A note on investment opportunities when the credit line is infinite

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Abstract

In the context of expected utility maximization for utilities defined on the whole real line, we define a new class of admissible strategies in terms of dynamic bounds on losses under the historical measure $\mathbb{P}$. More precisely, the loss control is given by a $\mathbb{P}$-martingale which is compatible with the preferences of the investor. The main result is the Ansel-Stricker-type Lemma 3.2 which shows that the admissible strategies are supermartingales under all sigma-martingale measures $\mathbb{Q}$ with finite relative entropy, therefore allowing for a duality theory for the optimization problem.

Key words: utility maximization – non locally bounded semimartingale – investment opportunity – supermartingale property – duality theory
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1 Introduction

We consider the "classic" frictionless model for the evolution of $d+1$ traded (liquid) assets. We assume that the risk-free asset (the money market account) is constant, or equivalently, that the interest rates are zero. This is a customary assumption in the utility maximization literature which does not restrict generality as long as interest rates are deterministic. The d risky assets, denoted by $S = (S^1, \dots, S^d)$, are assumed to be a (vector-valued, càdlàg) semimartingale on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ which verifies the usual assumptions of completeness and right-continuity.

The trading period is the interval $[0, T]$. A continuous-time trading (or investment) strategy is a predictable, S-integrable and $\mathbb{R}^d$-valued process H, where $H_t = (H_t^1, \dots, H_t^d)$ represents the number of shares of each risky asset held by the agent in the infinitesimal interval $[t, t+dt]$. If the strategy is (self)-financed by borrowing/investing in the risk-free money market, given the initial endowment x the (càdlàg) wealth process X of the investor evolves according to

$$X_t = x + \int_0^t H_s dS_s, \quad 0 \le t \le T.$$

We denote by $\mathcal{X}(x)$ the set of wealth processes with initial capital x, eventually subject to some restrictions depending on the preferences of the agent. The main goal of the present note is to define the precise restrictions needed in the definition of $\mathcal{X}(x)$, in the context of optimal investment. We model the preferences of an investor with time horizon T by expected utility from terminal wealth. A utility function is a function $U : (a, \infty) \to \mathbb{R}$ for *some* $-\infty \le a < \infty$, which is nondecreasing and concave. While further additional assumptions are usually made on the utility function U, they are not needed for our purposes. The optimal investment problem can be formulated as

$$u(x) := \sup_{X\in\mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right], \tag{1}$$

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where u denotes the value function, also known as indirect utility. In what follows, $X^*(x)$ or simply X^* indicates the optimal wealth process (if it exists) and H^* the optimal trading strategy, $X^* = x + \int H^* dS$.

As it is well known, doubling strategies or similar arbitrage schemes should be excluded from the class of investment strategies. Therefore, the very definition of the domain $\mathcal{X}(x)$ is a delicate issue. The nature of the utility function U itself leads to two different cases:

If $-\infty < a < \infty$, the agent tolerates only to lose a fixed amount of money (no more than $x - a$). Typical examples are the logarithmic or power utility. The natural admissibility condition is to require wealth processes to be uniformly bounded below by a, which means that the agent never exceeds his/her finite credit line (see e.g. [KS99], [BF08] and the discussions and bibliography cited there). Under additional hypotheses on the utility function U, in this case the optimization problem (1) can be solved by duality methods and the optimal strategy is admissible.

The case $a = -\infty$ corresponds to an infinite credit line. The archetypal example is the exponential utility, $U(x) = \frac{1}{\gamma} e^{-\frac{1}{\gamma}x}$. Here the definition of admissibility is a much more delicate issue, and this is *exactly where the contribution of the present note lies.* In Section 3, admissibility is defined in terms of a loss control which is a martingale under the historical measure $\mathbb{P}$ and satisfies a natural compatibility condition with the preferences of the investor. To the best of our knowledge, this is the first attempt to use a $\mathbb{P}$-martingale as a bound on losses of admissible strategies.

