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Data Envelopment Analysis with outputs uncertainty

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Abstract

This paper presents two different models for Data Envelopment Analysis with uncertain outputs. The corresponding deterministic formulations are obtained via a scenario approach which does not depend on the underlying distribution of data. These models are compared with similar formulations based on an expected value approach and a full computational test is carried out in order to validate them.

Key words: Data Envelopment Analysis, data uncertainty, efficiency measures.
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1 Introduction

Data Envelopment Analysis (DEA) has been first proposed in the pioneering paper by Charnes, Cooper and Rhodes (CCR) [4]. It is a nonparametric method for estimating the efficiency of decision-making units (DMUs), such as firms or public sector agencies. In the classic DEA model there are n DMUs to be evaluated. Each DMU consumes various inputs to produce different outputs. No production function needs to be specified.

In the classic DEA model (see [4]), which has been given in the form of a linear fractional program, the efficiency of the j^{th} DMU is defined by the ratio between the weighted sum of outputs and the weighted sum of inputs. In fact, since multiple inputs are used to produce multiple outputs then the individual inputs quantities and the individual outputs quantities need to be aggregated into a composite input and a composite output. One pratical approach uses market prices of inputs and outputs for aggregation. When market prices cannot or should not be used, we need to construct shadow prices for aggregation. DEA evaluates the efficiency of each DMU through the better system of weights (or shadow prices) for the considered DMU, identifying the best one. Stating the benchmark DMU, DEA classifies the remaining DMUs from the most efficient to the less one. The pioneer model (CCR) measures technical efficiency of a DMU which exhibits Constant Returns to Scale (CRS) everywhere on the production frontier. In an important extension of this approach, Banker, Charnes and Cooper [1] generalized the original DEA approach formulating a model (BCC) for exhibiting Variable Returns to Scale (VRS) at different points on the production frontier.

The classical models assume a deterministic framework with no uncertainty and this seems not suitable for concrete applications, due to the presence of errors and noise in the estimation of inputs and outputs values.
In this paper, starting from the generalized input-oriented (BCC) model, two different models with uncertain outputs and deterministic inputs are proposed. Various applications, in fact, are affected by random perturbations in output values estimation (see for instance, [3, 12, 13]).
A large number of papers, based on different approaches, can be found in the literature concerning DEA with outputs uncertainty. In particular, chance-constrained programming is the most used technique to include noise variations in data and to solve data envelopment analysis problems with uncertainty in data. Chance-constrained programming admits random data variations and permits constraint violations up to specified probability limits, allowing linear deterministic equivalent formulations in the case a normal distribution of the data uncertainty is assumed (see for all [5, 6, 8, 9, 10, 11]). The formulations proposed in this paper move away the classical chance-constrained method with the aim to obtain a more accurate DMU ranking whatever situation occurs. In particular, two different models are proposed where uncertainty is managed with a scenario generation approach. For the sake of completeness, these models are compared with two further ones based on an expected value approach, that is to say that the uncertainty is managed by means of the expected values of random factors both in the objective function and in the constraints.

Deeply speaking, the main difference between the two proposed models and the expected value approaches lies in their mathematical formulation. In the models based on the scenario generation approach the constraints concerning efficiency level are expressed for each scenario, while in the expected value models they are satisfied in expected value. As a consequence, the first kind of models result to be more selective in finding a ranking of efficiency, thus becoming useful strategic management tools aimed to determine a restrictive efficiency score ranking.
In Sections 2 and 3 two different models with uncertain data and assuming Variable Returns to Scale (VRS) are presented as a pair of Primal-Dual problems. Their corresponding deterministic formulations are obtained via a scenario approach. In the first model formulation (VRS1), the specific optimal weight composition is searched for each realization of the random factor ξ. On the other hand, the second model (VRS2) is formulated with the aim of optimizing an unique vector of weights whatever scenario occurs. Two further models based on the expected value approach are considered in order to be compared with the previously introduced ones. In Section 4 a constant returns to scale version of the proposed models is given and some theoretical results are presented. Finally, in Section 5 the results of a complete computational test are collected in order to compare the scenario generation models with the expected value approaches.

2 DEA models with outputs uncertainty

In this section two DEA models with variable returns to scale and uncertain data are presented (see Section 4 for corresponding constant returns to scale models). Both a primal and a dual versions are developed for each formulation and some properties are remarked. The output parameters for both formulations are assumed to be uncertain depending on a random factor ξ. Inputs parameter are deterministic. In the following subsections these models are described and compared.

