Local and global indeterminacy in an overlapping generations model with consumption externalities

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Abstract

We analyze an overlapping generations model where individuals' well-being is negatively affected by the economy-wide average consumption level. Each individual aims at increasing his consumption level relatively to that of the others. We show that this "positional competition" can generate (local and global) indeterminacy and chaos.

Mathematics Subject Classifications (2000): 37N40, 37M20, 39A11

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1 Introduction

In this paper we analyze economic dynamics in an overlapping generations framework. In each period of time t, $t = 1, 2, 3, ..., \infty$, two generations of individuals coexist, the "*young*" and the "*old*". Each individual lives for two periods of time, works when young and consumes when old[1]. The consumption good is produced by a population of perfectly competitive firms. In each period t, the representative young individual has to allocate his time endowment $\overline{L}$ between labour supply L_t to the representative firm and leisure $\overline{L} - L_t$. Labour L_t is remunerated at the wage rate W_t. The remuneration L_tW_t is entirely invested in productive capital K_{t+1} (i.e. $K_{t+1} = L_tW_t$) that the individual will rent to the representative firm in time $t+1$ at the interest factor R_{t+1}. The sum obtained, $W_tL_tR_{t+1}$, allows him to buy and consume the quantity $C_{t+1} = W_tL_tR_{t+1}$ of the good produced by the firm[2].

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[1] This assumption is adopted in several overlapping generations models (see, among the others, Zhang 1999; Duranton 2001). It simplifies our analysis by abstracting from the consumption-saving choices of agents.

[2] $W_t \cdot L_t$ and $W_t \cdot L_t \cdot R_{t+1}$ are expressed in unities of the consumption good.

Well-being of individuals born in period t is represented by a function which is increasing in leisure $\overline{L} - L_t$ and consumption C_{t+1} but decreasing in the economy-wide average consumption level $\overline{C}_{t+1}$; that is, $\overline{C}_{t+1}$ generates a negative externality that reduces the benefits deriving from individual consumption C_{t+1}. In particular, according to the seminal work of Fred Hirsch (1978), we assume that the benefits deriving from C_{t+1} are reduced by "congestion effects":

'The satisfaction derived from an auto or a country cottage depends on the conditions in which they can be used, which will be strongly influenced by how many other people are using them. This factor, which is social in origin, may be a more important influence on my satisfaction than the characteristics of these items as "private goods" (on the speed of the auto, the spaciousness of the cottage, and so forth). Beyond some point that has long been surpassed in crowded industrial societies, conditions of use tend to deteriorate as use becomes more widespread.' (Hirsch (1978, p.3)

An example in which congestion effects play a key role in conditioning individuals' well-being is urban sprawl (see e.g. Hirsch (1978, pp. 36-41)); the choice involving a house move to an area outside a degraded urban centre, generally motivated by environmental reasons (traffic noise, urban air pollution, desire for greater living spaces), has led to an increasing extension of the towns over time with a consequent invasion of the countryside. This phenomenon may end up increasing the problems that originally induced people to flee away from town, such as the reduction of green and public areas, the traffic congestion with the consequent increase of polluting emissions and of the time spent commuting to work (see, for example, Ciscel (2001) and Antoci et al. (2008)):

'A lot of people thought they were moving out to the quiet and beauty of the countryside-but the city followed them, with all its problems of traffic, housing, and congestion' (U.S. Department of Housing and Urban Development, Third Annual Report 1967)[3].

Very similar congestion effects are observed in tourism (see, for example, Hirsch (1978), pp. 37-38). Other congestion problems have not a physical manifestation. For example, positional competition is subject to congestion; when an individual increases his private consumption level in order to reach a higher status in the social hierarchy, then the results of his effort are undermined by the increase of other individuals' consumption level. (see, among the others, Alpizar et al. (2005), Ferrer-i-Carbonell (2005), Luttmer (2005), Johansson-Stenman and Martinsson (2006), Carlsson et al. (2007)).