The paper is organized as follows: Section 2 reviews existing definitions of admissibility for $a = -\infty$, pointing out the technical difficulties related to this case. This review plays a double role: it motivates the introduction of our new admissible class $\mathcal{X}^{ad}(x)$ and allows us to compare it with the previously defined admissible classes. Section 3 contains the definition of $\mathcal{X}^{ad}(x)$ and its properties: the main result is Lemma 3.2 stating the so-called "supermartingale property". Section 4 concludes with a thorough comparison of the newly defined class with the literature. For the sake of completeness, some needed Lemmas are presented in the Appendix.

2 Existing definitions of admissible strategies when $a = -\infty$

The literature is roughly split into two main branches: economically reasonable definitions of strategies, stated in terms of bounds on the losses, versus the good mathematical definition, given through the so-called "supermartingale property" of the wealth processes. Our goal is to define a class of strategies/integrals which is a good compromise between these two directions, analyzed hereafter.

2.1 Admissibility defined by bounds on losses

The first direction focuses on the economic interpretation of admissibility of the strategies H, by introducing a pathwise loss control on the wealth processes $X = x + \int H dS$. In case S is locally bounded, Schachermayer [Sch01] defines as admissible the wealth processes (and relative strategies) which are uniformly bounded from below by some constant, in analogy to the definition when a is finite:

$$\mathcal{X}^{bb}(x) = \{X \mid (\exists)\ c > 0\ X_t \geq -c, \quad 0 \leq t \leq T\}.$$

This amounts to having a finite credit line, which is dependent on the strategy. The analysis in [Sch01], based on the above definition of admissibility, actually works for the more general case of S being *sigma-bounded*, namely when there exist a scalar positive integrand φ such that $\int \varphi dS$ is (well defined and) bounded. The concept of *sigma localization* was introduced in [Kal02], and sigma-boundedness was used in the context of utility maximization in [KS06].

Biagini and Frittelli [BF05, BF08] allow for possibly non locally bounded S, so the set $\mathcal{X}^{bb}(x)$ may be trivial. Therefore, if one is willing to invest in such a market, the admissibility condition must be relaxed. The solution proposed there is the following. Consider a control random variable W which is compatible with the preferences, that is it verifies the U-integrability condition

$$\mathbb{E}[U(-\alpha W)] > -\infty, \tag{2}$$

either for all constants $\alpha > 0$ or for (the weaker requirement of) some $\alpha > 0$. Denote by $\mathcal{W}$ the set of random variables W which satisfy (2). For a *fixed* $W \in \mathcal{W}$, define the class of admissible wealth processes

$$\mathcal{X}^W(x) = \{X \mid (\exists)\ c > 0 \ \ X_t \geq -cW \text{ for all } t \in [0, T]\}. \tag{3}$$

When $W = 1$, $\mathcal{X}^W(x) = \mathcal{X}^{bb}(x)$. As may be easily derived from the mathematical details provided in the Appendix, a combination of Ansel-Stricker Lemma [AS94] and Fenchel inequality shows that wealth processes $X \in \mathcal{X}^W(x)$ are supermartingales under any probability measure $\mathbb{Q}$ which is a sigma-martingale measure for S

$$\mathbb{Q} \in \mathcal{M}_\sigma = \{\mathbb{Q}' \ll \mathbb{P} \mid S \text{ is a sigma-martingale under } \mathbb{Q}'\}$$

and has finite V-entropy

$$\mathbb{Q} \in \mathcal{P}_V = \{\mathbb{Q} \ll \mathbb{P} \mid (\exists) y > 0, \quad \mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})] < \infty\}.$$

Here $V(y) = \sup_x\{U(x) - xy\}$ is the convex conjugate of U. The set $\mathcal{M}_\sigma \cap P_V$ is actually the dual domain of the optimization problem (1). If, for the same *fixed* $W \in \mathcal{W}$, some restrictions in terms of W are imposed on the jumps of S, (there exist a non trivial set of W-controlled strategies), then the dual domain $\mathcal{M}_\sigma \cap P_V$ satisfies a closure property, allowing for a complete duality theory.

In spite of the higher flexibility, compared to wealth bounded below by a constant, there is still no guarantee about the admissibility of the optimal process X^* - it may or may not be controlled from below by cW, see the examples in [BF05].