It is worth noticing that many models in the literature are introduced only in dual form. Actually, the dual form does not provide an intuitive interpretation of them. For this very reason, in this paper the models are presented as a Primal-Dual formulation: the primal formulation is suitable for the model interpretation, while the dual one allows a direct comparison with the results in the literature.

Finally, notice that chance-constrained programming is the most used technique to include noise variations in data and to solve Data Envelopment Analysis problems with uncertainty in data. Chance-constrained programming permits constraint violations up to specified probability limits and generally assumes normal data distribution. Scenario generation based approaches remove the hypothesis of normal data distribution and use a scenario generation approach to include data perturbations. In DEA literature, as far as we know, there are no similar scenario approaches, except the one of Bruni et al. [3] where scenarios generation is used to linearize the problem formulated through the chance-constrained approach.

2.1 The VRS1 model

In this model both weights and outputs depend on the random factor ξ. In other words, a different efficiency level is computed for each realization of the uncertain factor ξ. In order to describe the Primal-Dual couple of model VRS1, the following parameters are introduced:

$y_{rj}(\xi) \in \mathbb{R}_+$:	r^{th} output quantity produced by the j^{th} DMU depending on the random factor ξ, $r = 1, \ldots, q, \quad j = 1, \ldots, n$;
$x_{ij} \in \mathbb{R}_+$:	i^{th} input quantity used by the j^{th} decision making unit $i = 1, \ldots, m, \quad j = 1, \ldots, n$.

By means of the previous notations, the following problem is defined:

$(\widehat{P1_V})$

$$\max_{u(\xi),u_0(\xi),v} \quad \mathbb{E}_\xi\left[\sum_{r=1}^{q} u_r(\xi)y_{rj_0}(\xi) + u_0(\xi)\right] \tag{1}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{2}$$

$$\sum_{r=1}^{q} u_r(\xi)y_{rj}(\xi) + u_0(\xi) - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1,\ldots,n \tag{3}$$

$$u_r(\xi) \geq 0 \quad r = 1,\ldots,q$$

$$v_i \geq 0 \quad i = 1,\ldots,m$$

$$u_0(\xi) \in \mathbb{R}$$

where the following further notations are used:

$u_r(\xi) \in \mathbb{R}_+$:	weight variable related to the r^{th} output depending on the random factor ξ, $r = 1,\ldots,q$;
$v_i \in \mathbb{R}_+$:	weight variable related to the i^{th} input, $i = 1,\ldots,m$;
$u_0(\xi) \in \mathbb{R}$:	scale variable depending on the uncertain factor ξ.

The corresponding dual formulation is as follows:

$(\widehat{D1_V})$

$$\min_{\theta,\lambda(\xi)} \quad \theta$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j(\xi) y_{rj}(\xi) \geq y_{rj_0}(\xi) \quad r = 1,\ldots,q \tag{4}$$

$$\mathbb{E}_\xi\left[\sum_{j=1}^{n} \lambda_j(\xi) x_{ij}\right] \leq \theta x_{ij_0} \quad i = 1,\ldots,m \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j(\xi) = 1 \tag{6}$$

$$\lambda_j(\xi) \geq 0 \quad j = 1,\ldots,n$$

where $\theta \in \mathbb{R}$, $\lambda_j(\xi) \in \mathbb{R}_+$ are the dual variables corresponding to the primal constraints.

Problem $(\widehat{P1_V})$ maximizes the expected efficiency for each DMU considering a different vector of weights for each realization of the uncertain factor ξ. Notice that a corresponding constant return to scale version of VRS1 can be obtained by deleting variable $u_0(\xi)$ in the primal formulation and by deleting constraints (6) in the dual one, as it will be shown in Section 4.

Notice also that in this model there is no need to make assumptions on the random variable distribution, thanks to the forthcoming use of scenarios for stating a corresponding deterministic formulation.

2.2 The VRS2 model

The model introduced in this section has the aim of finding an unique weights composition for the primal problem not depending on the realizations of the random factor ξ. At the same time, the corresponding dual problem presents less tightening constraints. The primal formulation of VRS2 is as follows:

$(\widehat{P2_V})$

$$\max_{u,u_0(\xi),v} \quad \mathbb{E}_\xi \left[\sum_{r=1}^{q} u_r y_{rj_0}(\xi) + u_0(\xi) \right] \tag{7}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{8}$$

$$\sum_{r=1}^{q} u_r y_{rj}(\xi) + u_0(\xi) - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \dots, n \tag{9}$$

$$u_r \geq 0 \quad r = 1, \dots, q$$

$$v_i \geq 0 \quad i = 1, \dots, m$$

$$u_0(\xi) \in \mathbb{R}$$

where:

$u_r \in \mathbb{R}_+$: is the weight variable related to the r^{th} output.

