Università degli Studi di Pisa
Dipartimento di Statistica e Matematica
Applicata all'Economia

Report n. 323

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Pisa, settembre 2009
- Stampato in Proprio –

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Admissible strategies in semimartingale portfolio selection

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September 10, 2009

Abstract

The choice of admissible trading strategies in mathematical modelling of financial markets is a delicate issue, going back to Harrison and Kreps [HK79]. In the context of optimal portfolio selection with expected utility preferences this question has been a focus of considerable attention over the last twenty years.

In this paper, we propose a novel notion of admissibility that has many pleasant features – admissibility is characterized purely under the objective measure P; the wealth of any admissible strategy is a supermartingale under all pricing measures; local boundedness of the price process is not required; strict monotonicity and/or strict concavity of the utility function are not necessary; the definition works both for utility functions defined on a half-line and for those that are finite on the whole $\mathbb{R}$; the definition encompasses both the classical mean-variance preferences and the monotone expected utility. Moreover, under very mild conditions *our class represents a minimal set containing simple strategies which also contains the optimizer.*

Acknowledgements Part of this research was conducted while the first author was visiting Collegio Carlo Alberto in Moncalieri, Turin, Italy in Spring 2009. Warm hospitality and financial support of the Collegio are gratefully acknowledged.

1 Introduction

A central concept of financial theory is the notion of a self-financing strategy H, whose wealth is expressed mathematically by the stochastic integral

$$x + H \cdot S_t := x + \int_0^t H_s dS_s,$$

where S is a semimartingale process representing discounted prices of d traded assets and x is the initial wealth. According to stochastic integration theory, there are some minimal requirements for the integral above to exist (see [Pr05]). The class of predictable processes H for which the integral exists is denoted by $L(S; P)$ or simply $L(S)$. It turns out, however, that for general S the class $L(S)$ is *not appropriate* for financial applications. Specifically, Harrison and Kreps [HK79] noted that when all trading strategies in $L(S)$ are allowed, arbitrage opportunities arise even in the

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standard Black-Scholes model. This is not a problem of the model – the reason is that the theory of stochastic integration operates with a set of integrands that is far too rich for such applications.

In the context of expected utility maximization a good set of trading strategies is a subset of $L(S)$ on which the utility maximization is well posed and which contains the optimizer. The search for a good definition of admissibility has proved to be a difficult task and it has evolved in two streams. For utility functions finite on a half-line, for example a logarithmic utility, the definition involves strategies whose wealth is bounded below by a constant, see [KS99, KS03]. For utility functions finite on the whole $\mathbb{R}$ this definition works only to a certain extent. Although the utility maximization over these strategies is well defined, some restrictions have to be imposed on S (S must be locally bounded) and the optimal wealth process itself may not be bounded from below.

A natural choice is to consider all strategies whose wealth is a martingale under all (suitably defined) pricing measures. This works well for exponential utility, see [DGRSSS02, KabStr02]. The seminal work of Schachermayer [Sch03] shows that in general the martingale class is too narrow and the optimal strategy only exists among strategies whose wealth is a *supermartingale* under all pricing measures. Thus, for utility functions finite on $\mathbb{R}$ the *supermartingale class* is now considered an appropriate notion of admissibility.

As transpires from the above discussion, admissibility is currently defined in a primal way for utility functions finite on $\mathbb{R}_+$ but for utilities finite on $\mathbb{R}$ the definition is dual, via pricing measures. A connection of sorts between the two approaches can be found in Bouchard et al. [BTZ04] who postulate that a strategy is admissible if the utility of its terminal wealth can be approximated in $L^1(P)$ by strategies whose wealth is bounded below. In this definition of admissibility not all strategies belong to the supermartingale class, but, crucially, the optimizer does. We note for completeness that the idea of $L^1(P)$ approximation of terminal utility was used already in Schachermayer [Sch01].

All of the papers cited above deal with *locally bounded price processes.* Biagini and Frittelli [BF07] show that Schachermayer's supermartingale class of strategies contains the optimizer also when S is not locally bounded. In a subsequent paper [BF08], they provide a unified treatment for utility functions finite on a half-line as well as those finite on the whole $\mathbb{R}$ in the unbounded case. The unified framework is based on strategies whose wealth is controlled below by an exogenously given random variable. The optimal strategy, however, may not be in this class and it is not clear whether it can be approximated by strategies with controlled losses. Another disadvantage is that the solution may in principle depend on the choice of the loss control.

The key point of the present paper is that we do not ask for approximation of terminal utility *only*, but we also require an approximation by simple integrands at intermediate times, in the spirit of Kallsen, cf. [ČK07, Definition 2.2]. The numerous advantages, mathematical and economic, of our definition have been anticipated in the abstract and they are thoroughly discussed in the main body of the paper. Here we mention only that our definition implies all admissible strategies are in the supermartingale class, that the optimizer belongs to this class under very mild conditions and, as a byproduct, we obtain an extremely compact proof of the supermartingale property of the optimal solution.

The paper is organized as follows. In Sections 2.1-2.3 we collect basic definitions from convex analysis, theory of Orlicz spaces and stochastic integration. Section 2.4 contains a new result on σ-localization. In Section 3.1 we discuss conditions imposed on the price process S. In Section 3.2 we define simple strategies and prove their "martingale property". In Section 3.3 we define the admissible strategies and prove their "supermartingale property". In Sections 4.1 and 4.2 we discuss the customary conditions of *reasonable asymptotic elasticity* and other related conditions used in the literature and we contrast them with a weaker Inada condition employed in this paper. The main result (Theorem 4.7) is stated in Section 4.3. In Section 5 we discuss the advantages of our framework compared to the existing literature.

2 Mathematical preliminaries

2.1 Utility functions

A utility function U is a proper, concave, non-decreasing, upper semi-continuous function. Its effective domain is

$$\operatorname{dom} U := \{x \mid U(x) > -\infty\}. \tag{1}$$

We assume that U is strictly increasing in a neighborhood of 0. Without loss of generality, suppose also $U(0) = 0$. Let $U(+\infty) := \lim_{x\to+\infty} U(x)$ and define

$$\overline{x} := \inf\{x \mid U(x) = U(+\infty)\}. \tag{2}$$

For strictly increasing utility functions $\overline{x} = +\infty$, but for truncated utility functions (which feature for example in shortfall risk minimization) one has $\overline{x} < +\infty$ and then $\overline{x}$ acts as a satiation point (bliss point) of the utility function.

The convex conjugate of U,

$$V(y) := \sup_{x\in\mathbb{R}}\{U(x) - xy\},$$

is a proper, convex, lower semi-continuous function, equal to $+\infty$ on $(-\infty, 0)$, and verifying $V(0) = U(+\infty)$. In the sequel we will often exploit the Fenchel inequality which is a simple consequence of the definition of V,

$$U(x) \le xy + V(y). \tag{3}$$

2.2 Orlicz spaces and the Orlicz space induced by U

Let Ψ be a *Young function*, that is an even, convex, lower semi-continuous, $[0, \infty]$-valued function with $\Psi(0) = 0$. Consider the corresponding Orlicz space

$$L^\Psi(P) = \{X \in L^0(P) \mid E[\Psi(c|X|)] < \infty \text{ for some } c > 0\}.$$

Orlicz spaces are generalizations of L^p spaces, since when $\Psi(x) = |x|^p, p \ge 1$, then $L^\Psi = L^p$ and if $\Psi(x) = +\infty I_{\{|x|>1\}}$ then $L^\Psi = L^\infty$.

The Morse subspace of L^Ψ, also called "Orlicz heart", is given by

$$M^\Psi(P) = \{X \in L^0(P) \mid E[\Psi(c|X|)] < \infty \text{ for all } c > 0\}.$$

In the context of this paper the Young function will be, from Section 3 onwards,

$$\hat{U}(x) := -U(-|x|),$$

meaning that the Orlicz space in consideration is generated by the lower tail of the utility function. For utility functions with lower tail which is asymptotically a power, say p, one has $L^{\hat{U}} = M^{\hat{U}} = L^p$. When U is exponential, say $U(x) = 1 - e^{-x}$, $\hat{U}(x) = e^{|x|} - 1$, and it is easy to check that $L^{\hat{U}} \supsetneq M^{\hat{U}} \supsetneq L^\infty$. For utility functions with half-line as their effective domain, such as $U(x) = \ln(1+x)$, one has $L^{\hat{U}} = L^\infty$ and $M^{\hat{U}} = \emptyset$.

Due to the link between U and $\hat{U}$ we have

$$X \in L^{\hat{U}} \text{ iff } E[U(-\alpha|X|)] > -\infty \text{ for some } \alpha > 0. \tag{4}$$

The reader interested in the general theory of Orlicz spaces is referred to the book by Rao and Ren [RR91].

2.3 Semimartingale distances

There are two standard norms in stochastic calculus. Let S be an $\mathbb{R}^d$-valued semimartingale on the filtered space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$ and let $S_t^* = \sum_{i=1}^d \sup_{0\le s\le t} |S_s^i|$ be the corresponding maximal process. For $p \in [1, \infty]$ let

$$\|S\|_{\mathscr{S}^p} := \|S_T^*\|_{L^p},$$

and denote the class of semimartingales with finite $\mathscr{S}^p$-norm also by $\mathscr{S}^p$. This definition is due to Meyer [M78]. We extend the definition slightly to allow for an arbitrary Orlicz space $L^\Psi(P)$,

$$\mathscr{S}^\Psi := \{\text{semimartingale } S \mid S_T^* \in L^\Psi\}.$$

Note for future use that $\mathscr{S}^\Psi$ is stable under stopping, that is if $S \in \mathscr{S}^\Psi$ and if τ is a stopping time, then the stopped process $S^\tau := (S_{\tau\wedge t})_t \in \mathscr{S}^\Psi$.