The effects of negative consumption externalities on economic dynamics have been analyzed by several theoretical works; see, among many others, Corneo and Jeanne (1998), de la Croix (1999), Chen and Hsu (2007), Frijters and Leigh (2008), Mino (2008), Wendner and Goulder (2008). The objective of our work is to analyze the role played by consumption negative externalities in generating indeterminacy and complex economic dynamics. The literature about indeterminacy distinguishes between two types of indeterminacy: *local indeterminacy* and *global indeterminacy*. Both types of indeterminacy can occur in

[3] Cited in Hirsch (1978), page 37, footnote 15.

optimization models where there are state variables (K_t in our model), whose initial values are predetermined, and control variables (L_t in our model), which instead are "jumping" variables in that their initial values are chosen by economic agents. In models with only one state variable and only one control variable, as the ours, local indeterminacy occurs when given the initial value K_0 of the state variable K, there exist a continuum of initial values L_0 of the control variable L such that the orbit starting from (K_0, L_0) approaches the same fixed point. When, starting from the same initial value K_0, different fixed points or ω-limit sets can be reached according to the initial choice of L, then we speak of *global indeterminacy* (see Brito and Venditti 2009). The scenario of global indeterminacy becomes more complex if one of the reachable ω-limit set is a chaotic attractor. In such a case, also the long run behavior of the orbits approaching the same (chaotic) attractor depends on the initial value of L.

When global indeterminacy occurs, "history" (represented by the initial values of the state variables) is not the unique factor determining the future evolution of the economy, but individuals' expectations (which determine the choices of the initial values of control variables) play a key role: economies starting from the same values of the state variables may follow orbits whose asymptotic behaviors are very different.

There exists an enormous literature on local indeterminacy in economic models; see, among many others, early studies of Benhabib and Farmer (1994) and Boldrin and Rustichini (1994) and the subsequent works of Zhang (2008), Mino et al. (2008), Nishimura et al. (2009). However, very few works on indeterminacy focus on global dynamics and stress the relevance of global analysis. See, for example, the seminal works of Krugman (1991) and Matsuyama (1991) followed by, among the others, Pintus et al. (2000), Raurich-Puigdevall (2000), Karp and Paul (2007), Mattana et al. (2008), Coury and Wen (2009), Brito and Venditti (2009), Antoci et al. (2010). The great majority of these works analyze economies in which the production side is conditioned by positive externalities. In our paper local and global indeterminacy is caused by negative externalities, in particular, by the negative effect on individuals' well-being due to the average consumption level $\overline{C}$.

There exist very few other works on indeterminacy that take into account the effects of negative externalities. For example, Raurich-Puigdevall (2000) and Chen and Lee (2007) analyze economies where congestion effects on public goods generate negative externalities. Itaya (2008) shows how pollution may affect indeterminacy results in a one-sector growth model. In Meng and Yip (2008) indeterminacy is produced by negative capital externalities. In Antoci and Sodini (2009) and Antoci et al. (2010) negative externalities may generate indeterminacy in an economy where private goods can be consumed as substitutes for free access environmental goods. Chen and Hsu (2007), in a continuous time model, and Mino (2008), in an overlapping generations framework, show that negative consumption externalities can be a source of local indeterminacy, however, they don't analyze global dynamics.

The present paper has the following structure. Sections 2 and 3 define the set-up of the model and the associated dynamic system. Sections 4 and 5 deal with local stability analysis of fixed points and local indeterminacy. Section 6 shows that global indeterminacy and chaos can occur in our model. Section 7 contains the conclusions.

2 The model

We analyze the growth dynamics of an overlapping generations economy where individuals live for two periods of time and two generations of individuals (*young* and the *old*) coexist in each period of time t, $t = 1, 2, 3, \dots\dots\dots, \infty$. Individuals work when they are young and consume when they are old.

The consumption good is produced by a population of perfectly competitive firms. In each period t, the representative young individual has to allocate his time endowment $\overline{L}$ ($\overline{L}$ is a fixed parameter) between labour supply L_t ($\overline{L} \geq L_t \geq 0$) to the representative firm and leisure $\overline{L} - L_t$. Labour L_t is remunerated at the wage rate W_t. The remuneration $L_t W_t$ is entirely invested in productive capital K_{t+1} (i.e. $K_{t+1} = L_t W_t$) that the individual will rent to the representative firm in time $t+1$ at the interest factor R_{t+1}. The sum obtained, $W_t L_t R_{t+1}$, allows him to buy and consume the quantity $C_{t+1} = W_t L_t R_{t+1}$ of the good produced by the firm[4].

The representative firm produces output via the following constant returns Cobb-Douglas technology:

$$Y = AF(K_t, L_t) = AL_t^{1-\alpha} K_t^{\alpha} = AL_t k_t^{\alpha}$$

where $k_t := K_t / L_t$, A is a strictly positive parameter representing (exogenous) technological progress and $1 > \alpha > 0$.