The corresponding dual formulations results to be:

$(\widehat{D2_V})$

$$\min_{\theta,\lambda(\xi)} \quad \theta$$

$$s.t. \quad \mathbb{E}_\xi \left[\sum_{j=1}^{n} \lambda_j(\xi) y_{rj}(\xi) - y_{rj_0}(\xi) \right] \geq 0 \quad r = 1, \dots, q \tag{10}$$

$$\mathbb{E}_\xi \left[\sum_{j=1}^{n} \lambda_j(\xi) x_{ij} \right] \leq \theta x_{ij_0} \quad i = 1, \dots, m \tag{11}$$

$$\sum_{j=1}^{n} \lambda_j(\xi) = 1$$

$$\lambda_j(\xi) \geq 0 \quad j = 1, \dots, n$$

As it was pointed out for VRS1, a corresponding CRS version for VRS2 can be easily obtained for both the primal and the dual formulations.

VRS2 model differs from VRS1 one in the task of finding an unique weights composition for uncertain outputs. In particular, in problem $(\widehat{P1_V})$ weights variables $u_r(\xi)$ depend on the random factor ξ, while in problem $(\widehat{P2_V})$ the weights are independent from its realizations. In the corresponding dual problems $(\widehat{D1_V})$ and $(\widehat{D2_V})$ the constraints are different. In problem $(\widehat{D2_V})$, in fact, constraints (10) have to be verified in expected value. These differences will be clarified in the next sections, where corresponding deterministic formulations for the proposed models are given.

3 A deterministic approach for VRS1 and VRS2

In order to manage the uncertainty of outputs and weights, a corresponding deterministic formulation for both VRS1 and VRS2 is obtained. Assuming that the random factor ξ is induced by a known probability distribution, this distribution can be discretized as follows. Let each scenario s represent a realization of the uncertain parameter ξ. In order to provide the corresponding deterministic formulation, the following notations are introduced:

$y^s_{rj} \in \mathbb{R}_+$:	r^{th} output quantity produced by the j^{th} DMU for each scenario s, $r = 1, \ldots, q$, $j = 1, \ldots, n$, $s = 1, \ldots, S$;
$p^s \in [0, 1]$:	realization probability of any scenario s, $s = 1, \ldots, S$.

3.1 Deterministic VRS1

The deterministic model corresponding to $(P1_V)$ is as follows:

$(P1_V)$

$$\max_{u,v,u_0} \sum_{s=1}^{S} p^s \left[\sum_{r=1}^{q} u^s_r y^s_{rj_0} + u^s_0 \right] \quad (12)$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \quad (13)$$

$$\sum_{r=1}^{q} u^s_r y^s_{rj} + u^s_0 - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \; j = 1, \ldots, n, \; s = 1, \ldots, S \quad (14)$$

$$u^s_r \geq 0 \quad r = 1, \ldots, q, \quad s = 1, \ldots, S$$

$$v_i \geq 0 \quad i = 1, \ldots, m$$

$$u^s_0 \in \mathbb{R} \quad s = 1, \ldots, S$$

where:

$u^s_r \in \mathbb{R}_+$:	is the weight variable related to the r^{th} output for each scenario s $r = 1, \ldots, q$, $s = 1, \ldots, S$;
$u^s_0 \in \mathbb{R}$:	is the scale variable for each scenario s, $s = 1, \ldots, S$.

The objective function (12) of the primal deterministic model $(\widehat{P1_V})$ represents the expected value of the efficiency of DMU j_0 with respect to all the possible realizations of the random factor ξ. In this model, all the scenarios are considered and for each scenario an optimal weight composition is established. In this light, by solving the model for each DMU, a global efficiency index is obtained taking into account the different reaction of each DMU to extreme scenarios. The scenario based method is very flexible, it does not require the use of a specific probability distribution and it allows concrete applications.

In particular, taking into account constraints (14), with simple calculations it results:

$$u_0^s \leq \min_j \left\{ \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{q} u_r^s y_{rj}^s \right\} \quad s = 1, \ldots, S$$

so that, since problem $(P1_V)$ is a maximization problem, the optimal solution $(u_0^s)^*$ assumes the following values:

$$(u_0^s)^* = \min_j \left\{ \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{q} u_r^s y_{rj}^s \right\} \quad s = 1, \ldots, S.$$

This result implies that, for each scenario s and for each DMU j, variable $(u_0^s)^* < 0$ can be interpreted as the maximum surplus (gap between weighted outputs and inputs for the most efficient unit), while in the case $(u_0^s)^* > 0$ it represents the minimum deficit (gap between weighted inputs and outputs for the most efficient unit). In other words, in each scenario, variable $(u_0^s)^*$ gives informations about the most efficient DMU in maximizing surplus or minimizing deficit.