Following Protter [Pr05], for any special semimartingale S with canonical decomposition into local martingale part M and predictable finite variation part A, $S = S_0 + M + A$, we define the following semimartingale norm,

$$\|S\|_{\mathscr{H}^p} = \|S_0\|_{L^p} + \|[M,M]_T^{1/2}\|_{L^p} + \|\mathrm{var}(A)_T\|_{L^p},$$

where $\mathrm{var}(A)$ denotes the absolute variation of process A. The class of processes with finite $\mathscr{H}^p$-norm is denoted by $\mathscr{H}^p$. As usual we let

$$\mathscr{M}^p := \mathscr{H}^p \cap \mathscr{M},$$

where $\mathscr{M}$ is the set of uniformly integrable P-martingales.

2.4 Localization and beyond: σ-localization and $\mathcal{I}$-localization

Recall that for a given semimartingale S on $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$, $L(S)$ indicates the class of predictable and $\mathbb{R}^d$-valued, S-integrable processes H *under* P, while $H \cdot S$ indicates the integral process. When H is a scalar predictable process belonging to $\cap_{i=1}^d L(S^i)$ we write, with a slight abuse of notation, $H \cdot S$ for the process $(H \cdot S^1, \dots, H \cdot S^d)$.

Now, let $\mathscr{C}$ be some fixed class of semimartingales. The following methods of extending $\mathscr{C}$ appear in the literature:

i) We write $S \in \mathscr{C}_{\text{loc}}$ and say S is *locally* in $\mathscr{C}$, if there is a sequence of stopping times τ_n increasing to $+\infty$ such that each of the stopped processes S^{τ_n} is in $\mathscr{C}$.

ii) We write $S \in \mathscr{C}_\sigma$ and say S is *σ-locally* in $\mathscr{C}$, if there is a sequence of predictable sets D_n increasing to $\Omega \times \mathbb{R}_+$ such that each of the processes $I_{D_n} \cdot S$ is in $\mathscr{C}$.

iii) We write $S \in \mathscr{C}_\mathcal{I}$ and say S is *$\mathcal{I}$-locally* in $\mathscr{C}$, if there is some scalar process $\varphi \in \cap_{i=1}^d L(S^i)$, $\varphi > 0$ such that $\varphi \cdot S$ is in $\mathscr{C}$.

The first two items are standard (cf. [JS03, I.1.33], [Ka04]) while the third item is an ad hoc definition. By construction, for an arbitrary semimartingale class $\mathscr{C}$ one has $\mathscr{C}_\sigma \supseteq \mathscr{C}_{\text{loc}} \supseteq \mathscr{C}$. However it is not *a priori* clear what inclusions hold for $\mathscr{C}_\mathcal{I}$, apart from the obvious $\mathscr{C}_\mathcal{I} \supseteq \mathscr{C}$. Émery [E80, Proposition 2] has shown that when $\mathscr{C} = \mathscr{M}^p$ or $\mathscr{H}^p$, the following equalities hold

$$\mathscr{M}^p_\sigma = \mathscr{M}^p_\mathcal{I}, \qquad \mathscr{H}^p_\sigma = \mathscr{H}^p_\mathcal{I}, \qquad \text{for } p \in [1, +\infty). \tag{5}$$

The name $\mathcal{I}$-localization ($\mathcal{I}$ standing for integral) is probably a misnomer, since no localization procedure is involved. But we choose it since in Emery's result, $\mathcal{I}$-localization coincides with σ-localization. In general, however, $\mathscr{C}_\mathcal{I} \neq \mathscr{C}_\sigma$. Intuition suggests that the two localizations coincide whenever the primary class $\mathscr{C}$ is defined via some sort of integrability properties, as in the case above: martingale property and its generalizations, boundedness or more generally Orlicz integrability conditions on the maximal process. The next result in this direction appears to be new.

Proposition 2.1. *For any Orlicz space L^Ψ and the corresponding semimartingale class $\mathscr{S}^\Psi$ we have $\mathscr{S}^\Psi_\sigma = \mathscr{S}^\Psi_\mathcal{I}$.*

Proof. i) Assume $S \in \mathscr{S}^\Psi_\sigma$. There are predictable sets D_n increasing to $\Omega\times\mathbb{R}_+$ such that $(I_{D_n} \cdot S)^*_T \in L^\Psi$ for all n. Thus there is $\alpha_n \in (0,1]$ such that $0 \le b_n := E[\Psi(\alpha_n (I_{D_n} \cdot S)^*_T)] < +\infty$. Let

$$\varphi := \sum_n \eta_n I_{D_n}, \qquad \eta_n := \frac{c}{2^n} \frac{\alpha_n}{1+b_n}, \qquad c := 1/(\sum_n 2^{-n}(1+b_n)^{-1}).$$

Then,

$$E[\Psi((\varphi \cdot S)^*_T)] \le E[\Psi(\sum_n \eta_n (I_{D_n} \cdot S)^*_T)] \le \sum_n \frac{\eta_n}{\alpha_n} E[\Psi(\alpha_n (I_{D_n} \cdot S)^*_T)]$$

$$= \sum_n \frac{\eta_n b_n}{\alpha_n} \le c < 2(1+b_1), \tag{6}$$

where the first inequality follows from monotonicity of Ψ and $(\varphi\cdot S)^*_T \leq \sum_n \eta_n (I_{D_n}\cdot S)^*_T$. The second inequality follows from $\sum_n \eta_n/\alpha_n = 1$ and from the following simple pointwise argument, which ensures that, when $\sum_{n\geq 1}\alpha_n x_n$ is summable, $\Psi(\sum_{n\geq 1}\alpha_n x_n) \leq \sum_{n\geq 1}\alpha_n \Psi(x_n)$ whenever the weights α_n are nonnegative and the series $\sum_{n\geq 1}\alpha_n = 1$. The argument is: from convexity and $\Psi(0)=0$, $\Psi(\sum_{n=1}^N \alpha_n x_n) \leq \sum_{n=1}^N \alpha_n \Psi(x_n)$. As $\sum_{n=1}^N \alpha_n x_n \to \sum_{n\geq 1}\alpha_n x_n$, from lower semicontinuity of Ψ we get $\Psi(\sum_{n\geq 1}\alpha_n x_n) \leq \liminf_n \Psi(\sum_{n=1}^N \alpha_n x_n)$ and the lim inf is clearly dominated by $\sum_{n\geq 1}\alpha_n\Psi(x_n)$. By construction, $\varphi > 0$ and inequality (6) implies $S \in \mathscr{S}^\Psi_{\mathcal{I}}$.

ii) To prove the opposite inclusion, $\mathscr{S}^\Psi_\sigma \supseteq \mathscr{S}^\Psi_{\mathcal{I}}$, consider $\varphi > 0$ such that $\varphi\cdot S \in \mathscr{S}^\Psi$. Set $D_n = \{\frac{1}{n} < \varphi < n\}$, and note that $(D_n)_n$ is a sequence of predictable sets increasing to $\Omega\times\mathbb{R}_+$. In addition, $I_{D_n}\cdot S \in \mathscr{S}^\Psi_{\text{loc}}$ for all n since for $\tau^n_k = \inf\{t \mid (I_{D_n}\cdot S)^* > k\}$ we have

$$(I_{D_n}\cdot S^{\tau^n_k})^*_T \leq (I_{D_n}\cdot S^{\tau^n_k})^*_{T-} + |\Delta(I_{D_n}\cdot S^{\tau^n_k})^*_T| \leq k + |\Delta((I_{D_n}\frac{1}{\varphi})\cdot(\varphi\cdot S^{\tau^n_k}))^*_T| \leq k + n(\varphi\cdot S)^*_T \in L^\Psi.$$

This shows that $S \in (\mathscr{S}^\Psi_{\text{loc}})_\sigma$ since $D_n \uparrow \Omega\times\mathbb{R}_+$. As $\mathscr{S}^\Psi$ is stable under stopping, a result by Kallsen [Ka04, Lemma 2.1] ensures $(\mathscr{S}^\Psi_{\text{loc}})_\sigma = \mathscr{S}^\Psi_\sigma$, which completes the proof. □

3 The strategies

3.1 Conditions on S and simple strategies

Let S be a d-dimensional semimartingale which models the discounted evolution of d underlyings. As hinted in the introduction, to accommodate popular models for S, including exponential Lévy, we do not assume that S is locally bounded. However, to make sure that there is a sufficient number of well-behaved simple strategies we impose the following condition on S:

Assumption 3.1. *There is a sequence of stopping times τ_n increasing to $+\infty$ and $\alpha_n > 0$ such that*

$$E[U(-\alpha_n S^*_{\tau_n\wedge T})] > -\infty \text{ for all } n, \tag{7}$$

or, equivalently given (4), $S \in \mathscr{S}^{\hat U}_{\text{loc}}$.

The localization in Assumption 3.1 provides a substantial amount of flexibility. For example, in the Black-Scholes model $S \notin \mathscr{S}^{\hat U}$ when U stands for the exponential utility. On the other hand S is continuous and therefore locally bounded which means $S \in \mathscr{S}^\infty_{\text{loc}} \subseteq \mathscr{S}^{\hat U}_{\text{loc}}$ for any utility function satisfying our assumptions, including the exponential.

When operating under Assumption 3.1 we opt for the following definition of simple strategies.