The representative individual's well-being depends positively on leisure $\overline{L} - L_t$ and consumption C_{t+1} but negatively on the economy-wide average consumption $\overline{C}_{t+1}$:

$$U(L_t, C_{t+1}, \overline{C}_{t+1}) = \ln(\overline{L} - L_t) + \frac{B}{1+\theta} \frac{(C_{t+1} - \rho \overline{C}_{t+1})^{1-\sigma}}{1-\sigma}$$

where $\frac{1}{1+\theta}$ is the discount factor, $\theta > 0$; $\sigma > 0$, $\sigma \neq 1$, denotes the the inverse of the intertemporal elasticity of substitution; B is a positive scale parameter which will be used to apply the "normalized fixed point" technique; ρ measures the congestion effect due to the economy-wide average consumption level $\overline{C}_{t+1}$. Following Hirsch (1978), we assume that the value of ρ is positively correlated with the average level of activity in the economy, represented by the economy-wide average level $\overline{L}_t$ of the labour input L_t (in our model, $\overline{L}_t$ is positively correlated with average consumption $\overline{C}_{t+1}$):

'.....as the level of average consumption rises, an increasing portion of consumption takes on a social as well as an individual aspect. That is to say, the

[4] $W_t \cdot L_t$ and $W_t \cdot L_t \cdot R_{t+1}$ are expressed in unities of the consumption good.

satisfaction that individuals derive from goods and services depends in increasing measure not only on their own consumption but on consumption by others as well.' (Hirsch (1978, p. 2)

In particular, we assume the following functional specification for ρ:

$$\rho := 1 - e^{-f\overline{L}_t}$$

where f is a strictly positive parameter which measures the congestion effect.

3 Agents' choices and economic dynamics

The representative individual has to choose the labour input L_t that solves the following maximization problem:

$$\underset{L_t}{Max}\left[\ln(\overline{L} - L_t) + \frac{B}{1+\theta}\frac{\left(C_{t+1} - \rho\overline{C}_{t+1}\right)^{1-\sigma}}{1-\sigma}\right]$$

under the constraints:

$$\begin{aligned} C_{t+1} &= R_{t+1}W_tL_t \\ L_t &\in \left[0, \overline{L}\right] \end{aligned}$$

Average consumption $\overline{C}_{t+1}$ and the value of ρ are considered as exogenously given by the representative individual in that he considers as negligible the impact of his choices on such values. Under these assumptions, the first order condition for an interior solution ($0 < L_t < \overline{L}$ always holds) is:

$$-\frac{1+\theta}{\overline{L} - L_t} + B[R_{t+1}W_tL_t - (1-\rho)\overline{C}_{t+1}]^{-\sigma}R_{t+1}W_t = 0 \tag{1}$$

In each period t, the representative firm maximizes the profit function:

$$AF(K_t, L_t) - W_tL_t - R_tK_t$$

taking the wage rate W_t and the interest factor R_t as exogenously given. As usual, this assumption gives rise to the following first order conditions:

$$W_t = A(1-\alpha)k_t^{\alpha} \tag{2}$$

$$R_t = A\alpha k_t^{\alpha-1} \tag{3}$$

Substituting (2) and (3) in (1), and taking into account that, being all individuals identical, (ex post) $\overline{C}_{t+1} = C_{t+1}$ and $\overline{L}_t = L_t$ (remember that $\rho = e^{-f\overline{L}_t}$) we obtain:

$$-\frac{1+\theta}{\overline{L} - L_t} + B[A^2\alpha(1-\alpha)k_{t+1}^{\alpha-1}k_tL_t(1-\rho)]^{-\sigma}A^2\alpha(1-\alpha)k_{t+1}^{\alpha-1}k_t = 0 \tag{4}$$

Furthermore, remembering that the remuneration L_tW_t is entirely invested in productive capital, that is, $K_{t+1} = L_tW_t$ and that $K_{t+1} = L_{t+1}k_{t+1}$, we can write (see (2)):

$$K_{t+1} = L_{t+1}k_{t+1} = L_tW_t = L_tA(1-\alpha)k_t^\alpha$$

that is:

$$k_{t+1}L_{t+1} = A(1-\alpha)k_t^\alpha L_t \tag{5}$$

The system (4)-(5) describes the time evolution of the variables L_t and k_t and, as we will see in the next section, can be written in normal form $L_{t+1} = m(L_t, k_t)$ and $k_{t+1} = n(L_t, k_t)$.

4 Fixed points

The system (4)-(5) defines k_{t+1} and L_{t+1} as functions of k_t and L_t. In this section, we study the stability of fixed points of such discrete dynamic system. We use the geometrical-graphical method developed by Grandmont and DeVilder (1999) that allows us to characterize the stability properties of the fixed points of this dynamic system. We impose some conditions on parameters under which a "normalized" fixed point (L^*, k^*), with $k^* = L^* = 1$, exists. This allows us to analyze the effects on stability due to changes in parameters' values being sure that the fixed point doesn't disappear. Without loss of generality, we pose $\overline{L} = 2$.