As in the case of model $(P1_V)$, the corresponding dual deterministic model can be obtained as follows:

$$\min_{\theta, \tilde{\lambda}} \quad \theta$$

$$s.t. \quad \sum_{j=1}^{n} \tilde{\lambda}_j^s y_{rj}^s \geq p^s y_{rj_0}^s \quad r = 1, \ldots, q, \quad s = 1, \ldots, S \tag{15}$$

$$\sum_{s=1}^{S} \sum_{j=1}^{n} \tilde{\lambda}_j^s x_{ij} \leq \theta x_{ij_0} \quad i = 1, \ldots, m \tag{16}$$

$$\sum_{j=1}^{n} \tilde{\lambda}_j^s = p^s \quad s = 1, \ldots, S$$

$$\tilde{\lambda}_j^s \geq 0 \quad j = 1, \ldots, n \quad s = 1, \ldots, S$$

In order to improve the dual formulation, with a simple variable substitution, we have the following equivalent version:

$(D1_V)$

$$
\begin{aligned}
\min_{\theta,\lambda} \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j^s y_{rj}^s \geq y_{rj_0}^s \quad r=1,\ldots,q, \quad s=1,\ldots,S && (17) \\
& \sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s x_{ij} \leq \theta x_{ij_0} \quad i=1,\ldots,m && (18) \\
& \sum_{j=1}^{n} \lambda_j^s = 1 \quad s=1,\ldots,S \\
& \lambda_j^s \geq 0 \quad j=1,\ldots,n \quad s=1,\ldots,S
\end{aligned}
$$

where $\tilde{\lambda}_j^s = p^s \lambda_j^s$.

Problem $(D1_V)$ optimizes the values of λ_j^s in order to compare each DMU j_0 with a reference efficient unit, obtained as an optimal combination between available inputs and outputs. Constraints (17) impose that the reference unit produces, for each kind of outputs and in each scenario, a quantity of outputs greater or equal to the one produced by the considered unit j_0. Constraints (18) mean that the reference unit uses, in expected value for each input, a total amount lower or equal to a fraction θ of inputs consumed by the considered unit j_0. The variable θ represents the reduction inputs factor, which states how much the inputs consumed by DMU j_0 can be reduced in order to improve its efficiency.

3.2 Deterministic VRS2

The problem formulation is stated below:

$(P2_V)$

$$
\begin{aligned}
\max_{u,v,u_0} \quad & \sum_{s=1}^{S} p^s \left[\sum_{r=1}^{q} u_r y_{rj_0}^s + u_0^s \right] && (19) \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ij_0} = 1 && (20) \\
& \sum_{r=1}^{q} u_r y_{rj}^s + u_0^s - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j=1,\ldots,n, \ s=1,\ldots,S && (21) \\
& u_r \geq 0 \quad r=1,\ldots,q \\
& v_i \geq 0 \quad i=1,\ldots,m \\
& u_0^s \in \mathbb{R} \quad s=1,\ldots,S
\end{aligned}
$$

where:
$u_r \in \mathbb{R}_+$: weight variable related to the r^{th} output, $r=1,\ldots,q$.

The dual corresponding model, after the same transformation stated in $(D1_V)$, results to be:

$(D2_V)$

$$\min_{\theta,\lambda} \theta$$

$$s.t. \quad \sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s y_{rj}^s \geq \sum_{s=1}^{S} p^s y_{rj_0}^s \quad r = 1, \ldots, q \tag{22}$$

$$\sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s x_{ij} \leq \theta x_{ij_0} \quad i = 1, \ldots, m \tag{23}$$

$$\sum_{j=1}^{n} \lambda_j^s = 1 \quad s = 1, \ldots, S$$

$$\lambda_j^s \geq 0 \quad j = 1, \ldots, n \quad s = 1, \ldots, S$$

The difference between $(P1_V)$ and $(P2_V)$, as mentioned before, concerns the choice of weights for the uncertain outputs. In $(P1_V)$ we find out for each scenario the optimal weight composition u_r^s maximizing the efficiency of the DMU denoted with j_0. The DMU j_0 global efficiency index is the expected value of each scenario efficiency index. In model $(P2_V)$ we find out a unique weight for each output that can handle the uncertainty, whatever scenario will occur. In the dual versions $(D1_V)$ and $(D2_V)$, the models differ in constraints (17) and (22). In model $(D1_V)$ constraints (17) hold for each scenario $s = 1 \ldots S$, that is to say for each output in each scenario; in model $(D2_V)$ constraints (22) for each output hold in expected value.