Definition 3.2. *H is a* simple integrand *for S if it is of the form $H = \sum_{k=1}^N H_k I_{]T_{k-1},T_k]}$ where $T_1 \leq \cdots \leq T_N$ are stopping times dominated by some τ_n satisfying (7) and where each H_k is $\mathcal{F}_{T_{k-1}}$-measurable and bounded. The vector space of all simple integrands is denoted by $\mathcal{H}$.*

Every simple integral represents a buy-and-hold strategy over finitely many trading dates and it is thus a very natural financial object.

3.2 σ-martingale measures, σ-localization and modified simple strategies

To discuss a further weakening of Assumption 3.1 we now introduce dual asset pricing measures.

Definition 3.3. *$Q \ll P$ is a σ-martingale measure for S iff S is a σ-martingale under Q. The set of all σ-martingales measures for S is denoted by $\mathcal{M}$.*

The concept of σ-martingale measure was introduced to mathematical finance by Delbaen and Schachermayer [DS98]. When S is (locally) bounded, it can be shown that $\mathcal{M}$ coincides with the absolutely continuous (local) martingale measures for S (see e.g. Protter [Pr05, Theorem 91]). Therefore, σ-martingales are a natural generalization of local martingales in the case when S is not locally bounded and the elements of $\mathcal{M}$ which are equivalent to P can be used as arbitrage-free pricing measures for the derivative securities whose payoff depends on S. The recent book [DS06] contains an extensive treatment of the financial applications of this mathematical concept.

Suppose that Assumption 3.1 is too restrictive and that we only have $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\mathrm{loc}}^{\hat{U}}$. By Proposition 2.1, $\mathscr{S}_\sigma^{\hat{U}} = \mathscr{S}_\mathcal{I}^{\hat{U}}$. Fix an $\mathcal{I}$-localizing strategy $\varphi \in \cap_{i=1}^d L(S^i; P)$, $\varphi > 0$ so that $\varphi \cdot S \in \mathscr{S}^{\hat{U}}$. Thanks to Émery's equality (5) the set of absolutely continuous σ-martingale measures for S is the same as the set of σ-martingale measures for $S' := \varphi \cdot S$. In fact, $Q \ll P$ is a σ-martingale measure for S by (5) if and only if there exists a Q-positive, predictable process $\psi_Q \in \cap_{i=1}^d L(S^i; Q)$ such that $\psi_Q \cdot S$ is a Q-martingale. And this happens if and only if $\psi'_Q \cdot (\varphi \cdot S)$ is a Q-martingale, where $\psi'_Q = \frac{\psi_Q}{\varphi}$.

Therefore, we could consider as the main building blocks in our utility maximization problem those strategies which are simple relative to the new underlying process S'. Since the sets of σ-martingale measures for S and S' are the same, this implies that the *dual* problem to the utility maximization is unchanged. Under suitable conditions (see the statement of the main Theorem 4.8), we would end up with the same optimizer, irrespective of a specific choice of the $\mathcal{I}$-localizing strategy φ. Using this construction we could replace Assumption 3.1 with the following:

Condition 3.4. *$S \in \mathscr{S}_\sigma^{\hat{U}}$. In other words, there exists a $\varphi \in \cap_{i=1}^d L(S^i; P)$, $\varphi > 0$ such that $(\varphi \cdot S)_T^* \in L^{\hat{U}}$.*

The $\mathcal{I}$-localizing strategy φ from Condition 3.4 plays an important role in the work of Biagini [Bia04] where the maximal process $(\varphi \cdot S)^*$ is taken as a dynamic loss control for the strategies in the utility maximization problem. Within setups of increasing generality in Biagini and Frittelli [BF05, BF08] φ gives rise to so-called *suitable* and *(weakly) compatible* loss control variables $W := (\varphi \cdot S)_T^*$.

When operating under the weaker Condition 3.4 we would apply Definition 3.2 to the $\mathcal{I}$-localized price process $\varphi \cdot S$ instead of S, with the advantage that in such case one would not need the stopping times τ_n at all. The price we pay for the extra flexibility in Condition 3.4 is that simple strategies can no longer be interpreted as buy-and-hold vis-à-vis the original price process S.

3.3 Properties of simple strategies

In either case, simple integrals have good mathematical properties with respect to σ-martingale measures with *finite relative entropy.*

Definition 3.5. *We say that probability Q has finite relative entropy, and write $Q \in P_V$, if there is $y_Q > 0$ such that*

$$v_Q(y_Q) := E[V(y_Q \frac{dQ}{dP})] < \infty. \tag{8}$$

This definition differs from the classic formulation of finite relative entropy, also called finite V-divergence, which requires $y_Q = 1$ (see Liese and Vajda [LV87], Kramkov and Schachermayer [KS99], Bellini and Frittelli [BeF02], Goll and Rüschendorf [GR01] and basically all the contemporary literature on utility maximization).

Lemma 3.6. *The wealth process $X = H \cdot S$ of every simple strategy $H \in \mathcal{H}$ is a uniformly integrable martingale under all $Q \in \mathcal{M} \cap P_V$.*

Proof. Since $H \in \mathcal{H}$, the maximal functional X^* verifies $X_T^* \leq cS^*_{\tau_n \wedge T}$ for some constant $c > 0$ and some τ_n of the localizing sequence of stopping times from Assumption 3.1. Now, this inequality together with (3.1) implies

$$E[U\left(-\frac{\alpha_n}{c} X_T^*\right)] > -\infty.$$

For any fixed $Q \in \mathcal{M} \cap P_V$, the Fenchel inequality $U(x) - xy \leq V(y)$ applied with $x = -\frac{\alpha_n}{c} X_T^*, y = y_Q \frac{dQ}{dP}$ gives

$$U(-\frac{\alpha_n}{c} X_T^*) + \frac{\alpha_n}{c} X_T^* y_Q \frac{dQ}{dP} \leq V(y_Q \frac{dQ}{dP}),$$

whence

$$0 \leq \frac{\alpha_n}{c} y_Q X_T^* \frac{dQ}{dP} \leq V(y_Q \frac{dQ}{dP}) - U(-\frac{\alpha_n}{c} X_T^*),$$

and therefore X_T^* is in $L^1(Q)$. As Q is a σ-martingale probability for S, X is also a Q-σ-martingale. Since its maximal process is integrable, X is in fact a Q-uniformly integrable martingale (see Protter [Pr05, Chapter IV-9]). □

In financial terms, each $Q \in \mathcal{M} \cap P_V$ represents a pricing rule that assigns a correct price to every simple self-financing strategy.

3.4 Admissible integrands and integrals

Since simple integrands are very basic tools, it is clear that their class may not contain the solution of the utility maximization problem. Therefore, we consider an extension given in terms of suitable limits of strategies in $\mathcal{H}$.

Definition 3.7. *$H \in L(S)$ is an* admissible integrand *if $U(H \cdot S_T) \in L^1(P)$ and if there exists an approximating sequence $(H^n)_n$ in $\mathcal{H}$ such that:*

i) $H^n \cdot S_t \to H \cdot S_t$ in probability for all $t \in [0, T]$;

ii) $U(H^n \cdot S_T) \to U(H \cdot S_T)$ *in* $L^1(P)$.

We denote the set of all admissible integrands by $\overline{\mathcal{H}}$.

The two requirements above are quite natural assumptions if considered *separately*. In fact, fix any integral $H \cdot S$. By the very definition of stochastic integral, one can always find a sequence of simple integrands that approximate $H \cdot S$ as required by item i): $H^n = \sum_{i=1}^n H_{T_i^n} I_{]T_i^n, T_{i+1}^n]}$ whenever $(T_i^n)_i$ is a finite random partition of $[0,T]$ via stopping times, with mesh going to zero in probability when $n \to +\infty$. Item ii), on the other hand ensures that utility of an admissible strategy can be approximated by simple strategies. Definition 3.7 combines these *two desirable approximation features* together.

While for $H \in \mathcal{H}$ the wealth process $H \cdot S$ is always a martingale, the following result shows that $\overline{\mathcal{H}}$ is a subset of the supermartingale class of strategies introduced by [Sch03]:

$$\mathcal{H}^{\sup} := \{H \in L(S) \mid H \cdot S \text{ is a local martingale and a supermartingale under any } Q \in \mathcal{M} \cap P_V\}. \tag{9}$$

Proposition 3.8. $\overline{\mathcal{H}} \subseteq \mathcal{H}^{\sup}$.

Proof. Let $X = H \cdot S$ for some $H \in \overline{\mathcal{H}}$ and let $(X^n := H^n \cdot S)_n$ with $H^n \in \mathcal{H}$ be its approximating sequence. Fix a $Q \in \mathcal{M} \cap P_V$ and a corresponding scaling y_Q as in Definition 3.5. Item *i)* applied at time T implies $(X_T^n)^-$ converge in P and therefore Q-probability to X_T^-. Moreover, Fenchel inequality gives

$$U(X_T^n) - V(y_Q \frac{dQ}{dP}) \leq X_T^n y_Q \frac{dQ}{dP}.$$

From item *ii)*, the left hand side above converges in $L^1(P)$, whence the family $(Y^n)_n, Y^n := (X_T^n)^- \frac{dQ}{dP}$ is P-uniformly integrable, so $((X_T^n)^-)_n$ is Q-uniformly integrable. Uniform integrability plus convergence in probability ensures $(X_T^n)^- \to X_T^-$ in $L^1(Q)$. By passing to a subsequence, we can construct

$$W^Q := \sum_n |(X_T^{n+1})^- - (X_T^n)^-| \in L^1(Q),$$

and the associated uniformly integrable Q-martingale Z^Q with $Z_T^Q = W^Q$. Since $X_T^n \geq -W^Q$ and process X^n is a Q-martingale for all n by Lemma 3.6 we obtain

$$X_t^n = E_Q[X_T^n \mid \mathcal{F}_t] \geq -E_Q[W^Q \mid \mathcal{F}_t] = Z_t^Q, \tag{10}$$

so that the sequence X^n is controlled from below by the Q-martingale Z^Q. Therefore by Delbaen and Schachermayer compactness result [DS99, Theorem D] (in the version stated in Section 5, [DS98]) there exists a limit càdlàg supermartingale $\widetilde{V}$ to which a sequence $K^n \cdot S$, where K^n is a suitable convex combinations of tails $K^n \in conv(H^n, H^{n+1}, \ldots)$, converges Q-almost surely for every rational time $0 \leq q \leq T$. By item *i)*, $(X^n)_t$ converges in P-probability to X_t for every t, thus $K^n \cdot S_t$ converges to X_t for every t as well. Therefore $\widetilde{V}$ coincides Q-a.s. with X on rational times, and since X is also càdlàg as it is an integral, X and $\widetilde{V}$ are indistinguishable, so that X is a Q-supermartingale. By assumption Q is a σ-martingale measure, so $X = H \cdot S = (\frac{1}{\varphi}H) \cdot (\varphi \cdot S)$ where $\varphi > 0$ and $(\varphi \cdot S)$ is a Q-martingale. As $X \geq Z^Q$, Ansel and Stricker lemma [AS94, Corollaire 3.5] implies that X is a local Q-martingale. □

The importance of the supermartingale property will become clear in Section 4.