Posing $k^* = k_{t+1} = k_t = 1$ and $L^* = L_{t+1} = L_t = 1$ in equation (5), we obtain:

$$1 = A(1-\alpha)$$

that is:

$$A = \frac{1}{1-\alpha} \tag{6}$$

Posing $A = \frac{1}{1-\alpha}$ and $k^* = k_{t+1} = k_t = 1$ and $L^* = L_{t+1} = L_t = 1$ in equation (4), we obtain:

$$B\left(\frac{\alpha}{1-\alpha}\right)^{1-\sigma} e^{f\sigma} + \theta - 1 = 0$$

which gives:

$$B = \frac{1-\theta}{\left(\frac{\alpha}{1-\alpha}\right)^{1-\sigma} e^{f\sigma}} \tag{7}$$

Using conditions (6)-(7), the dynamic system (4)-(5) can be explicitly written as:

$$k_{t+1} = \left[e^{f\sigma(L_t-1)} k_t^{\alpha(1-\sigma)} \frac{(2-L_t)}{L_t^\sigma} \right]^{\frac{1}{(1-\alpha)(1-\sigma)}} \tag{8}$$

$$L_{t+1} = L_t k_t^\alpha \left[\frac{L_t^\sigma}{e^{f(L_t-1)\sigma} k_t^{\alpha(1-\sigma)} (2-L_t)} \right]^{\frac{1}{(1-\alpha)(1-\sigma)}} \tag{9}$$

Notice that $k = 1$ always holds at the fixed points of system (8)-(9) (see (5)) while the fixed point values of L are given by the solutions of the equation (see (8)):

$$1 = g(L) := \left[e^{f\sigma(L-1)} \frac{2-L}{L^\sigma} \right]^{\frac{1}{(1-\alpha)(1-\sigma)}} \tag{10}$$

Obviously, $L = 1$ is a solution of equation (10). The following proposition can be easily checked.

Proposition 1 *In case $\sigma < 1$ (respectively, $\sigma > 1$), the normalized fixed point (L^*, k^*), with $k^* = L^* = 1$, is the unique fixed point of system (8)-(9) if $f < (1+\sigma)/\sigma$ (respectively, $f > (1+\sigma)/\sigma$) while other two fixed points exist if $f > (1+\sigma)/\sigma$ (respectively, $f < (1+\sigma)/\sigma$), one with $L < 1$ and the other with $L > 1$.*

In the numerical example in Figure 1 we have posed $\alpha = 0.421$, $\sigma = 0.6$ and we have plotted the graphs of $g(L)$ corresponding to three values of the parameter f: 1.92, 2.6667, 3.2. Notice that for $f = 1.92$ only the normalized fixed point exists while for $f = 3.2$ three fixed points exist; $f = 2666\,7$ is the bifurcation value.

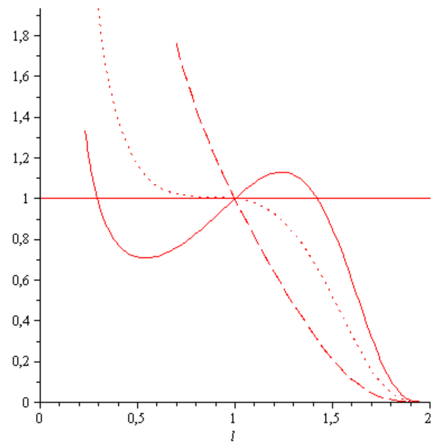

Figure 1: *Graphs of $g(L)$ in case $\sigma < 1$*

Figure 2 shows the graphs of $g(L)$ obtained posing $\alpha = 0.421$, $\sigma = 1.6$ and $f = 1.12$ (three fixed points), $f = 1.625$ (the bifurcation value), $f = 1.82$ (one fixed point).

