leq 0$, $\forall\, x \in \Re^n_+$. From Theorem 4.2 and Theorem 4.1 we have $\Re^n_+ \subseteq T$

and this implies that $v^1 \in \Re^n_+$. On the other hand, $s^T v^1 = \frac{1}{\lambda_1} s^T Q v^1 = -\frac{1}{\lambda_1} q^T v^1 \leq 0$, and consequently, $(v^1)^T(x-s) \geq 0$, $\forall\, x \in \Re^n_+$, so that $\Re^n_+ \subseteq T$.

Assume now that $Q(x)$ is merely quasiconvex on $\Re^n_+$. From Theorem 3.7 we have $\nu_-(Q) = 1$, $\exists s \in \Re^n : Qs + q = 0$, and $\Re^n_+ \subseteq s + T$, while from Theorem 4.3 and from Theorem 4.2, we have $Q \leq 0$ and $\Re^n_+ \subseteq T$. It remains to be proven that $q \leq 0$ and $q^T s \geq 0$. The inclusion $\Re^n_+ \subseteq s + T$ implies that $0 \in s + T$, i.e., $s \in -T$. Consequently, $s^T Q s \leq 0$ and since $q^T s = -s^T Q s$, we have $q^T s \geq 0$. Furthermore, from Lemma 2.3, $Qs \in T^+$, i.e., $q \in T^-$; it follows that $q^T x \leq 0\ \forall\, x \in T$ and, in particular, $q^T x \leq 0\ \forall\, x \in \Re^n_+$ so that $q \leq 0$.

The proof is complete. □

5 Pseudoconvexity of a quadratic function on a closed set

In this section we shall characterize the maximal domains of pseudoconvexity of a non-convex quadratic function. In particular, we are interested in analyzing pseudoconvexity on the nonnegative orthant $\Re^n_+$ since many extremum quadratic problems have a feasible region contained in $\Re^n_+$ and not just in the open set $int\Re^n_+$. Since $\Re^n_+$ is a closed set, we shall refer to the following definition of pseudoconvexity at a point.

Definition 5.1 *Let f be a differentiable function defined on an open set of $\Re^n$ containing the convex set S. f is pseudoconvex at $x_0 \in S$ if*

$$x \in S,\ f(x) < f(x_0) \Rightarrow (x - x_0)^T \nabla f(x_0) < 0.$$

Obviously, a function is pseudoconvex on a convex set if and only if is pseudoconvex at every point of the set.

Since a non-convex quadratic function is quasiconvex on a convex set S if only if it is pseudoconvex on $intS$, we must further investigate the study of the pseudoconvexity on the boundary of S, starting from the maximal domains of quasiconvexity of a quadratic form.

Regarding this, the following lemma holds.

Lemma 5.1 *Consider the quadratic form* $Q_0(x) = \frac{1}{2}\, x^T Q x$, *and assume* $\nu_-(Q) = 1$. *Then,* $Q_0(x)$ *is pseudoconvex at* $x_0 \in \pm T$ *if and only if* $\nabla Q_0(x_0) = Qx_0 \neq 0$.

Proof Since $Q_0(x)$ is merely pseudoconvex on $\pm intT$ (see Theorem 3.3 and Theorem 3.2), we must investigate the boundary of cones T and $-T$. We shall consider ∂T since the other case is analogous.

$Q_0(x)$ is pseudoconvex at $x_0 \in \partial T$ if and only if (5.9) holds:

$$x_0 \in \partial T,\ x \in T,\ Q_0(x) < Q_0(x_0) \Rightarrow (x - x_0)^T Q x_0 < 0. \tag{5.9}$$

Obviously, (5.9) implies that $\nabla Q_0(x_0) = Qx_0 \neq 0$.

Conversely, let $x_0 \in \partial T$; we necessarily have $Q_0(x_0) = 0$, so that $Q_0(x) < Q_0(x_0) = 0$ implies that $x \in intT$. On the other hand, $Qx_0 \neq 0$ implies that $Qx_0 \in T^- \setminus \{0\}$ so that $x^T Q x_0 < 0,\ \forall x \in intT$. Since $(x - x_0)^T Q x_0 = x^T Q x_0$, the thesis is achieved. □

The following theorems, which are a direct consequence of Lemma 5.1, characterize the maximal domains of pseudoconvexity of a nonconvex quadratic form.