Remark 2.1. Even when S is locally (or sigma)-bounded, an alternative to $\mathcal{X}^{bb}(x)$ is

$$\mathcal{X}^U(x) = \left\{X \mid (\exists)\alpha > 0, \quad \mathbb{E}[U(-\alpha \sup_{0 \le t \le T}(X_t^-))] > -\infty\right\} = \cup_{W \in \mathcal{W}} \mathcal{X}^W(x). \tag{4}$$

Duality still works, since $\mathcal{M}_\sigma \cap P_V$ is closed. In addition, $\mathcal{X}^U(x)$ has a much better chance of capturing the optimal strategy than $\mathcal{X}^{bb}(x)$. As a matter of fact, this is exactly the case for the optimal investment problem in the Black-Scholes model with exponential utility: the maximal loss of the optimal strategy has a finite exponential moment but it is not bounded below. An evident drawback of the class $\mathcal{X}^U(x)$ is that the control is *not* dynamic, as the maximal loss is known only at the terminal time T.

2.2 Admissibility defined through the supermartingale property

The second direction focuses more on giving a mathematically good definition, in the sense that the optimal process will surely be admissible. The admissible strategies are defined as those strategies for which the wealth process is a supermartingale under all the probability measures in $\mathcal{M}_\sigma \cap P_V$ (or a martingale when U is exponential). For locally (or sigma) bounded S, the "supermartingale strategies" are used in the Six Authors' paper [6AU] and Kabanov and Stricker [KS02] for exponential U; in Schachermayer [Sch03] for general U and in the more recent Owen and Zitkovic [OZ07] for the case when there is random endowment. We recall that when S is locally bounded, $\mathcal{M}_\sigma$ are simply the local martingale measures for S, so that $\mathcal{M}_\sigma \cap P_V$ are the local martingale measures with finite entropy. The supermartingale strategies can be used also when S non locally bounded, as shown by Biagini and Frittelli [BF07].

The main point in favor of the supermartingale strategies is that the optimal process X^* belongs to this class, but there is a price to pay. Such a definition is not easy to interpret economically, as it is given in terms of the dual set of probability measures $\mathcal{M}_\sigma \cap P_V$. Moreover, realistic market models are incomplete and thus the description of the whole $\mathcal{M}_\sigma \cap P_V$ is often impossible. Consequently, checking admissibility with respect to this definition is practically unfeasible.

3 The new class of admissible strategies $\mathcal{X}^{ad}(x)$

As already mentioned, our goal is to define a new class of strategies/integrals which reconciles as much as possible these two different directions of the current literature. We are thus interested in a class of admissible strategies which:

1. has a clear financial interpretation, therefore is of the type in Subsection 2.1 (control on losses)
2. the control on losses is adapted (dynamic)

3. is larger than any economically reasonable class defined so far (for example in [Sch01], [BF05], [BF08]) to capture as much as possible the optimizer. In other words, we want a class of admissible strategies larger than the class $\mathcal{X}^U(x)$ defined in (4), even in the locally bounded case.

It turns out that **the good notion of admissibility is a loss control which is a martingale under the historical measure** $\mathbb{P}$, plus a compatibility condition with the preferences in the spirit of Biagini and Frittelli.

Definition 3.1. *The set of admissible processes $\mathcal{X}^{ad}(x)$ is the set of wealth processes*

$$X_t = x + \int_0^t H_s dS_s,$$

such that there exists a (positive) martingale W under the historical measure $\mathbb{P}$ such that

$$X_t \geq -W_t, \quad 0 \leq t \leq T,$$

and for some $\alpha > 0$

$$\mathbb{E}[U(-\alpha W_T)] > -\infty. \tag{5}$$

The martingale property of the dynamic loss control $W = (W_t)_t$ means that bounds on losses at earlier times are just conditional expectations under *historical measure* $\mathbb{P}$ of the bound on terminal loss. Condition (5) is the compatiblity with preferences mentioned above.