Remark 3.9. Proposition 3.8 would go through if one replaced $\mathcal{H}$ in Definition 3.7 with the set

$$\mathcal{H}' = \{H \in L(S) \mid H \cdot S \geq c \text{ for some } c \in \mathbb{R}\}, \tag{11}$$

as in Schachermayer [Sch01], when S is locally bounded, or more generally the set of those $H \in L(S)$ whose losses are in some sense well controlled as in Biagini and Frittelli [BF05, BF08] or Biagini and Sîrbu [BS09].

An application of the Ansel and Stricker lemma [AS94, Corollaire 3.5] shows that wealth processes for strategies in $\mathcal{H}'$ are local martingales and supermartingales under any $Q \in \mathcal{M} \cap P_V$ – but not martingales in general. However, $\mathcal{H}' \subseteq \mathcal{H}^{\text{sup}}$ is exactly what is needed from a *mathematical* point of view in the utility maximization problem.

We summarize the advantages of $\overline{\mathcal{H}}$ over current definitions of admissibility in the following list:

a) Definition 3.7 is *primal*. No pricing measures come into play, and admissibility can thus be checked under P.

b) The present definition is *dynamic*, that is the whole wealth process, rather than just its terminal value, is involved in the definition of $\overline{\mathcal{H}}$. As a result all admissible strategies are in the supermartingale class. In contrast, in [BTZ04, Section 4] only *the optimal strategy* is guaranteed to be in the supermartingale class.

c) The loss controls required in the proof of the supermartingale property are generated endogenously. This provides a great deal of flexibility and ensures that the optimizer is in $\overline{\mathcal{H}}$ under very mild conditions, milder than the conditions assumed to obtain the supermartingale property of the optimizer in [Sch03, BF07].

d) Approximation by simple strategies is built into the definition of admissibility, it does not have to be deduced separately (cf. [Str03]).

e) The desirable properties above hold without any technical assumptions on U. It can be finite on $\mathbb{R}$ or only on a half-line; bounded above or not, or even truncated; neither strict monotonicity, strict convexity nor differentiability are required.

f) Our definition is compatible with the existing definition of admissibility for non-monotone quadratic preferences, see Remark 3.10 below. We have therefore found a good notion of admissibility which encompasses both the classical mean-variance preferences *and* monotone expected utility.

Remark 3.10. For the purpose of this remark only, we admit non-monotone U. Let $U(x) := x - x^2/2$, which represents a normalized quadratic utility. In such case, $H \in \overline{\mathcal{H}}$ if and only if there is a sequence of $H^n \in \mathcal{H}$ such that: 1) $H^n \cdot S_t \to H \cdot S_t$ in probability for all $t \in [0,T]$ and 2) $H^n \cdot S_T \to H \cdot S_T$ in $L^2(P)$. In other words, when U is quadratic the admissibility criterion in Definition 3.7 coincides with the notion of admissibility pioneered by Kallsen in [ČK07, Definition 2.2], which inspired our work. Since 1) above and i) in Definition 3.7 coincide, the only thing to prove is that ii) in our definition is equivalent to 2) above:

- $\Rightarrow$ Suppose first $H \in \overline{\mathcal{H}}$. The $L^1(P)$ convergence of utilities implies $E[U(X_T^n)] \to E[U(X_T)]$ so that X_T^n are uniformly bounded in $L^2(P)$. Since $L^2(P)$ is a reflexive space there is a sequence of convex combinations of tails of $(X_T^n, X_T^{n+1}, \ldots)$, say $\widetilde{X}_T^n$, which converges in $L^2(P)$ to a square integrable random variable which necessarily is $X_T = H \cdot S_T$ thanks to Def 3.7-i). By considering the corresponding convex combinations of strategies, which are again simple, we obtain the existence of an approximating sequence à la Kallsen for H.
- $\Leftarrow$ Conversely, let $X = H \cdot S$ be an integral approximated à la Kallsen by simple integrals $(X^n)_n$. $L^1(P)$ convergence of the utilities $U(X_T^n)$ to $U(X_T)$ is then a consequence of the Cauchy-Schwartz inequality.

4 Optimal trading strategy is in $\overline{\mathcal{H}}$

The optimal investment problem can be formulated over $\mathcal{H}$ or over $\overline{\mathcal{H}}$, respectively,

$$u_{\mathcal{H}}(x) := \sup_{H \in \mathcal{H}} E[U(x + H \cdot S_T)], \tag{12}$$

$$u_{\overline{\mathcal{H}}}(x) := \sup_{H \in \overline{\mathcal{H}}} E[U(x + H \cdot S_T)]. \tag{13}$$

Alongside, we consider an auxiliary complete market utility maximization problem, fixing an arbitrary $Q \in \mathcal{M} \cap P_V$,

$$u_Q(x) := \sup_{X \in L^1(Q), E_Q[X] \le x} E[U(X)]. \tag{14}$$

The value functions $u_{\mathcal{H}}(x), u_{\overline{\mathcal{H}}}(x), u_Q(x)$ are also known as indirect utilities. The next lemma is an easy consequence of the definition of $\overline{\mathcal{H}}$, of the Fenchel inequality (3) and of the supermartingale property of the strategies in $\overline{\mathcal{H}}$. The proof is omitted.

Lemma 4.1. *For any $x > \inf \operatorname{dom} U$ and for any $Q \in \mathcal{M} \cap P_V$*

$$u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x) \le u_Q(x). \tag{15}$$

This innocent Lemma is quite important. In fact, $\mathcal{H}$ is a vector space and this means that it is stable for all the financial operations like: investing in long or short position over (arbitrary) quantities of a certain strategy; risk diversification, i.e. building a portfolio by taking convex combinations of a given set of strategies. On the other hand, the set $\overline{\mathcal{H}}$ is not a vector space in general, unless U has lower tail which is a power as e.g. in Remark 3.10. The reason lies in the specific convergence required in item *ii)* of Definition 3.7. This is the price to pay for *a primal characterization* of a minimal set of admissible strategies capturing the optimizer.

4.1 Reasonable asymptotic elasticity and Inada conditions

It is well known in the literature that the existence of an optimizer is not guaranteed yet, nor in the set $\overline{\mathcal{H}}$, larger than $\mathcal{H}$, nor in $\mathcal{H}^{\text{sup}}$, the larger set of all. An additional condition has to be imposed, essentially to ensure that the expected utility functional $X \to E[U(X)]$ is upper semicontinuous with respect to some weak topology on terminal wealths.

Kramkov and Schachermayer were the first to address this issue in [KS99, Sch01] for regular U, that is utilities that are strictly increasing, strictly concave and differentiable in the interior of their effective domain. To the end of recovering an optimizer they introduced the celebrated reasonable asymptotic elasticity condition on U (RAE(U)),

$$\limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1, \tag{16}$$

$$\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1, \text{ when } U \text{ is finite on } \mathbb{R}, \tag{17}$$

as a necessary and sufficient condition to be imposed on the utility U *only*, regardless of the probabilistic model. This condition is now very popular, see [OŽ09, RS05, Sch03, B02] just to mention a few contributions.

In subsequent work, in the context of utilities finite on $\mathbb{R}_+$, Kramkov and Schachermayer [KS03] put forward less restrictive conditions*, *imposed jointly on the model and on the preferences*, in order to recover the optimal terminal wealth. Here they work under assumptions which are equivalent to the Inada conditions on the indirect utility $u_{\mathcal{H}'}$, where the class $\mathcal{H}'$ is defined in (11):

$$\lim_{x\to+\infty} u_{\mathcal{H}'}(x)/x = 0 \qquad \text{and} \qquad \lim_{x\to 0_+} u_{\mathcal{H}'}(x)/x = +\infty. \tag{18}$$

We follow the same logic, but we only impose the Inada condition at $+\infty$ on the complete market indirect utility u_Q, for *one* $Q \in \mathcal{M} \cap P_V$.

Assumption 4.2. *There exists $Q \in \mathcal{M} \cap P_V$ such that the corresponding indirect utility u_Q verifies the Inada condition at $+\infty$,*

$$\lim_{x\to+\infty} u_Q(x)/x = 0. \tag{19}$$

We provide a detailed discussion of Assumption 4.2 and its relation to RAE(U) (16, 17) and the Inada condition (18) in Section 5.1. The results of the next section go in that direction.