Theorem 5.1 *Consider the quadratic form* $Q_0(x)$ *and assume that* $\nu_-(Q) = 1$. *Then the following properties hold:*

i) $Q_0(x)$ *is merely pseudoconvex on the maximal domains* $T \setminus kerQ, -T \setminus kerQ$;

ii) $Q_0(x)$ *is merely pseudoconvex on* $T \setminus \{0\}, -T \setminus \{0\}$ *if and only if* Q *is nonsingular.*

Theorem 5.1 may be re-stated as follows.

Theorem 5.2 *A quadratic form* $Q_0(x)$ *is merely pseudoconvex on a convex set* S *with nonempty interior if and only if*

i) $\nu_-(Q) = 1$;

ii) $S \subseteq T \setminus kerQ$, *or* $S \subseteq -T \setminus kerQ$.

Taking into account i) of Theorem 2.1, the maximal domains $T \setminus kerQ, -T \setminus kerQ$ can be characterized by means of the inequalities $(v^1)^T x > 0$, $(v^1)^T x < 0$, respectively. More exactly, we have the following theorem.

Theorem 5.3 *Consider the quadratic form* $Q_0(x) = \frac{1}{2}\, x^T Qx$ *and assume* $\nu_-(Q) = 1$. *Then, the maximal domains of pseudoconvexity of* $Q_0(x)$ *are given by*

$$T \setminus kerQ = \{x \in \Re^n : x^T Qx \leq 0,\ (v^1)^T x > 0\}$$

$$-T \setminus kerQ = \{x \in \Re^n : x^T Qx \leq 0,\ (v^1)^T x < 0\}.$$

The relation between the pseudoconvexity of a quadratic function and the pseudoconvexity of the corresponding quadratic form is specified in the following theorem.

Theorem 5.4 *Consider the quadratic function* $Q(x) = \frac{1}{2}\, x^T Qx + q^T x$, *and assume the existence of* $s \in \Re^n$ *such that* $Qs + q = 0$. *Then,* $Q(x)$ *is pseudoconvex on* $s \pm T$ *if and only if* $Q_0(x) = \frac{1}{2}\, x^T Qx$ *is pseudoconvex on* $\pm T$.

Proof $Q(x)$ is pseudoconvex at $x_0 \in s + T$ if and only if

$$x \in s + T,\ Q(x) < Q(x_0) \Rightarrow \nabla Q(x_0)^T (x - x_0) < 0. \tag{5.10}$$

Set $y_0 = x_0 - s \in T$, $y = x - s \in T$. We have $Q(x) = \frac{1}{2}(x-s)^T Q(x-s) - \frac{1}{2}s^T Qs = Q_0(y) - \frac{1}{2}s^T Qs$, $Q(x_0) = Q_0(y_0) - \frac{1}{2}s^T Qs$. It follows that $Q(x) < Q(x_0)$ if and only if $Q_0(y) < Q_0(y_0)$. Furthermore, $\nabla Q(x_0) = Qx_0 + q = Q(x_0 - s) = Qy_0 = \nabla Q_0(y_0)$, so that $\nabla Q(x_0)^T (x - x_0) < 0$ if and only if $\nabla Q_0(y_0)^T (y - y_0) < 0$. Consequently, (5.10) is equivalent to

$$y_0 \in T,\ y \in T,\ Q_0(y) < Q_0(y_0) \Rightarrow \nabla Q_0(y_0)^T (y - y_0) < 0$$

i.e., the pseudoconvexity of $Q(x)$ on $s + T$ is equivalent to the pseudoconvexity of $Q_0(y)$ on T.

Analogously, the pseudoconvexity of $Q(x)$ on $s - T$ is equivalent to the pseudoconvexity of $Q_0(y)$ on $-T$. □

The following lemma extends Lemma 5.1 to a quadratic function.

Lemma 5.2 *Consider the quadratic function* $Q(x) = \frac{1}{2}\, x^T Qx + q^T x$ *with* $\nu_-(Q) = 1$, *and assume the existence of* $s \in \Re^n$ *such that* $Qs + q = 0$. *Then,* $Q(x)$ *is merely pseudoconvex at* $x_0 \in s \pm T$ *if and only if* $\nabla Q(x_0) = Qx_0 + q \neq 0$.