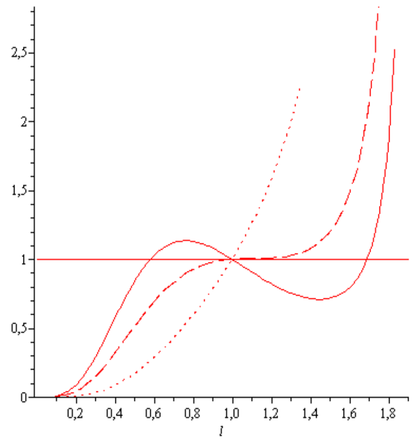

Figure 2: *Graphs of $g(L)$ in case $\sigma > 1$*

5 Stability of fixed points and local indeterminacy

In our model, productive capital K_t represents a state variable, so its initial value K_0 is given. Differently from K_t, the variable L_t is a "jumping" variable in that it represents the representative individual's labor input, chosen taking into account of the average labour input and consumption level in the economy. Consequently, individuals have to choose the initial value L_0 (and consequently the initial value of $k_t = K_t/L_t$). If the normalized fixed point is a saddle and K_0 is near enough to 1, then there exists an unique initial value of L_t, L_0, such that the orbit passing through (k_0, L_0) approaches the fixed point. When the fixed point is a sink, given the initial value K_0, then there exists a continuum of initial values L_0 such that the orbit passing through (k_0, L_0) approaches the fixed point; consequently, the orbit the economy will follow is "indeterminate" in that it depends on the choice of the initial value L_0.

The Jacobian matrix of (8)-(9), evaluated at the normalized fixed point, is:

$$J(L^*, k^*) = \begin{pmatrix} \frac{\alpha}{1-\alpha} & \frac{\sigma f-\alpha-1}{(1-\alpha)(1-\sigma)} \\ -\frac{\alpha^2}{1-\alpha} & -\frac{\sigma(f-\alpha)+\alpha-2}{(1-\alpha)(1-\sigma)} \end{pmatrix}$$

with:

$$Det(J^*) = \frac{2\alpha}{(1-\alpha)(1-\sigma)} - \frac{\alpha\sigma f}{(1-\alpha)(1-\sigma)} \tag{11}$$

$$Tr(J^*) = \frac{2}{(1-\alpha)(1-\sigma)} - \frac{\sigma f}{(1-\alpha)(1-\sigma)} \tag{12}$$

Note that, varying (ceteris paribus) the parameter f, the point (see equations (11)-(12)):

$$(P_1, P_2) := \left(\frac{2}{(1-\alpha)(1-\sigma)} - \frac{\sigma f}{(1-\alpha)(1-\sigma)}, \frac{2\alpha}{(1-\alpha)(1-\sigma)} - \frac{\alpha\sigma f}{(1-\alpha)(1-\sigma)}\right)$$

describes in the plane $(Tr(J), Det(J))$ a half-line T_1 with slope α $(0 < \alpha < 1)$ (see Figures 3 and 4) starting from the point (obtained posing $f = 0$ in (11)-(12)):

$$(\overline{P}_1, \overline{P}_2) := \left(\frac{2\alpha}{(1-\alpha)(1-\sigma)}, \frac{2}{(1-\alpha)(1-\sigma)}\right) \tag{13}$$

The point $(\overline{P}_1, \overline{P}_2)$, varying α, describes in the plane $(Tr(J), Det(J))$ a half-line T_2 with slope 1 starting from the point $\left(\frac{2}{1-\sigma}, 0\right)$. We now distinguish between the cases $\sigma \in (0, 1)$ and $\sigma > 1$.

If $\sigma \in (0, 1)$, then $(\overline{P}_1, \overline{P}_2) \to (+\infty, +\infty)$ for $\alpha \to 1$ and $(P_1, P_2) \to (-\infty, -\infty)$ for $f \to +\infty$. In such a context, only two cases are possible:

1. If $\alpha \in (0, 1/2)$ (see Figure 3), then the starting point point $(\overline{P}_1, \overline{P}_2)$ of the half-line T_1 lies in the region "saddle" and consequently the normalized fixed point (L^*, k^*) is saddle-point stable; increasing the value of the parameter f, the point (P_1, P_2) moves along T_1 and (L^*, k^*) becomes a sink via a pitchfork bifurcation (occurring for $f = \frac{1+\sigma}{\sigma}$) that gives rise to two other fixed points, which are saddle-point stable. If f continues to increase, then the point (P_1, P_2) leaves the region "sink" and enters the region "saddle" giving rise to a supercritical flip bifurcation (occurring for $f = [3\alpha + 1 + \sigma(1-\alpha)] / \sigma(1+\alpha)$) which generates a periodic orbit of period 2; further increases in f leads to chaotic behavior via period doubling bifurcations, as Figure 4 shows.

2. If $\alpha \in (1/2, 1)$ (see Figure 3), then the starting point point $(\overline{P}_1, \overline{P}_2)$ of the half-line T_1 lies in the region "saddle" and consequently the normalized fixed point (L^*, k^*) is saddle-point stable; increasing the value of the parameter f, the point (P_1, P_2) moves along T_1 and (L^*, k^*) becomes first a source (via a flip bifurcation) and successively a sink (via a subcritical Neimak-Hopf bifurcation).