*The interested reader is referred also to the recent Biagini and Guasoni [BG09] for counterexamples and a different, *relaxed* framework that allows optimal terminal wealth to be a measure and not only a random variable.

4.2 Complete market duality

In this section we study a complete market $Q \in P_V$ and hence no model for S is required. Here we provide, among other results, an alternative characterization of the Inada condition (19) in terms of the relative entropy of Q.

Lemma 4.3. *Consider the function v_Q defined in* (8). *For any fixed $Q \in P_V$ and $x > \inf \operatorname{dom} U$,*

$$u_Q(x) = \min_{y \geq 0}\{xy + v_Q(y)\} < +\infty. \tag{20}$$

The proof of Lemma 4.3 follows standard arguments and it is given in Section 4.4.

Corollary 4.4. *Fix $Q \in P_V$. The following statements are equivalent:*

i) u_Q verifies the Inada condition at $+\infty$: $\lim_{x\to+\infty} u_Q(x)/x = 0$;

ii) there is $y_Q > 0$ such that

$$v_Q(y) = E[V(y\frac{dQ}{dP})] < +\infty \textit{ for all } y \in (0, y_Q]. \tag{21}$$

Proof. ($\Leftarrow$) Suppose that $v_Q(y)$ is finite in a right neighborhood of 0. By Fenchel inequality, $E[U(X)] - E[y\frac{dQ}{dP}X] \leq E[V(y\frac{dQ}{dP})]$ for all $X \in L^1(Q)$ so that $u_Q(x) \leq xy + v_Q(y)$ for all $y > 0$. Fixing y one obtains $\lim_{x\to+\infty} u_Q(x)/x \leq y$ and on letting $y \to 0$ equation (19) follows.
($\Rightarrow$) For a given $x > \inf \operatorname{dom} U$, select one dual minimizer in (20) and denote it by y_x. Since $u_Q(x) = xy_x + v_Q(y_x)$, $v_Q(y_x)$ is finite. Now, the chain of inequalities

$$u_Q(x) = xy_x + v_Q(y_x) \overset{Jensen}{\geq} xy_x + V(y_x) \geq xy_x \geq 0$$

holds for any $x > 0$ as V is nonnegative. Dividing by x and sending x to $+\infty$, (19) implies $\lim_{x\to+\infty} y_x = 0$. Finiteness of v_Q over the set $\{y_x\}_x$, whose closure contains 0, and convexity of v_Q finally imply v_Q is finite in the interval $(0, y_Q]$, with y_Q from (8). □

The next proposition contains a novel characterization of the "no utility-based arbitrage" condition, $u_Q(x) < U(+\infty)$ when Q has finite entropy. Bliss utility $U(+\infty)$ cannot be reached if the initial capital x is smaller than the satiation point $\bar{x}$ and viceversa.

Proposition 4.5. *For $Q \in P_V$ and $x > \inf \operatorname{dom} U$ the following statements are equivalent:*

i) $x < \bar{x}$;

ii)

$$u_Q(x) = \min_{y>0}\{xy + v_Q(y)\} < U(+\infty). \tag{22}$$

Proof. ($\Leftarrow$) $U(x) \le u_Q(x) < U(+\infty)$ implies $x < \bar{x}$.
($\Rightarrow$) $x < \bar{x}$ implies $U(x) < U(+\infty)$. Let $Z := y_Q dQ/dP$, with y_Q from (8). When $U(+\infty) = V(0) = +\infty$ there is nothing to prove in view of (20). Consider therefore the remaining case $0 < U(+\infty) = V(0) < +\infty$. Function $f(y) := V(y) + xy$ is convex and by Rockafellar [R70, Theorem 23.5] it attains its minimum at $\hat{y} := U'_-(x) > 0$ with $f(\hat{y}) = V(\hat{y}) + x\hat{y} = U(x)$. Convexity then gives

$$f(y) \le f(0) - y\frac{f(0) - f(\hat{y})}{\hat{y}} = U(+\infty) - y\frac{U(+\infty) - U(x)}{\hat{y}} \qquad \text{for } y \in [0, \hat{y}],$$
$$f(y/k) \le f(0) + \frac{f(y) - f(0)}{k} \le U(+\infty) + \frac{f(y)}{k} \qquad \text{for } k \ge 1, y \ge 0.$$

For $k \ge 1$ these estimates imply

$$\begin{aligned} E[f(Z/k)] &= E[f(Z/k)1_{\{Z \le k\hat{y}\}}] + E[f(Z/k)1_{\{Z > k\hat{y}\}}] \\ &\le U(+\infty) - \frac{1}{k}\left(\frac{U(+\infty) - U(x)}{\hat{y}} E[Z1_{\{Z \le k\hat{y}\}}] - E[f(Z)1_{\{Z > k\hat{y}\}}]\right), \end{aligned}$$

whereby for sufficiently large k we have $E[f(Z/k)] < U(+\infty) = V(0)$ which completes the proof, since then any dual minimizer y_x in (22)will verify $E[V(y_x \frac{dQ}{dP})] \le E[f(Z/k)] < V(0) = U(+\infty)$. □

Remark 4.6. Proposition 4.5 should be contrasted with an example by Schachermayer [Sch01, Lemma 3.8], where the author constructs an arbitrage-free complete market with unique pricing measure Q for which $u_Q(x) \equiv U(+\infty)$, while U is strictly increasing and bounded above (and therefore it satisfies the Inada condition at $+\infty$). This is possible because the measure Q in question does not belong to P_V.

4.3 The main result

For U finite on a half line it would be rather straightforward to recover the known results of [KS03] (primal optimizer, etc.) given the current assumptions. The results would follow exactly as in [KS03] modulo the "unified framework" translation terms as shown in [BF08]. Therefore, from now on, we focus on the case where

U is finite on $\mathbb{R}$.

Assumption 4.7. *For any x such that $u_{\mathcal{H}}(x) < U(+\infty)$, the following dual relation holds:*

$$u_{\mathcal{H}}(x) = \min_{Q \in \mathcal{M} \cap P_V} u_Q(x) = \min_{y \ge 0, Q \in \mathcal{M} \cap P_V} \{xy + v_Q(y)\}. \tag{23}$$

This requirement amounts to ask for the existence of a dual representation of the utility maximization problem which consists of a minimization over probability measures. It is made to keep the presentation simple. In fact, while the existence of a dual representation follows quite easily (see [BF08] and Remark 4.9), in general the right hand side of (23) (the dual problem) may contain singular parts. Specifically, the dual problem consists of a constrained minimization over the dual space $(L^{\hat{U}})^*$. Any $z \in (L^{\hat{U}})^*$ admits a unique decomposition $z = z_r + z_s$, where the regular term

z_r is a measure absolutely continuous with respect to P but the singular term z_s is *not* a measure, not even a finitely additive one. In fact, z_s is null over L^∞: $z_s(f) = 0$ if $f \in L^\infty$. Thanks to Assumption 4.7, such unpleasant singular parts do not show up. In Remark 4.9 below there are also very mild sufficient conditions that enable a dual minimization over probability measures only.

Theorem 4.8. *Under Assumptions 4.2 and 4.7, for any initial wealth x such that $u_{\mathcal{H}}(x) < U(+\infty)$:*

a) $u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x) = \min_{y>0, Q\in\mathcal{M}\cap P_V}\{xy + E[V(y\frac{dQ}{dP})]\}$.

Any optimal dual couple is indicated by $(\hat{y}, \hat{Q})$, dependence on x is understood. The lack of uniqueness of the optimal dual couple is due to the lack of strict convexity of V, which in turn is due to the lack of strict concavity of U.

b) There exists a claim $\hat{f}$, not unique in general, that realizes the optimal expected utility, in the sense that

$$E[U(\hat{f})] = u_{\mathcal{H}}(x).$$

In case $U(+\infty) < +\infty$, there exists a unique modification of $\hat{f}$, still denoted in the same way, such that

$$\hat{f} \leq \bar{x},$$

where $\bar{x} \in (0, +\infty]$ is the satiation point of U.

c) $\hat{f}$ has the following properties:

i) $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} = V(\hat{y}\frac{d\hat{Q}}{dP})$ *and* $\{\hat{f} = \bar{x}\} = \{\frac{d\hat{Q}}{dP} = 0\}$. *Moreover,*

$$\hat{f} \in L^1(Q) \text{ and } E_Q[\hat{f}] \leq x \text{ for all } Q \in \mathcal{M}\cap P_V$$

while $E_{\hat{Q}}[\hat{f}] = x$ for any dual optimizer $\hat{Q}$. In case U is strictly concave, V is strictly convex and both the solutions of primal and dual problem are unique. If in addition U is differentiable, the unique solutions $\hat{f}, \hat{y}, \hat{Q}$ satisfy $\hat{y}\frac{d\hat{Q}}{dP} = U'(\hat{f})$.

ii) $\hat{f}$ can be approximated by terminal values $X_T^n = (H^n \cdot S)_T$, $H^n \in \mathcal{H}$, in the sense that

$$U(x + X_T^n) \overset{L^1(P)}{\rightarrow} U(\hat{f}) \tag{24}$$

and

$$X_T^n \overset{L^1(\hat{Q})}{\rightarrow} \hat{f} \tag{25}$$

iii) There exists an integral representation $\hat{f} = x + \widehat{H}\cdot S_T$ with $\widehat{H} \in L(S;\hat{Q})$, and $\widehat{H}\cdot S$ is a $\hat{Q}$-martingale. When $\hat{Q} \sim P$, this representation holds also under P, $\widehat{H} \in \overline{\mathcal{H}}$ and thus the utility maximization problem $u_{\overline{\mathcal{H}}}(x)$ admits a maximum

$$u_{\overline{\mathcal{H}}}(x) = \max_{H\in\overline{\mathcal{H}}} E[U(x + H\cdot S_T)] = E[U(x + \widehat{H}\cdot S_T)].$$