Proof From Theorem 5.4, $Q(x)$ is merely pseudoconvex at $x_0 \in s \pm T$ if and only if Q_0 is merely pseudoconvex at $y_0 = x_0 - s \in \pm T$.
Since $Qy_0 = Q(x_0 - s) = Qx_0 + q$, the thesis follows from Lemma 5.1. □

As a direct consequence of the previous results, we have the following characterization of the maximal domains of pseudoconvexity of a quadratic function.

Theorem 5.5 *The quadratic function $Q(x) = \frac{1}{2}\, x^T Q x + q^T x$ is merely pseudoconvex on a convex set S with nonempty interior if and only if*
i) $\nu_-(Q) = 1$;
ii) there exists $s \in \Re^n$ *such that* $Qs + q = 0$;
iii) $S \subseteq s + (T \setminus kerQ) = \{x \in \Re^n : (x-s)^T Q (x-s) \le 0,\ (v^1)^T(x-s) > 0\}$, *or* $S \subseteq s - (T \setminus kerQ) = \{x \in \Re^n : (x-s)^T Q (x-s) \le 0,\ (v^1)^T(x-s) < 0\}$.

5.1 Pseudoconvexity on the nonnegative orthant

The above criteria can be specified to the case where S is the nonnegative orthant.

Theorem 5.6 *Let $Q_0(x) = \frac{1}{2}\, x^T Q x$ be merely quasiconvex on $\Re^n_+$. Then, $Q_0(x)$ is merely pseudoconvex on $\Re^n_+ \setminus \{0\}$ if and only if Q does not contain a column (or a row) of zeros.*

Proof From Lemma 5.1, $Q_0(x)$ is pseudoconvex on $\Re^n_+ \setminus \{0\}$ if and only if $Qx \neq 0\ \forall x \in \Re^n_+ \setminus \{0\}$. By denoting with q^j the j-th column of Q, $j = 1, ..., n$, we have $Qx = \sum_{j=1}^n x_j q^j, x_j \ge 0$. Since $Q \le 0$ (see Theorem 4.2), $Qx = 0$ if and only if for some j we have $q^j = 0$. □

Theorem 5.7 *Let $Q(x) = \frac{1}{2}\, x^T Q x + q^T x$ be merely quasiconvex on $\Re^n_+$. Then, $Q(x)$ is merely pseudoconvex on $\Re^n_+$ if and only if $q \neq 0$.*

Proof From Lemma 5.2, $Q(x)$ is pseudoconvex on $\Re^n_+$ if and only if $Qx + q \neq 0\ \forall x \in \Re^n_+$.
Since $Q \le 0$, $q \le 0$ (see Theorem 4.4), the thesis follows. □

By applying Theorem 4.2 and Theorem 5.6 to a 2×2 matrix we obtain the following criteria of quasiconvexity and of pseudoconvexity of a quadratic form.

Theorem 5.8 *Consider the matrix* $Q = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$. *Then, the quadratic form* $Q_0(x) = \frac{1}{2}x^T Qx$ *is merely quasiconvex on* $\Re^2_+$ *if and only if*

i) $\alpha \leq 0, \ \beta \leq 0, \ \gamma \leq 0, \ (\alpha, \beta, \gamma) \neq (0, 0, 0)$;

ii) $detQ = \alpha\gamma - \beta^2 \leq 0$.

Furthermore, $Q_0(x)$ *is pseudoconvex on* $\Re^2_+ \setminus \{(0, 0)\}$ *if and only if in addition to i) and ii) we have* $(\alpha, \beta) \neq (0, 0)$ *and* $(\beta, \gamma) \neq (0, 0)$.

Example 5.1 *Consider the matrices*

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \alpha < 0; \ B = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, \beta < 0;$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}, \gamma < 0; \ D = \begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}, \alpha < 0, \beta < 0.$$

The quadratic forms associated with all the matrices are quasiconvex on $\Re^2_+$ *but only the quadratic forms associated with the matrices B and D are pseudoconvex on* $\Re^2_+ \setminus \{(0, 0)\}$.