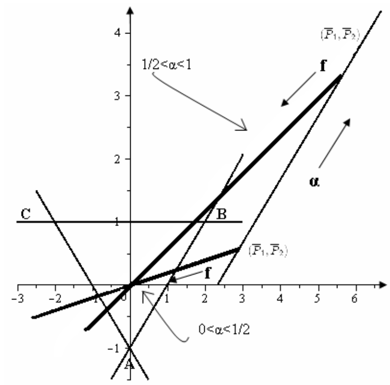

Figure 3: *Stability analysis of the normalized fixed point in case* $\sigma < 1$

If $\sigma > 1$, then $(\overline{P}_1, \overline{P}_2) \to (-\infty, -\infty)$ for $\alpha \to 1$ and and $(P_1, P_2) \to (-\infty, -\infty)$ for $f \to +\infty$. Therefore, as in the case $\sigma < 1$, only two cases are possible:

1. a) If $\alpha \in (0, 1/2)$ (see Figure 4), then the normalized fixed point (L^*, k^*) may be initially (for $f = 0$) a saddle or a sink; in the former case, when f increases, it becomes a sink via a subcritical flip bifurcation; in both cases, a further increase in f leads to pitchfork bifurcation according to which the normalized fixed point becomes saddle-point stable and two attractive fixed points arise; after a further increase in f, these points lose their stability and chaotic behavior occurs via period doubling bifurcations (see the next section).

2. b) If $\alpha \in (1/2, 1)$ (see Figure 4), then the line T_1 intersects the segment BC. As in case (a), the normalized fixed point is initially a saddle or a sink, according to parameter values. Starting from the context in which the normalized fixed point is a sink, an increase in f gives rise to a Neimak-Hopf bifurcation (the bifurcation value is $f = [2 - (1 - \alpha)(1 - \sigma)/\sigma]/\sigma)$ according to which the normalized fixed point becomes a source and an attractive invariant cycle (a closed curve mapped onto itself) appears.

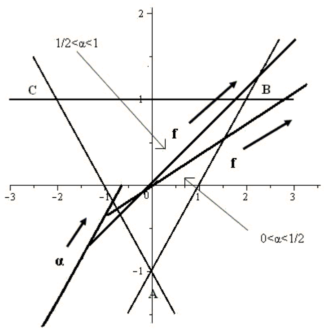

Figure 4: *Stability analysis of the normalized fixed point in case $\sigma > 1$*

6 Global indeterminacy and chaos

In this section we show that an increase in the parameter f, which measures the magnitude of consumption negative externalities, may generate global indeterminacy and chaos. Remember that global indeterminacy occurs when starting from the same initial value K_0 of the state variable K, different fixed points or other ω-limit sets can be reached according to the initial choice L_0 of the jumping variable L. The scenario of global indeterminacy becomes more complex if one of the reachable ω-limit set is a chaotic attractor. In such a case, also the long run behavior of the orbits approaching the same (chaotic) attractor depends on L_0.

We obtain some insights about the dynamics of our dynamic system by means of numerical simulations and the theory of the critical curves, a powerful tool for the analysis of the global properties of noninvertible two-dimensional maps (see e.g. Mira et al. (1996) for details). We first consider the case $\sigma \in (0, 1)$. In the numerical example in Figure 5, the normalized fixed point is a saddle for $f < 3.5$; when f crosses the value 3.5, it becomes a sink and coexists with two saddles, arisen via a pitchfork bifurcation. For $f = 6.2$ we obtain the strange attractor showed on the right of Figure 5. This is an example of global indeterminacy in that starting from the same initial value of K, the economy may approach the normalized fixed point (when it is a saddle or a sink) or another $\omega-$limit set (a fixed point or a strange attractor).

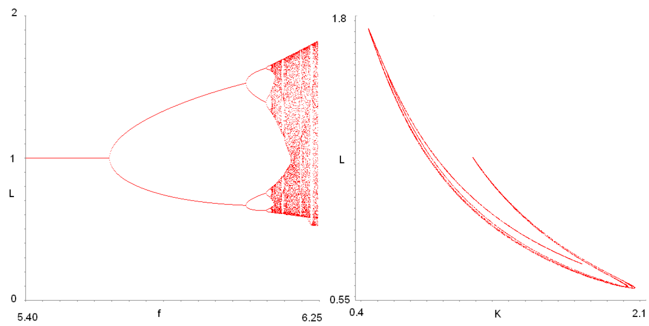

Figure 5a: *Bifurcation diagram with $\alpha = .4$, $\sigma = .4$ (on the left)* **Figure 5b:** *strange attractor for $f = 6.2$ (on the right)*