Proof. Item a) follows directly from Assumption 4.7, from the straightforward identity $u_{\mathcal{H}}(x) = u_{\hat{Q}}(x)$, from $u_{\mathcal{H}}(x) < U(+\infty)$ and from the dual formula (22) in Proposition 4.5. Let us fix a couple $\hat{y}, \hat{Q}$ of dual minimizers. The proof of item b) is articulated in several points.

b1) Select a maximizing sequence $(k_n)_n, k_n = K^n \cdot S_T, K^n \in \mathcal{H}$ so that $E[U(x+k_n)] \uparrow u_{\mathcal{H}}(x)$.

b2) The sequence $(k_n)_n$ is bounded in $L^1(\hat{Q})$. In fact, as $(k_n)_n$ is maximizing, $(E[U(k_n)])_n$ is bounded below. And since $(k_n)_n$ are terminal wealths from strategies in $\mathcal{H}$ and $Q \in \mathcal{M}$, $E_Q[k_n] = 0$. Now $L^1(\hat{Q})$-boundedness would be easy to prove, using Fenchel inequality (3), in case U bounded from above. In the general case it follows from the auxiliary Proposition 4.11, which is in turn a consequence of Assumption 4.2.

b3) $L^1(\hat{Q})$ boundedness of $(k_n)_n$ enables the application of Komlòs Theorem, so that there exists a sequence of convex combinations of tails of $(k_n)_n$: $(f_n)_n$, with $f_n \in conv(k_n, k_{n+1}, \ldots)$, that converges $\hat{Q}$ a.s. to a certain r.v. $f \in L^1(\hat{Q})$. As $\mathcal{H}$ is a vector space, these f_n are terminal values of simple integrals $X^n = H^n \cdot S$, $f_n = H^n \cdot S_T$. By concavity, the f_n are still maximizers, i.e. $E[U(x+f_n)] \uparrow u_{\mathcal{H}}(x)$.

b4) If $\hat{Q} \sim P$, then the f obtained above is a well defined element of $L^0(\Omega, \mathcal{F}_T, P)$. Unfortunately, when U is bounded above, it may happen that $\hat{Q}$ is only absolutely continuous, with $P(d\hat{Q}/dP = 0) > 0$. If this is the case, redefine f uniquely in the following way

$$fI_{\{\frac{d\hat{Q}}{dP}>0\}} + \bar{x}I_{\{\frac{d\hat{Q}}{dP}=0\}}$$

We now show that $f = \hat{f}$, i.e. the claim just obtained is an optimizer in the sense that $u_{\mathcal{H}}(x) = E[U(f)]$. To simplify notation, here and in the rest of the proof we assume w.l.o.g. $x = 0$. Fix a y such that $v_{\hat{Q}}(y) < +\infty$. For fixed n, Fenchel pointwise inequality gives

$$U(f_n) - f_n y\frac{d\hat{Q}}{dP} \leq V(y\frac{d\hat{Q}}{dP})$$

Let us split the analysis on the two sets, $A = \{\frac{d\hat{Q}}{dP} > 0\}$ and its complement A^c. On A, $f_n \to f$ P-a.s. and therefore $(U(f_n) - f_n y\frac{d\hat{Q}}{dP})I_A$ converges P-a.s. to $(U(f) - fy\frac{d\hat{Q}}{dP})I_A$. On I_{A^c},

$$(U(f_n) - f_n y\frac{d\hat{Q}}{dP})I_{A^c} = U(f_n)I_{A^c} \leq U(+\infty)I_{A^c} = U(\bar{x})I_{A^c} = (U(f) - yf\frac{d\hat{Q}}{dP})I_{A^c}$$

where we use the convention $+\infty \cdot 0 = 0$.

Taking these two results into account, and given the integrability of $V(y\frac{d\hat{Q}}{dP})$, Fatou Lemma applies and

$$u_{\mathcal{H}}(0) = \limsup_n E[U(f_n) - f_n y\frac{d\hat{Q}}{dP}] \leq E[\limsup_n (U(f_n) - f_n y\frac{d\hat{Q}}{dP})] \leq E[U(f) - fy\frac{d\hat{Q}}{dP}] \quad (26)$$

The latter term by Fenchel inequality is also dominated by $E[V(y\frac{d\hat{Q}}{dP})]$. Therefore, taking the infimum over $y > 0$

$$u_{\mathcal{H}}(0) \leq \inf_{y>0}\{E[U(f)] - yE_{\hat{Q}}[f]\} \leq E[V(\hat{y}\frac{d\hat{Q}}{dP})] \tag{27}$$

but the inner inf is finite iff $E_{\hat{Q}}[f] \leq 0$, in which case it equals $E[U(f)]$. In the previous displayed relation the inequalities are in fact equalities, so $E[U(f)] = u_{\mathcal{H}}(0)$ whence f is the desired $\hat{f}$. Since $\limsup_n(U(f_n) - f_n y\frac{d\hat{Q}}{dP}) \leq (U(\hat{f}) - \hat{f}y\frac{d\hat{Q}}{dP})$ and the inequalities in (26) are also equalities for $y = \hat{y}$, on A^c one has $\limsup_n U(f_n) = U(\hat{f}) = U(+\infty)$. Therefore, modulo the passage to a subsequence converging to the limsup, still denoted by $U(f_n)$, $U(f_n)I_{A^c} \to U(\hat{f})I_{A^c}$, whence globally $U(f_n) \to U(\hat{f})$ P-a.s.

Item c).

i) The Fenchel optimal relation $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} = V(\hat{y}\frac{d\hat{Q}}{dP})$ follows from (27). There the inequalities are in fact equalities, and this plus Fenchel inequality $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} \leq V(\hat{y}\frac{d\hat{Q}}{dP})$ in turn implies that two sides must coincide. By construction $\hat{f} = \bar{x}$ when $\frac{d\hat{Q}}{dP} = 0$, but the Fenchel optimal relation and $\hat{y} > 0$ easily imply that the converse is also true: when $\hat{f} = \bar{x}$, then $\frac{d\hat{Q}}{dP} = 0$, so $\{\hat{f} = \bar{x}\} = \{\frac{d\hat{Q}}{dP} = 0\}$ (equality holds P-a.s.). As a consequence, $E_Q[\hat{f}] = 0$.

To prove that $\hat{f} \in L^1(Q)$ and $E_Q[\hat{f}] \leq 0$ for all $Q \in \mathcal{M} \cap P_V$, we only need a slight modification of the arguments in item b4). Given that $U(f_n) \to U(\hat{f})$ and $f_n I_A \to \hat{f} I_A$ P-a.s. and that U is continuous on $\mathbb{R}$, necessarily $\liminf_{n\to+\infty} f_n I_{A^c} \geq \bar{x} I_{A^c} = \hat{f} I_{A^c}$. Fix $Q \in \mathcal{M} \cap P_V$. Then, $(f_n)_n$ is $L^1(Q)$ bounded: $E_Q[f_n] = 0$ and $(E[U(f_n)])_n$ is bounded below, so Proposition 4.11 apllies. The Fatou lemma yields $E[|f|] \leq E[\liminf_n |f_n|] \leq \liminf_n E[|f_n|] \leq const$, so $f \in L^1(Q)$. By the Fenchel inequality,

$$U(f_n) - f_n y\frac{dQ}{dP} \leq V(y\frac{dQ}{dP})$$

so that the Fatou lemma can be applied for any y such that $v_Q(y) < \infty$ and

$$\begin{aligned}\limsup_n E[U(f_n) - f_n y\frac{dQ}{dP}] &\leq E[\limsup_n(U(f_n) - f_n y\frac{dQ}{dP})] \\ &= E[U(\hat{f}) - \liminf_n f_n y\frac{dQ}{dP}] \leq E[U(\hat{f}) - \hat{f}y\frac{dQ}{dP}].\end{aligned}$$

Now exactly as in the last part of item b4) we obtain an analogy of (27) with Q instead of $\hat{Q}$ and even if in this case the inequalities do not boil down to equalities, the conclusion $E_Q[\hat{f}] \leq 0$ is still valid.