4 A particular case: Constant Returns to Scale

The pioneer model of Data Envelopment Analysis by Charnes at al. [4] determine the most efficient DMU under the assumption of constant returns to scale.
Constant returns to scale means that the producers are able to linearly scale the inputs and outputs without increasing or decreasing efficiency. In other words, if we assume that CRS holds, we assume that if (x, y) is couple of input-output feasible, then for any $k \geq 0$, (kx, ky) is also feasible. This is a very tightening assumption. For this reason, CRS tends to lower the efficiency scores while VRS tends to raise efficiency scores.

In this section the corresponding couples of VRS1 and VRS2 models arising from the assumption of constant returns to scale are presented and a theoretical result on the optimal solution for the corresponding CRS2 is proved. The deterministic CRS1 formulation corresponding to VRS1 is as follows:

$(P1_C)$

$$\max_{u,v} \quad \sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r^s y_{rj_0}^s \tag{24}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{25}$$

$$\sum_{r=1}^{q} u_r^s y_{rj}^s - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \ldots, n, \quad s = 1, \ldots, S \tag{26}$$

$$u_r^s \geq 0 \quad r = 1, \ldots, q, \quad s = 1, \ldots, S$$

$$v_i \geq 0 \quad i = 1, \ldots, m$$

$(D1_C)$

$$\min_{\theta,\lambda} \quad \theta$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j^s y_{rj}^s \geq y_{rj_0}^s \quad r = 1, \ldots, q, \quad s = 1, \ldots, S \tag{27}$$

$$\sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s x_{ij} \leq \theta x_{ij_0} \quad i = 1, \ldots, m \tag{28}$$

$$\lambda_j^s \geq 0 \quad j = 1, \ldots, n \quad s = 1, \ldots, S$$

Notice that the difference between constant and variable returns to scale is in the absence of variables u_0^s in the primal formulation and in the absence of the constraints $\sum_{j=1}^{n} \lambda_j^s = 1$, $s = 1, \ldots, S$, in the corresponding dual model. These differences hold also for the couple CRS2 of primal and dual of VRS2 which is defined as follows:

$(P2_C)$

$$\max_{u,v} \quad \sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r y_{rj_0}^s \tag{29}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{30}$$

$$\sum_{r=1}^{q} u_r y_{rj}^s - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \ldots, n, \quad s = 1, \ldots, S \tag{31}$$

$$u_r \geq 0 \quad r = 1, \ldots, q$$

$$v_i \geq 0 \quad i = 1, \ldots, m$$

$(D2_C)$

$$\min_{\theta,\lambda} \quad \theta$$

$$s.t. \quad \sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s y_{rj}^s \geq \sum_{s=1}^{S} p^s y_{rj_0}^s \quad r = 1, \ldots, q \tag{32}$$

$$\sum_{s=1}^{S} p^s \sum_{j=1}^{n} \lambda_j^s x_{ij} \leq \theta x_{ij_0} \quad i = 1, \ldots, m \tag{33}$$

$$\lambda_j^s \geq 0 \quad j = 1, \ldots, n \quad s = 1, \ldots, S$$

The following theorem shows that, in the case of CRS2, the most efficient units are characterized by an efficiency equal to one if and only if their efficiency is equal to one for each single scenario.

Theorem 4.1 *Consider problem* $(P2_C)$ *and assume* $\sum_{i=1}^{S} p^s = 1$, $p^s > 0\ \forall s$. *Then,*

$$\sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r y_{rj_0}^s = 1 \Longleftrightarrow \sum_{r=1}^{q} u_r y_{rj_0}^s = 1 \quad \forall s = 1, \ldots, S$$

Proof $(\Leftarrow)$ Taking into account that $\sum_{r=1}^{q} u_r y_{rj_0}^s = 1 \quad \forall s = 1, \ldots, S$. the thesis follows trivially.
$(\Rightarrow)$ From the model $(P2_C)$, taking into account constraints (30)-(31), we obtain $\sum_{r=1}^{q} u_r y_{rj_0}^s \leq \sum_{i=1}^{m} v_i x_{ij_0} = 1$. Suppose, by contradiction, that there exist a scenario $\overline{s}$ and a value $\epsilon > 0$ such that

$$\sum_{r=1}^{q} u_r y_{rj_0}^{\overline{s}} \leq 1 - \epsilon$$

This implies that

$$\begin{aligned} \sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r y_{rj_0}^s &= \sum_{s \neq \overline{s}} p^s \sum_{r=1}^{q} u_r y_{rj_0}^s + p^{\overline{s}} \sum_{r=1}^{q} u_r y_{rj_0}^{\overline{s}} = \\ &= 1 - p^{\overline{s}} + p^{\overline{s}}(1-\epsilon) = 1 - p^{\overline{s}}\epsilon < 1 \end{aligned}$$

□

In the light of Theorem 4.1, the following simple example proves that there exist cases in which no DMU reaches the level of efficiency equal to one.