6 A special case

The necessary and sufficient conditions stated in the previous sections are, in general, not easy to use for testing the quasiconvexity (pseudoconvexity) of a quadratic function. Nevertheless, when $Q(x)$ has a particular structure, it is possible to obtain a characterization that is easy to test. In this section we shall consider the following class of functions:

$$f(x) = (a^T x + a_0)(b^T x + b_0) + c^T x. \tag{6.11}$$

Theorem 6.1 *Consider the function f in (6.11) and assume that the vectors a and b are linearly independent. Then, f is merely quasiconvex on a convex set* $S \subset \Re^n$ *with* *nonempty interior if and only if*

i) there exist $\alpha, \beta \in \Re$ such that $c = \alpha a + \beta b$;

ii) $S \subseteq \{x \in \Re^n : a^T x + a_0 + \beta \geq 0, b^T x + b_0 + \alpha \leq 0\}$ or
$S \subseteq \{x \in \Re^n : a^T x + a_0 + \beta \leq 0, b^T x + b_0 + \alpha \geq 0\}$.

Proof We have $f(x) = \frac{1}{2} x^T Q x + q^T x + q_0$, where

$$Q = ab^T + ba^T, \ q = b_0 a + a_0 b + c, \ q_0 = a_0 b_0.$$

The linear independence of a and b implies that $dim(ImQ) = 2$, where $ImQ = \{z = \mu_1 a + \mu_2 b, \mu_1, \mu_2 \in \Re\}$; consequently, $dim(kerQ) = n - 2$. Taking into account that the quadratic form $x^T Q x = 2a^T x b^T x$ is not constant in sign, we necessarily have a unique negative eigenvalue, i.e., $\nu_-(Q) = 1$. From Theorem 3.7, f is quasiconvex on S if and only if there exists $s \in \Re^n$ such that $Qs + q = 0$ and $S \subseteq s + T$ or $S \subseteq s - T$. We have $Qs + q = 0$ if and only if $q \in ImQ$ or, equivalently, if and only if there exist $\alpha, \beta \in \Re$ such that $c = \alpha a + \beta b$, i.e., if and only if $i)$ holds. Furthermore, $Qs + q = (b^T s + b_0 + \alpha)a + (a^T s + a_0 + \beta)b$ so that $Qs + q = 0$ if and only if $b^T s = -(b_0 + \alpha)$ and $a^T s = -(a_0 + \beta)$. By means of simple calculations we have $(x - s)^T Q (x - s) = 2(a^T x + a_0 + \beta)(b^T x + b_0 + \alpha)$ and thus $(x - s)^T Q (x - s) \leq 0$ if and only if ii) holds.
The proof is complete. □

Remark 6.1 *When a and b are linearly dependent, f is convex on $\Re^n$ or it is concave on $\Re^n$. In this last case f turns out to be quasiconvex on a convex set S if and only if $c = \alpha a$ and S is contained in one of the two half-spaces associated with the hyperplane given by the set of critical points of the function.*

Corollary 6.1 *Consider the function f in (6.11) and assume that a and b are linearly independent. Then, f is merely pseudoconvex on a convex set $S \subset \Re^n$ with nonempty interior if and only if*

i) there exist $\alpha, \beta \in \Re$ such that $c = \alpha a + \beta b$;

ii) $S \subseteq \{x \in \Re^n : a^T x + a_0 + \beta > 0, b^T x + b_0 + \alpha \leq 0\} \cup \{x \in \Re^n : a^T x + a_0 + \beta \geq$

$0, b^T x + b_0 + \alpha < 0\}$ *or*
$S \subseteq \{x \in \Re^n : a^T x + a_0 + \beta < 0, b^T x + b_0 + \alpha \geq 0\} \cup \{x \in \Re^n : a^T x + a_0 + \beta \leq 0, b^T x + b_0 + \alpha > 0\}$.

Proof The thesis follows from Lemma 5.2, taking into account that $\nabla f(x_0) = 0$ if and only if $x_0 \in \{x \in \Re^n : a^T x + a_0 + \beta = 0, b^T x + b_0 + \alpha = 0\}$. □

In order to characterize the quasiconvexity of f on $\Re^n_+$, we shall state, firstly, the following lemma.