Let us now consider the case $\sigma > 1$, with $\alpha \in (0, 1/2)$. In the numerical example in Figure 6, the normalized fixed point is a saddle for $f < 1.306$ and becomes a sink for $1.306 < f < 1.714$; a further increase in f generates a pitchfork bifurcation that gives rise to two attractive fixed points which coexist with the normalized fixed point which, after the bifurcation, becomes a saddle. Finally, for higher values of f, the two attractive fixed points become sources (not simultaneously) and two chaotic attractors arise; these attractors coexist with the normalized fixed point, which remains a saddle. This is an interesting example in that global indeterminacy occurs in a context in which there is not local indeterminacy (that is, no fixed point is attractive), as in Coury and Wen (2008) and Brito and Venditti (2009).

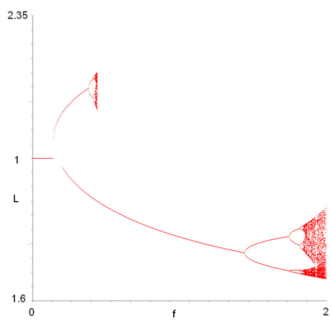

Figure 6: *Bifurcation diagram with $\alpha = .4$, $\sigma = 1.4$*

More interesting is the case $\sigma > 1$, with $\alpha \in (1/2, 1)$. Remember that, is such case, the line T_1 intersects the segment BC (see Figure 4) and when this happens, a Neimak-Hopf bifurcation occurs (see the previous section). In the numerical simulations showed in the following figures, we posed $\alpha = 0.582$ and $\sigma = 1.4$. For $f < 1.3531$ the normalized fixed point is a saddle. As f increases, for $f = 1.63375$, a Neimak-Hopf bifurcation occurs according to which the normalized fixed point becomes a source and an attractive invariant curve (a closed curve mapped onto itself) Γ appears which gradually expands as f increases (see Figure 7).

For = 1.6627 the attractive invariant curve dies and a 12-periodic cycle exists until $f = 1.6648$ (see Figure 8). For $f > 1.665$ an invariant attractive curve reappears through a secondary Neimak-Hopf bifurcation.

For f near to 1.667 the invariant curve is tangent to the critical curve[5] LC_{-1}. In Figure 8, for $f = 1.669$, the curve crosses LC_{-1} in two different points. This collision induces the wavy shape of Γ and near $f = 1.68$ and after that the attractor loses its stability and two attractive limit cycles with very long period coexist. When $f = 1.685$ there is an explosion to a chaotic attractor that persists until $f = 1.718$. If we let f increase further, we obtain the effect of destroying the attractor and almost all orbits diverge to infinity.

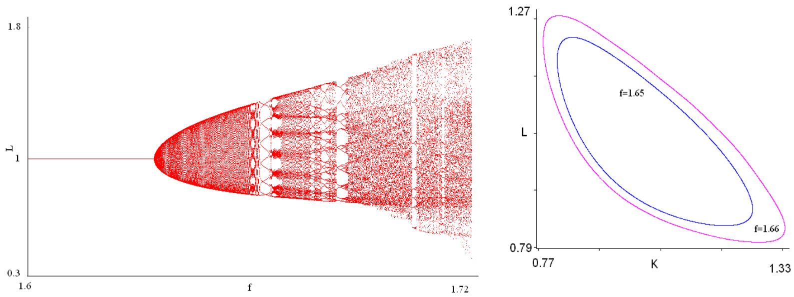

Figure 7: *Bifurcation diagram with $\alpha = 0.582$, $\sigma = 1.4$ (on the left) and enlargement of the invariant curve (on the right)*

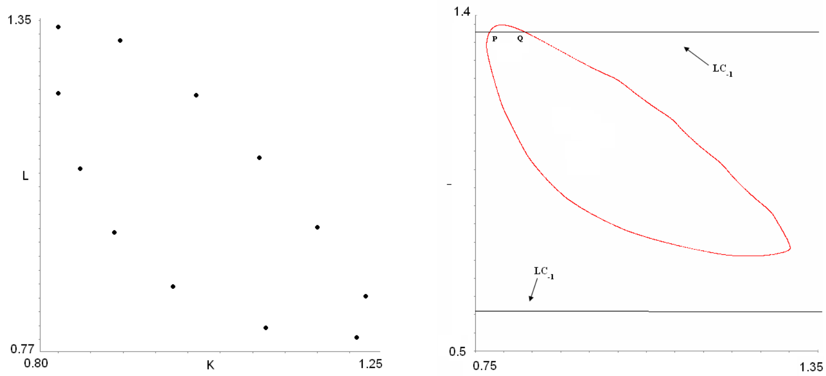

Figure 8: *Period-12 cycle obtained for $f = 1.6627$ (on the left) and wavy shape of Γ given by the collision of Γ with the critic line LC_{-1} for $f = 1.669$*

[5] We recall that the basic critical curve LC is the set of points with "coincident preimages", that is the set of points having nearby points with different numbers of preimages. LC_{-1} is defined as the preimage of LC.