Example 4.1 We suppose to have a simple instance in which it is:
$n = 2, \quad m = 1, \quad q = 1, \quad S = 3, \quad p^s = \frac{1}{3}\ \forall s.$
Suppose also that $y_{11}^1 > y_{11}^2 > y_{11}^3 > 0$.
The corresponding Problem $(P2_C)$ for DMU1 is:

$$\begin{aligned}
\max_{u_1} \quad & \frac{1}{3} u_1 \sum_{s=1}^{3} y_{11}^s \\
s.t. \quad & v_1 x_{11} = 1 && (34) \\
& u_1 y_{11}^s \le v_1 x_{11}, \quad s = 1, 2, 3 && (35) \\
& u_1 y_{12}^s \le v_1 x_{12}, \quad s = 1, 2, 3 \\
& u_1 \ge 0 \\
& v_1 \ge 0
\end{aligned}$$

Taking into account constraints (34)-(35), we obtain $u_1 y_{11}^s \le 1\ \forall s$ which yields

$$u_1 \le \min\left\{\frac{1}{y_{11}^1}, \frac{1}{y_{11}^2}, \frac{1}{y_{11}^3}\right\}$$

Since $y_{11}^1 > y_{11}^2 > y_{11}^3 > 0$ we have that $u_1 \le \frac{1}{y_{11}^1}$ and hence

$$\frac{1}{3} u_1 \sum_{s=1}^{3} y_{11}^s \le \frac{1}{3} \frac{1}{y_{11}^1} \left(y_{11}^1 + y_{11}^2 + y_{11}^3\right) = \frac{1}{3}\left(1 + \frac{y_{11}^2}{y_{11}^1} + \frac{y_{11}^3}{y_{11}^1}\right) < 1$$

In the light of Theorem 4.1, the efficiency equal to one is not reached since it is impossible to have $u_1 = \frac{1}{y_{1j}^s}$ for all $s = 1, 2, 3$.
The same simple calculations hold for DMU2, so that no DMU reaches the level of efficiency equal to one.

5 Models effectiveness and computational results

The aim of this section is to analyze the behaviour of the proposed models in order to point out their effectiveness. This will be done by means of computational tests which compare the proposed models with two further deterministic models based on the expected value approach.

5.1 Further deterministic models

The basic idea of the proposed models VRS1 and VRS2 is that the efficiency inequality constraints (14) and (21) have to be verified for each scenario $s = 1, \ldots, S$, that is to say that the efficiency index has to be smaller or equal to one whatever scenario will occur. Actually, in the literature, this hypothesis is often relaxed by considering efficiency constraints in expected value. In this light, it is worth comparing the proposed models VRS1 and VRS2 with another model, named VRSqEV, having the following Primal-Dual formulation:

$(PqEV_V)$

$$\max_{u,v,u_0} \quad \sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r^s y_{rj_0}^s + u_0 \tag{36}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{37}$$

$$\sum_{s=1}^{s} p^s \sum_{r=1}^{q} u_r^s y_{rj}^s + u_0 - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \dots, n \tag{38}$$

$$u_r^s \geq 0 \quad r = 1, \dots, q, \quad s = 1, \dots, S$$

$$v_i \geq 0 \quad i = 1, \dots, m$$

$$u_0 \in \mathbb{R}$$

$(DqEV_V)$

$$\min_{\theta,\lambda} \quad \theta$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j y_{rj}^s \geq y_{rj_0}^s \quad r = 1, \dots q, \quad s = 1, \dots, S \tag{39}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ij_0} \quad i = 1, \dots, m \tag{40}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad j = 1, \dots, n$$

In Problems $(P1_V)$ and $(P2_V)$, constraints (14) and (21) define, for each scenario s, an optimal compositions of weights feasible for all DMUs and so that the maximum level of efficiency is one for each DMU and for each scenario s. In $(PqEV_V)$ the linear constraints (38) are satisfied in expected value on all possible scenarios. This means, for example, that in some scenarios the efficiency level can be greater than one, but it is smaller or equal to one in expected value. As a consequence, the dual formulations of VRS1 and VRS2 differ from $(DqEV_V)$ in the corresponding dual variables λ: in $(D1_V)$ and $(D2_V)$ models variables λ depend on the selected scenario s.