Lemma 6.1 *Consider the matrix* $Q = ab^T + ba^T$. *Then,* $Q \leq 0$ *if and only if* $a \geq 0, b \leq 0$ *or* $a \leq 0, b \geq 0$.

Proof Obviously, if $a \geq 0, b \leq 0$ or $a \leq 0, b \geq 0$, then $Q \leq 0$. Conversely, since the thesis is trivial if $a = 0$ or $b = 0$, we shall consider the case $a \neq 0, b \neq 0$. Assume by contradiction, the existence of i, j such that $a_i > 0, a_j < 0$ and consider the submatrix $Q_{ij} = \begin{bmatrix} 2a_i b_i & a_i b_j + a_j b_i \\ a_i b_j + a_j b_i & 2a_j b_j \end{bmatrix}$. If $b_i b_j \neq 0$, $a_i b_i \leq 0$, $a_j b_j \leq 0$ imply that $b_i < 0, b_j > 0$, respectively, so that $a_i b_j + a_j b_i > 0$ and this is absurd. If $b_i = 0$ and $b_j \neq 0$, we have $Q_{ij} = \begin{bmatrix} 0 & a_i b_j \\ a_i b_j & 2a_j b_j \end{bmatrix}$, so that $a_j b_j \leq 0$ implies that $b_j > 0$ while $a_i b_j \leq 0$ implies that $b_j < 0$ and, once again, we get a contradiction. The case $b_j = 0$, $b_i \neq 0$ is analogous, so that the case $b_i = 0$, $b_j = 0$ remains to be considered. Let k be such that $b_k \neq 0$ and consider the submatrix $\begin{bmatrix} a_i b_k + a_k b_i & a_i b_j + a_j b_i \\ 2a_k b_k & a_k b_j + a_j b_k \end{bmatrix} = \begin{bmatrix} a_i b_k & 0 \\ 2a_k b_k & a_j b_k \end{bmatrix}$; $a_i b_k < 0$ implies that $b_k < 0$ while $a_j b_k < 0$ implies that $b_k > 0$ and this is absurd. Consequently, we have $a \geq 0$ or $a \leq 0$. For symmetric reasons, the components of b also have the same sign so that, necessarily, $a \geq 0,\ b \leq 0$ or $a \leq 0,\ b \geq 0$. The thesis is achieved. □

Theorem 6.2 *Consider the function* f *in (6.11) and assume that* a *and* b *are linearly independent. Then,* f *is merely quasiconvex on* $\Re^n_+$ *if and only if there exist* $\alpha, \beta \in \Re$ *such that* $c = \alpha a + \beta b$ *and one of the following conditions holds:*

i) $a \geq 0, \;\; b \leq 0, \;\; \alpha \leq -b_0, \;\; \beta \geq -a_0$*;*

ii) $a \leq 0, \;\; b \geq 0, \;\; \alpha \geq -b_0, \;\; \beta \leq -a_0$.

Proof From Theorem 4.4 we have $Q = ab^T + ba^T \leq 0$, while from Lemma 6.1 we have $a \geq 0$, $b \leq 0$ or $a \leq 0$, $b \geq 0$. The thesis follows from Theorem 6.1. □

Corollary 6.2 *Consider the function f in (6.11) and assume that a and b are linearly independent. Then, f is merely pseudoconvex on $\Re^n_+$ if and only if there exist $\alpha, \beta \in \Re$ such that $c = \alpha a + \beta b$ and one of the following conditions holds:*

i) $a \geq 0, \;\; b \leq 0$ *and* $\alpha < -b_0, \;\; \beta \geq -a_0$ *or* $\alpha \leq -b_0, \;\; \beta > -a_0$ *;*

ii) $a \leq 0, \;\; b \geq 0$ *and* $\alpha > -b_0, \;\; \beta \leq -a_0$ *or* $\alpha \geq -b_0, \;\; \beta < -a_0$.

Proof Referring to Theorem 5.7, it is sufficient to note that $b_0 a + a_0 b + c \neq 0$ if and only if $a_0 + \beta \neq 0$ or $b_0 + \alpha \neq 0$. □

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e-mail of the authors:

acambini@ec.unipi.it; lmartein@ec.unipi.it