An immediate consequence of Definition 3.1 is that $\mathcal{X}^{ad}(x)$ is a convex cone of processes. This definition is dynamic, as the control is a process, and admissibility can be checked directly under $\mathbb{P}$. In addition, $\mathcal{X}^{ad}(x)$ is stable under stopping (an easy consequence of Jensen's inequality) and

$$\mathcal{X}^U(x) \subset \mathcal{X}^{ad}(x),$$

namely $\mathcal{X}^{ad}(x)$ includes any class described in Section 2.1. The result below states that $\mathcal{X}^{ad}(x)$ also enjoys the "super-martingale property":

Lemma 3.2. *Any $X \in \mathcal{X}^{ad}(x)$ is a supermartingale under all $\mathbb{Q} \in \mathcal{M}_\sigma \cap P_V$.*

Proof. Fix $\mathbb{Q} \in \mathcal{M}_\sigma \cap P_V$ and denote by Z its density process. Let $y > 0$ such that $\mathbb{E}[V(yZ_T)] < \infty$. An application of the Fenchel inequality at time t gives

$$U(-\alpha W_t) + \alpha y W_t Z_t \leq V(yZ_t),$$

so that

$$W_t Z_t \leq \frac{V(yZ_t) - U(-\alpha W_t)}{\alpha y}$$

Denoting by $M_t = \frac{V(yZ_t)-U(-\alpha W_t)}{\alpha y}$, Jensen's inequality implies that $M := (M_t)_{0 \leq t \leq T}$ is a submartingale under $\mathbb{P}$ that controls WZ:

$$0 \leq WZ \leq M.$$

Now, for each stopping time $\tau \leq T$,

$$\mathbb{E}_{\mathbb{Q}}[W_\tau 1_{\{W_\tau > c\}}] = \mathbb{E}[Z_\tau W_\tau 1_{\{W_\tau > c\}}] \leq \mathbb{E}[M_\tau 1_{\{W_\tau > c\}}].$$

Since $\mathbb{P}(W_\tau > c) \leq \mathbb{E}[W_\tau]/c \leq \mathbb{E}[W_T]/c$ and the set of random variables $(M_\tau)_\tau$ is uniformly integrable we obtain:

$$\lim_{c \to \infty} \sup_\tau \mathbb{E}_{\mathbb{Q}}[W_\tau 1_{\{W_\tau > c\}}] = 0.$$

In other words, $(W_\tau)_\tau$ is a uniformly integrable family under $\mathbb{Q}$. Since X is bounded below by W, $(X_\tau^-)_{\tau \leq T}$ is also a uniformly integrable family under $\mathbb{Q}$. Lemma 5.1 in the Appendix implies that X is a local martingale and a supermartingale under $\mathbb{Q}$. ☐

Remark 3.3. In case $\mathbb{Q} \sim \mathbb{P}$, the proof above can be made even shorter, as $\frac{M}{Z}$ is well defined, and a $\mathbb{Q}$-submartingale.

The supermartingale property is very desirable because of its economic budget implications.

Corollary 3.4. *The class $\mathcal{X}^{ad}(x)$ verifies a budget constraint with respect to all sigma-martingale measures compatible with the preferences, i.e. for all $\mathbb{Q} \in \mathcal{M}_\sigma \cap P_V$ and all $\tau \leq T$ stopping times,*

$$\mathbb{E}_{\mathbb{Q}}[X_\tau] \leq x. \tag{6}$$

The budget constraint (6), which is a consequence of our definition of admissibility, allows for an easy proof of the duality inequality

$$u(x) \leq \inf_{y>0}\{v(y) + xy\},$$

where v is the dual value function defined by

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{M}_\sigma} \mathbb{E}\left[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})\right], \quad y > 0.$$

which holds without any extra assumption on the utility function U, like Inada conditions on asymptotic marginal utility U' and Reasonable Asymptotic Elasticity of Kramkov and Schachermayer. However, in order to make the above duality work completely and to recover a primal optimizer, one must impose such conditions on U *plus* some restrictions on the jumps of S. These restrictions may be for example the requirement of local /sigma boundedness on S, or the existence of a "very integrable" loss control, i.e. one satisfying condition (2) for all $\alpha > 0$ (see [BF05], [BF08]).