For the sake of completeness, it is finally worth comparing the proposed models VRS1 and VRS2 with the classical expected value model VRSEV, obtained by substituting the random factors with their expected values:

(PEV_V)

$$\max_{u,v,u_0} \quad \sum_{r=1}^{q} u_r \overline{y}_{rj_0} + u_0 \tag{41}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1 \tag{42}$$

$$\sum_{r=1}^{q} u_r \overline{y}_{rj} + u_0 - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1, \ldots, n \tag{43}$$

$$u_r \geq 0 \quad r = 1, \ldots, q$$

$$v_i \geq 0 \quad i = 1, \ldots, m$$

$$u_0 \in \mathbb{R}$$

(DEV_V)

$$\min_{\theta,\lambda} \quad \theta$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j \overline{y}_{rj} \geq \overline{y}_{rj_0} \quad r = 1, \ldots q, \tag{44}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ij_0} \quad i = 1, \ldots, m \tag{45}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad j = 1, \ldots, n$$

where $\overline{y}_{rj} = \sum_{s=1}^{S} p^s y_{rj}^s$.

From a computational point of view, as it will be pointed out in the following subsections, since constraints (14) and (21) have to be satisfied for all units and for all scenarios, VRS1 and VRS2 models provide a ranking of the units efficiency more accurate than VRSqEV and VRSEV.

5.2 Comparing VRS1, VRS2, VRSqEV and VRSEV

Models VRS1 and VRS2 have been implemented in order to test them in comparison with the models based on the expected value approach. In all the simulations, different outputs distributions have been considered. In particular, the results concerning normal and beta distribution are presented. The data generation procedure has been implemented in MatLab 2008b by using Halton Sequences. The optimal solution of the models is obtained by solving the linear formulation with AMPL+CPLEX v.11. Different instances have been tested varying the number of DMUs, the number of inputs and outputs. For each instance the input data have been generated in the interval [0,1500] by using the "rand" MatLab function. Output uncertain parameters have been generated taking into account different distributions and different mean and standard deviation values.

A first computational test compares the optimal solutions of VRS1, VRS2, VRSqEV and VRSEV models. In particular, 32 different classes of problems have been considered varying the data distribution function and the number of inputs, outputs and DMUs. For each class of problems we fix the number of scenarios equal to 500 and we generate 1000 random instances. In Table 1 the results concerning the mean, minimum and maximum number of efficient units for each class of problems are collected. Models VRS1 and VRS2 results to be very selective in identifying the most efficient units. On the other hand, model VRSqEV results to be unable to provide a valuable ranking of the units, as it is pointed out by the average number of efficient units which is close to the half of the number of units themselves.

I/O	DMU	VRS1						VRS2						VRSqEV						VRSEV					
		N			B			N			B			N			B			N			B		
		$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M	$\overline{m}$	m	M
4 I 5 O	**8**	1.34	1	3	1.58	1	5	1.26	1	3	1.51	1	5	5.21	1	8	5.35	2	8	2.38	1	5	2.38	1	5
	10	1.34	1	4	1.62	1	4	1.27	1	4	1.55	1	4	6.15	3	10	6.23	3	10	2.52	1	5	2.52	1	5
	15	1.31	1	3	1.60	1	5	1.25	1	3	1.54	1	5	8.52	3	13	8.39	4	13	2.79	1	6	2.80	1	6
	30	1.33	1	3	1.67	1	5	1.25	1	3	1.61	1	5	14.18	4	25	12.62	2	24	3.21	1	8	3.31	1	8
4 I 10 O	**8**	1.34	1	3	1.76	1	4	1.26	1	3	1.70	1	4	5.25	2	8	5.37	2	8	2.36	1	5	2.36	1	5
	10	1.32	1	3	1.78	1	5	1.23	1	3	1.70	1	5	6.24	2	10	6.31	2	10	2.52	1	6	2.51	1	6
	15	1.32	1	3	1.84	1	5	1.25	1	3	1.77	1	5	8.52	4	14	8.38	4	13	2.85	1	7	2.85	1	7
	30	1.35	1	3	1.90	1	6	1.28	1	3	1.81	1	6	14.08	4	23	12.45	2	20	3.15	1	7	3.28	1	7
2 I 5 O	**8**	1.30	1	4	1.56	1	4	1.24	1	3	1.50	1	4	5.24	2	8	5.36	2	8	2.35	1	5	2.36	1	5
	10	1.34	1	3	1.62	1	6	1.24	1	3	1.56	1	4	6.22	2	10	6.30	3	10	2.50	1	7	2.50	1	7
	15	1.31	1	3	1.63	1	5	1.26	1	3	1.57	1	5	8.49	4	14	8.35	4	14	2.79	1	7	2.80	1	7
	30	1.34	1	3	1.67	1	5	1.27	1	3	1.60	1	5	14.06	3	22	12.48	2	20	3.20	1	7	3.26	1	7
2 I 10 O	**8**	1.34	1	3	1.78	1	4	1.28	1	3	1.71	1	4	5.28	2	8	5.41	2	8	2.39	1	5	2.39	1	5
	10	1.29	1	3	1.77	1	5	1.22	1	3	1.71	1	5	6.24	2	10	6.34	2	10	2.49	1	5	2.49	1	5
	15	1.32	1	3	1.80	1	5	1.26	1	3	1.74	1	5	8.44	4	14	8.34	4	13	2.81	1	6	2.81	1	6
	30	1.34	1	4	1.93	1	6	1.26	1	3	1.86	1	6	13.91	6	21	12.23	2	20	3.25	1	7	3.39	1	7