In the context of a generic smooth map, the fundamental critical curve LC_{-1} is a subset of the set J_0 of critical points of the Jacobian matrix, that is points at which the determinant of the Jacobian matrix vanishes. For this specific map, from straightforward computation we find that J_0 is given by the lines:

$L = \frac{\pm\sigma f + \sqrt{\sigma^2 f^2 - 2\sigma f}}{\sigma f}$

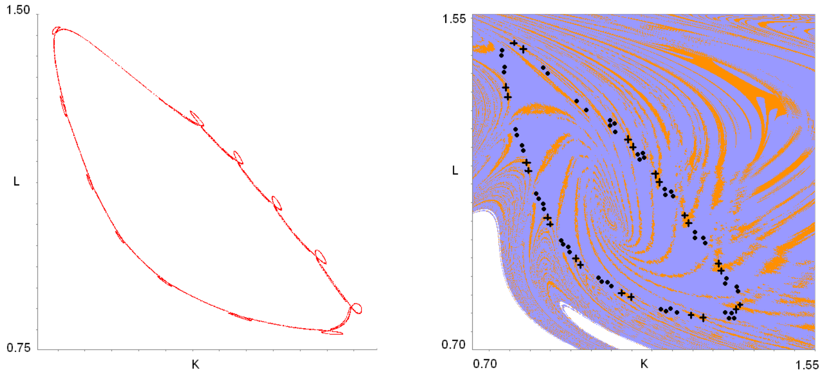

Figure 9: *Shape of the attractor for* $f = 1.682$ *(on the left). For* $f = 1.685$ *the attractor is vanished and the right figure shows the very intricate structure for the basins of attraction of the two coexisting limit cycles*

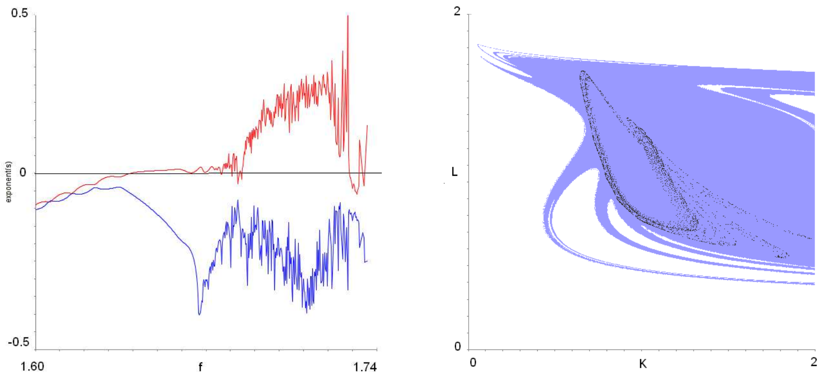

Figure 10: Lyapunov exponents on the left. The exploded attractor on the right for $f = 1.7$

7 Conclusions

Our paper shows that positional competition aimed at increasing the difference between individual consumption and the economy-wide average consumption may be a source of local and global indeterminacy. The parameter f, which measures the magnitude of consumption negative externalities, plays a key role in determining indeterminacy. In section 3 we showed that an increase in f initially generates local indeterminacy (i.e. the local attractivity of the normalized fixed point). The numerical simulations in section 4 showed that further increases of f may produce global indeterminacy and chaotic behavior.

It is worth to stress that, in our model, global indeterminacy and chaos can occur even if all the existing fixed points are saddle-point stable (i.e. locally determinate) or repulsive. That is, global indeterminacy can be observed also in a context in which there is not local indeterminacy.

These results are analogous to those obtained in Coury and Wen (2009) and in Brito and Venditti (2009). Coury and Wen study a real business cycle model in discrete time, where there exists a unique stationary state, locally determinate (i.e. saddle-point stable), "surrounded" by stable periodic orbits, so that the economy can approach either the stationary state or the periodic orbits, starting from the same initial values of the state variable. Brito and Venditti analyze a two-sector growth model in continuous time with accumulation of human and physical capital that admits two balanced growth paths (BGPs). They find that global indeterminacy can occur in the following contexts: a) both BGPs are locally determinate; b) both BGPs are locally indeterminate; c) one BGP is locally indeterminate while the other is determinate.

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