Finally, the results when U is strictly concave and differentiable follow now from the pointwise identity $U(x) - xU'(x) = V(U'(x))$.

ii) Since $U(f_n) \to U(\hat{f})$ P-a.s., the L^1 convergence of the utilities is equivalent to show uniform integrability of $(U(f_n))_n$. Given the convergence of the expected utility, $E[U(f_n)] \uparrow E[U(\hat{f})]$, an argument "à la Scheffé" shows* that the uniform integrability of $(U(f_n))_n$ is equivalent to uniform integrability of any of the two families $(U^-(f_n))_n$, $(U^+(f_n))_n$. $U(0) = 0$ and strict monotonicity of U in a neighborhood of 0 imply $U^-(f_n) = -U(-f_n^-)$ and $U^+(f_n) = U(f_n^+)$. Suppose by contradiction that the family $(U^+(f_n))_n \equiv (U(f_n^+))_n$ is *not* uniformly integrable, and proceed as in [KS03, Lemma 1]. Given the supposed lack of uniform integrability, there exist disjoint measurable sets $(A_n)_n$ and a constant $\alpha > 0$ such that

$$E[U(f_n^+)I_{A_n}] \geq \alpha.$$

Set $g_n = \sum_{i=1}^n f_i^+ I_{A_i}$ and fix a $Q \in \mathcal{M} \cap P_V$ satisfying Assumption 4.2. $(f_n)_n$ is $L^1(Q)$ bounded by Proposition 4.11 and clearly $E_Q[g_n] \leq nC$ where C is a positive bound on the $L^1(Q)$ norms of the sequence $(f_n)_n$. In addition, $E[U(g_n)] \geq n\alpha$ because the $(A_n)_n$ are disjoint. Therefore,

$$\frac{u_Q(nC)}{nC} \geq \frac{E[U(g_n)]}{nC} \geq \frac{\alpha}{C} > 0$$

and passing to the limit when $n \uparrow \infty$ the conclusion contradicts the Inada condition (19). So the family $(U^+(f_n))_n$ is uniformly integrable, and $(U(f_n))_n$ as well, which means $U(f_n)$ tends in $L^1(P)$ to $U(\hat{f})$.
To see that $f_n \to \hat{f}$ in $L^1(\hat{Q})$, from $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} = V(\hat{y}\frac{d\hat{Q}}{dP}) \geq U(f_n) - f_n\hat{y}\frac{d\hat{Q}}{dP}$ the difference $U(\hat{f}) - U(f_n) - (\hat{f} - f_n)\hat{y}\frac{d\hat{Q}}{dP}$ is nonnegative and has P-expectation which tends to zero. Henceforth such difference is $L^1(P)$ convergent to 0, which, thanks to $L^1(P)$ convergence of $U(f) - U(f_n)$, yields $L^1(P)$ convergence to 0 of $(\hat{f} - f_n)\frac{d\hat{Q}}{dP}$.

iii) Recall that X^n are all $\hat{Q}$ *uniformly integrable* martingales by Lemma 3.6. Moreover, $\hat{Q}$ is a σ-martingale measure for S, so $X^n = H^n \cdot S = (H^n \frac{1}{\varphi_{\hat{Q}}}) \cdot (\varphi_{\hat{Q}} \cdot S)$, where $M = \varphi_{\hat{Q}} \cdot S$ is a $\hat{Q}$ martingale and $\varphi_{\hat{Q}} > 0$ $\hat{Q} - a.s.$. The convergence (25) permits a straightforward application of a celebrated result by Yor [Yor78] on the closure of stochastic integrals, which gives an integral representation with respect to M of the limit $\hat{f}$ under $\hat{Q}$, $\hat{f} = H^* \cdot M_T = \widehat{H} \cdot S_T$ and $\widehat{X} := \widehat{H} \cdot S$ is also a $\hat{Q}$-uniformly integrable martingale.

When $\hat{Q} \sim P$ then the representation $\widehat{X} = \widehat{H} \cdot S$ holds also under P. To show $\widehat{H} \in \overline{\mathcal{H}}$, note we have already proved (24) so we only need convergence in P-probability of the wealth process at intermediate times. The convergence in (25) and the martingale property of $X^n, \widehat{H} \cdot S$ under $\hat{Q}$ imply

$$E_{\hat{Q}}[|X_t^n - \widehat{H} \cdot S_t|] = E_{\hat{Q}}\left[|E[X_T^n - \widehat{H} \cdot S_T \mid \mathcal{F}_t]|\right] \overset{Jensen}{\leq} E_{\hat{Q}}[|X_T^n - \widehat{H} \cdot S_T|].$$

Therefore, for any t, $X_t^n \to \widehat{H} \cdot S_t$ in $L^1(\hat{Q})$ and henceforth in $\hat{Q}$-probability, which is equivalent to convergence in P-probability. Thus, $\widehat{H} \in \overline{\mathcal{H}}$ follows.

*The a.s. convergence and convergence of expected values do not necessarily imply convergence in L^1 if the variables are not positive (or uniformly bounded below by a r.v. in L^1): take e.g. $g_n, n \geq 2$, defined on $[0,1]$ as n on $[0, 1/n[$, $-n$ on $]1 - 1/n, 1]$ and 0 in $[1/n, 1 - 1/n]$. $g_n \to 0$ dx a.s. and $\int g_n(x)dx = 0$, but $(g_n)_n$ do not converge in $L^1(dx)$. So, some extra work is needed here.

□

Remark 4.9. [Bia04], [BF05] and [BF08] provide a mild sufficient condition joint on S and U for (23) to hold. Re-formulated in our terms, it requires that the maximal process S^* is locally in the Orlicz heart $M^{\hat{U}}$, meaning the sequence of stopping times τ_n in Assumption 3.1 can be chosen so that (7) holds for all $\alpha > 0$ - or, if Condition 3.4 is used, $(\varphi \cdot S)^*_T \in M^{\hat{U}}$. *Note that this is not a restriction at all if U has lower tail behaving like a power, since then $L^{\hat{U}} = M^{\hat{U}}$*. However, in general $L^{\hat{U}} \supsetneq M^{\hat{U}}$, so the consequence of such stronger compatibility between S and U is that terminal values of integrals from $\mathcal{H}$ lie in the Orlicz heart $M^{\hat{U}}$ (see also [BF08] for more details). In turn this implies that the quantities in the minimization domain on the right hand side in (23) can be identified with random variables - and, suitably normalized, are probabilities.

4.4 Auxiliary results

Proof of Lemma 4.3. $u_Q(x) < +\infty$ follows from Fenchel inequality and from finite entropy of Q: if X satisfies $E_Q[X] \le x$, $E[U(X)] \le xy_Q + v_Q(y_Q)$, with y_Q from Definition 3.5.

The utility maximization problem $\sup_{E_Q[X]\le x} E[U(X)]$ can be rewritten over the utility-induced Orlicz space $L^{\hat{U}}(P)$ defined in (2.2). This can be done because: i) the supremum will be reached over those X such that $E[U(X)]$ is finite, so that $-X^- \in L^{\hat{U}}(P)$; ii) if X^- satisfies $E[U(-X^-)]$ is finite, then the truncated sequence $X_n = X \wedge n$ is also in the Orlicz space and by Fatou Lemma in the limit it delivers the same expected utility from X; iii) $L^{\hat{U}}(P) \subseteq L^1(Q)$, which follows from $Q \in P_V$, from (4) and Fenchel inequality (this also implies Q is in the topological dual of $L^{\hat{U}}$). Therefore, $u_Q(x) = \sup_{X\in L^{\hat{U}}, E_Q[X]\le x} E[U(X)]$. On $L^{\hat{U}}$, the concave functional $I_U(X) := E[U(X)]$ is proper:

$$X \in L^{\hat{U}} \Rightarrow X \in L^1(P) \text{ so that } E[U(X)] \overset{\text{Jensen}}{\le} U(E[X]) < +\infty.$$

Moreover, I_U has a continuity point which belongs to the maximization domain $D = \{X \in L^{\hat{U}} \mid E_Q[X] \le x\}$. This is more subtle to check, but it can be proved that the set

$$\mathcal{B} := \{X \in L^{\hat{U}} \mid E[U(-(1+\epsilon)X^-)] > -\infty \text{ for some } \epsilon > 0\},$$

coincides with the interior of the proper domain of I_U (see [BFG08, Lemma 4.1] modulo a sign change), where I_U is automatically continuous by the Extended Namioka Theorem (see e.g. [BF09]). Then, as $x > \inf \operatorname{dom} U$, the constant x is in $\mathcal{B} \cap D$.
The dual formula (20) is thus a consequence of Fenchel Duality Theorem [Bre83, Chapter 1], of the fact that the polar set of the constraint $C := \{X \mid E_Q[X] \le x\} \supseteq -L^\infty_+$, i.e. the set $\{\mu \in (L^{\hat{U}})^* \mid \mu(X) \le x \,\forall X \in C\}$, by the Bipolar Theorem is the positive ray $\{yQ \mid y \ge 0\}$, and of the expression of the convex conjugate $(I_U)^*$ of I_U over the variables $y\frac{dQ}{dP}$: $(I_U)^*(y\frac{dQ}{dP}) = E[V(y\frac{dQ}{dP})] = v_Q(y)$. □

Lemma 4.10. *Suppose f is a concave proper increasing function satisfying the Inada condition $\lim_{x\to+\infty} f(x)/x = 0$. Then for every $\epsilon > 0$ there is $\lambda \in [0,\epsilon)$ and $c \in \mathbb{R}$ such that $f(x) \le c + \lambda x$ for all $x \in \mathbb{R}$.*

Proof. Wlog assume $f(0) > -\infty$. By [R70, Theorem 24.1] f'_- is a decreasing function and therefore $\xi := \lim_{x\to\infty} f'_-(x) \geq 0$ exists. For $x \geq 0$, by [R70, Corollary 24.2.1], $f(x) = f(0) + \int_0^x f'_-(s)ds \geq f(0) + \xi x$ whereby the assumption $\lim_{x\to\infty} f(x)/x = 0$ implies $\xi = 0$. Consequently, for any $\epsilon > 0$ there exists $\widetilde{x} > 0$ with $0 \leq \lambda := f'_-(\widetilde{x}) < \epsilon$. By concavity $f(x) \leq f(\widetilde{x}) + \lambda(x - \widetilde{x})$ and the proof is completed on taking $c = f(\widetilde{x}) - \lambda\widetilde{x}$. □

Proposition 4.11. *Suppose $\widetilde{Q} \in P_V$ is such that $u_{\widetilde{Q}}$ verifies the Inada condition* (19). *Suppose $(k_n)_n$ is a sequence of random variables such that $E[U(k_n)]$ is bounded below and $E_{\widetilde{Q}}[k_n]$ is bounded above. Then, $(k_n)_n$ is $L^1(\widetilde{Q})$-bounded. Moreover, $(k_n)_n$ is $L^1(Q)$-bounded for any other $Q \in P_V$ (not necessarily satisfying the Inada condition) for which the sequence of expectations $(E_Q[k_n])_n$ is bounded above.*