We would like to point out that there is an extra advantage in considering the dynamic loss control in $\mathcal{X}^{ad}(x)$ compared to the static loss control used by Biagini and Frittelli (3). As it can be seen from Remark 2.1, the static control, apart from the measurability issues, imposes a stronger condition on the losses. In other words, usually the class $\mathcal{X}^{ad}(x)$ is stricly larger than $X^U(x)$. This kind of drawback is present also in Biagini [Bia04], where the problem of the admissible strategies for general, possibly non locally bounded, underlyings S was first addressed. There the loss control W^B is dynamic, but it is fixed, equal for all strategies, and pathwise increasing. An integral X is thus admissible in the sense of [Bia04] if for some $c > 0$

$$X_t \geq -cW_t^B$$

It is easy to check that such processes do belong to $\mathcal{X}^{ad}(x)$, since

$$X_t \geq -cW_t^B \geq -\mathbb{E}_t[cW_T^B]$$

because $W_T^B \geq W_t^B$ given the property of the control.

4 Conclusions

We have constructed a new class of admissible strategies which:

a- has a clear financial interpretation, with restrictions imposed in terms of dynamic pathwise loss controls

b- is larger than previous economically meaningful classes, namely $\mathcal{X}^U(x) \subset \mathcal{X}^{ad}(x)$

c- is (still) smaller than the whole class of "supermartingale strategies", so it may not contain the optimal strategy

d- in some important examples (like the Black-Scholes model with exponential utility, described in Remark 2.1) the optimizer is actually admissible.

The message of item c- should not be regarded as a negative result due to our specific choice of admissibility. In a general semimartingale model it is impossible to completely reconcile an economically meaningful definition of admissibility with a mathematically good definition, even if the traded assets are locally bounded. This point is made clear in a (counter)example of Schachermayer. In [Sch03] there is a financial market with two locally bounded risky assets, such that if admissibility in the sense of Subsection 2.2 is used, then the optimal strategy is to buy and hold the first asset, completely disregarding the second. However, if the second asset is withdrawn from the market then the old optimal strategy is not admissible any more, although it involves only the surviving asset. Therefore, we believe that Definition 3.1 goes the furthest possible in reconciling "economically meaningful" with "mathematically good" definitions of admissibility for the case of infinite credit line.

5 Appendix: an Ansel-Stricker-type lemma and consequences

Lemma 5.1. *(similar to Strasser [Stra])*
Let $(X_t)_{0\leq t\leq T}$ be a sigma martingale. The following are equivalent

1. *$(X_t)_{0\leq t\leq T}$ is a supermartingale (true, not local)*
2. *$(X^-_\tau)_{0\leq \tau\leq T}$ is u.i.*

If either of these conditions holds, then $(X_t)_{0\leq t\leq T}$ is also a local martingale

Proof. One direction is obvious, since the negative part of a supermartingale is a submartingale. In order to prove the other implications, we define $T_n = \inf\{t \mid |X_t| > n\}$ and consider the stopped process X^{T_n}. As the jump process ΔX^{T_n} of X^{T_n} verifies

$$\Delta X^{T_n}_t = X_{t\wedge T_n} - X_{t\wedge T_n -} \geq \begin{cases} -2n & \text{if } T_n > t \\ -(X_{T_n})^- - n & \text{if } T_n \leq t \end{cases}$$

it is clear that X^{T_n} is bounded from below by an integrable random variable, say $\Theta_n = -2n - (X_{T_n})^-$ (the integrability of the latter follows from the of uniform integrability of the negative parts of X along stopping time). By a version of the Ansel-Stricker Lemma (attached below as Lemma 5.2 for completeness), this means that X^{T_n} is a local martingale. It is well known that a local local martingale is a local martingale, so X is, indeed, a local martingale. Now, since the negative part of X is of class (D), it is an easy exercise involving Fatou's lemma for conditional expectations to show that X is actually a supermartingale. □

Lemma 5.2. ***(Variant of the Ansel-Stricker Lemma)*** *If X is a sigma-martingale and $X_t \geq -W, 0 \leq t \leq T$ where $W > 0$ is a random variable such that $\mathbb{E}[W] < \infty$ then*

1. *$(X_t)_{0\leq t\leq T}$ is a local martingale*
2. *$(X_t)_{0\leq t\leq T}$ is a super martingale*

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