Proof. In this proof $c_i, i = 1, \dots 4$, refers to a real constant. By hypothesis there is $y_{\widetilde{Q}} > 0$ such that $v_{\widetilde{Q}}(y_{\widetilde{Q}}) = c_1 < +\infty$ whereby the Fenchel inequality implies

$$E[U(-k_n^-)] \leq c_1 - y_{\widetilde{Q}} E_{\widetilde{Q}}[k_n^-]. \tag{28}$$

Since $u_{\widetilde{Q}}$ satisfies the Inada condition, by Lemma 4.10 there are constants $0 \leq \lambda < y_{\widetilde{Q}}, c_2$ such that $u_{\widetilde{Q}}(x) \leq c_2 + \lambda x$ for all x, whence letting $x_n := E_{\widetilde{Q}}[k_n^+]$ we have for all n

$$E[U(k_n^+)] \leq u_{\widetilde{Q}}(x_n) \leq c_2 + \lambda x_n. \tag{29}$$

On combining (28) and (29) we obtain

$$E[U(k_n)] \leq (c_1 + c_2) + \lambda E_{\widetilde{Q}}[k_n] - (y_{\widetilde{Q}} - \lambda) E_{\widetilde{Q}}[k_n^-]. \tag{30}$$

By assumption, $(E_{\widetilde{Q}}[k_n])_n$ is bounded above and $(E[U(k_n)])_n$ is bounded below, whereby one concludes from (30) and from $y_{\widetilde{Q}} - \lambda > 0$ that $(E_{\widetilde{Q}}[k_n^-])_n$ is bounded. Boundedness from above of the expectations $(E_{\widetilde{Q}}[k_n])_n$ ensures $(E_{\widetilde{Q}}[k_n^+])_n$ is also bounded, whence $L^1(\widetilde{Q})$-boundedness follows.

If Q is another probability in P_V, then (28) holds also for Q, *mutatis mutandis*:

$$E[U(-k_n^-)] \leq c_3 - y_Q E_Q[k_n^-]$$

Moreover, from (29) and boundedness in $L^1(\widetilde{Q})$ of k_n, $E[U(k_n^+)] \leq c_4$. Therefore, an inequality simpler than (30) shows up

$$E[U(k_n)] \leq (c_3 + c_4) - y_Q E_Q[k_n^-],$$

whence $(E_Q[k_n^-])_n$ is bounded. If $(E_Q[k_n])_n$ is additionally bounded above, as before we get $(E_Q[|k_n|])_n$ is also bounded. □

5 Connections to literature

5.1 Sufficient conditions

It may seem surprising that no condition is imposed on the behavior of u_Q (or U) at the left extremum of the domain. However, a closer inspection of proofs in [KS03] for U whose effective domain is a half-line shows that no condition at the left extremum of the domain of the indirect utility is needed for the existence of a primal optimizer. What seems indeed surprising is that the same holds true also for utilities U finite on the whole $\mathbb{R}$.

Remark 5.1. Since $u_Q(x) \geq u_{\mathcal{H}}(x) \geq U(x)$, and $U(x) > 0$ for $x > 0$, Assumption 4.2 implies an identical Inada condition both on the indirect utility $u_{\mathcal{H}}$ and also on the original utility U at $+\infty$.

In view of Remark 5.1 Assumption 4.2 is stronger than the condition $\lim_{x\to+\infty} u_{\mathcal{H}'}(x)/x = 0$ in [KS03] when U is finite on a half line.

At the same time, to the best of our knowledge Assumption 4.2 is strictly weaker than any other assumption used in the current literature for U finite on $\mathbb{R}$. In fact, in the cited works the typical assumption is RAE (16), (17), which implies $v_Q(y) < +\infty$ for all $y > 0$ and for all $Q \in P_V$ by [Sch01, Corollary 4.2], whence Assumption 4.2 necessarily holds. In the non-smooth utility case studied by Bouchard et al., equivalent asymptotic elasticity conditions are imposed on the Fenchel conjugate V,

$$\lim_{y\to 0_+} \frac{|V'_-(y)|y}{V(y)} < +\infty, \qquad \lim_{y\to +\infty} \frac{|V'_+(y)|y}{V(y)} < +\infty. \tag{31}$$

These again imply $v_Q(y) < +\infty$ for all $y > 0$ and for all $Q \in P_V$, see [BTZ04, Lemma 2.3].

On the other hand, Biagini and Frittelli [BF05, BF08] do not require RAE, but instead assume that only for the $Q \in \mathcal{M} \cap P_V$ $v_Q(y)$ is finite, for all $y > 0$, which is weaker than RAE but clearly stronger than Assumption 4.2 as shown in Corollary 4.4. Therefore Assumption 4.2 seems the best for a unified treatment of utility maximization problems, regardless of the domain of U.

5.2 Characterization of the optimal solution

As a general comment, in items b) and c) above we provide a nice and natural approximation result both for the value function $u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x)$ and for the optimizer $\hat{f}$. This approximation holds under very mild conditions both on U (which may lack strict monotonicity and strict concavity) and on the underlying price process S, which is allowed to be non locally bounded. In this respect we extend and substantially simplify the proof of results in Bouchard et al. [BTZ04] who work with fairly general U but require S locally bounded.

5.2.1 $\hat{Q}$ equivalent to P

In the mathematically pleasant case $\hat{Q} \sim P$ (see Remark 5.2 for a list of sufficient conditions for $\hat{Q} \sim P$) our approximation improves the current literature: [Sch01], [KabStr02], [Str03], [OŽ09], [BTZ04] all assume S locally bounded. Moreover, when $\hat{Q} \sim P$ we provide a sequence of simple strategies which approximate the optimal strategy. Approximation by simple strategies has so far been shown only for exponential utility. In fact, Kabanov and Stricker [KabStr02] show approximation by strategies with wealth bounded below while Stricker [Str03, Theorem 5] shows approximation by simple strategies, in both cases for locally bounded S and for expected utility only – not in the stronger sense of $L^1(P)$ convergence given here. In contrast, Schachermayer [Sch01] proves an approximation similar to (24) for the optimal solution via integrals bounded from below. He considers locally bounded S and regular U, satisfying RAE and this is the first article to show in such generality an integral representation for $\hat{f} = \widehat{H} \cdot S_T$ when $\hat{Q} \sim P$. Moreover, in [Sch03] $\widehat{H}$ is shown to be in the supermartingale class of strategies through a (hard) contradiction argument.

In a more general setup, we show here the supermartingale property of $\widehat{H}$ in a very natural way, as $\widehat{H} \in \overline{\mathcal{H}}$, and provide its approximation via simple integrands.

Remark 5.2 *(On sufficient conditions for $\hat{Q} \sim P$).* The following two conditions are well-established in the literature. First, utility function unbounded above implies $\hat{Q} \sim P$. This is straightforward, as $V(0) = U(+\infty) = +\infty$ while $E[V(\hat{y}\frac{d\hat{Q}}{dP})]$ must be finite. Second, when U is strictly monotone but bounded, a sufficient condition is the existence of a $Q^* \sim P$ in $\mathcal{M} \cap P_V$. From c-i) and strict monotonicity of U in fact one has $\hat{f} = +\infty$ on $\frac{d\hat{Q}}{dP} = 0$ and since $E_{Q^*}[\hat{f}] \leq x$ and $Q^* \sim P$, the set $\{\frac{d\hat{Q}}{dP} = 0\}$ must be (Q^* and P)-negligible.

5.2.2 $\hat{Q}$ not equivalent to P

In the less appealing case when $\hat{Q}$ is not equivalent to P, which may happen only when U is bounded above, we also improve the known literature. Note that all the results in the Theorem still hold true, apart the last part of item c-iii). If U is strictly monotone (a typical example is when the utility is exponential) an approximation result for $\hat{f}$ via terminal values of integrals under P was first shown by Acciaio [A05], under the following technical conditions:

- U is differentiable, monotone, strictly concave, and it satisfies RAE (16, 17),
- S is locally bounded,
- the stopping times of the filtration are predictable.

Acciaio builds a sequence of integrals $\widetilde{H}_n \cdot S_T$, whose expected utility tends to the optimum, and which satisfies $(x + \widetilde{H}_n \cdot S_T) \to \hat{f}$ P-a.s.

Our setup allows us to remove the technical conditions above and to prove the stronger convergence in (24), $U(x + f_n) \overset{L^1(P)}{\to} U(\hat{f})$, with $f_n = H^n \cdot S_T, H^n \in \mathcal{H}$, which clearly implies convergence of expectations. And when U is invertible, as in [A05], by passing to a subsequence if necessary, $x + f_n \to \hat{f}$ P-a.s.

When U is not strictly monotone, that is $\overline{x} < +\infty$, the sufficient conditions for $\hat{Q} \sim P$ known in the monotone case do not work; here typically $\hat{Q}$ is *not* equivalent to P even when there are equivalent probabilities in $\mathcal{M} \cap P_V$. And as U is not strictly monotone, we are unable to approximate $\hat{f}$ pointwise under P, but still (24) and (25) hold true, plus the integral representation of $\hat{f}$ under $\hat{Q}$. We thus extend results of Bouchard et al. [BTZ04] to non-locally bounded S under a weaker condition from Assumption 4.2 instead of RAE (31), while considerably simplifying the required proofs.

Remark 5.3. On $u_{\overline{\mathcal{H}}} < U(+\infty)$

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- ☐ n. 54 Riccardo Cambini, Alcune condizioni di ottimalità relative ad un insieme stellato.
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- ☐ **n. 179** E. Barucci - M.E. Mancini - Roberto Renò, Volatility Estimation via Fourier Analysis
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