Table 1: Test on DMU efficiency: 500 scenarios, 1000 instances, Mean ($\overline{m}$), minimum (m) and maximum (M) number of efficient DMUs

Morerover, by considering the minimum and maximum number of efficient DMUs the results show that models VRS1 and VRS2, both in the case of normal and beta distribution, have a minimum number of efficient DMUs equal to 1 and have a maximum number of efficient DMUs smaller or equal than the expected values based models. This implies that the results are stable in terms of efficiency and confirms the selectiveness of the proposed models VRS1 and VRS2. As regards to the two models based on the expected value approach, in VRSqEV the maximum number of efficient DMUs often reaches the total number of considered DMUs. Generally speaking, the behaviour of model VRSqEV suggests a high overestimation of efficient DMUs.

In order to deeply analyze the behaviour of the proposed models, the results concerning a single instance are fully reported in Table 2 by varying the number of scenarios. We consider 4 inputs, 5 outputs and 8 DMUs; the number of considered scenarios varies between 5 and 500 scenarios. Taking into account Table 2, it can be observed that models VRS1 and VRS2 are very selective in identifying the most efficient units and the hierarchy between the different DMUs. In particular, the most efficient unit DMU4 with score one is the most efficient for all the four models. Due to the high selectiveness of our models, even the smallest difference between DMUs can be recognized. Specifically speaking, it is worth noticing that both VRS1 and VRS2 models are able to classify DMU5, DMU7 and DMU1 as second, third and fourth, respectively, while for both VRSEV and VRSqEV models these three DMUs have the same maximum score. Notice also that, for normal distribution, this accuracy can be observed even in the case of a small number of scenario. It can be observed also that VRSqEV model seems even less selective in identifying efficient units than VRSEV; in fact, in VRSqEV model the DMUs reaching the maximum score are the ones having the maximum score in VRSEV model plus the DMU2 unit. Another aspect to point out is that the inefficient DMUs for expected value based models are also inefficient in the scenario based models.

DMU2 results to be one of the most efficient DMUs in VRSqEV. Nevertheless, it reaches in VRS1 and VRS2 formulations a low position in ranking score. In order to understand this result, Table 3 shows, in the case of DMU2, the constraints of the primal formulation of $(PqEV_V)$ model corresponding to constraints (14) for $(P1_V)$ and (21) for $(P2_V)$. In the first four columns of Table 3 there are the constraints (38) of $(PqEV_V)$ model in the case of 5 scenarios and it results that the constraints are satisfied according to their expected value based formulation. On the other hand, if the constraints are considered scenario by scenario it comes out that some of the constraints are not satisfied in all scenarios, that is to say that an efficiency level greater than one holds in some scenario s. This paradoxical situation can not be accepted in VRS1 and VRS2. It can then be concluded that VRS1 and VRS2 are more selective in identifing the best efficients units.

FVal	DMU1		DMU2		DMU3		DMU4		DMU5		DMU6		DMU7		DMU8	
	Norm	Beta	Norm	Beta	Norm	Beta	Norm	Beta	Norm	Beta	Norm	Beta	Norm	Beta	Norm	Beta
5 scen																
VRS1	0.5617	0.9106	0.4822	0.5582	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	0.5138	0.8692	0.0211	0.0211
VRS2	0.4505	0.7876	0.2637	0.4313	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	0.3369	0.8553	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
VRSEV	1.0000	1.0000	0.5863	0.6020	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
10 scen																
VRS1	0.6534	0.9158	0.5000	0.5641	0.0188	0.0190	1.0000	1.0000	0.8949	1.0000	0.0939	0.0948	0.7569	0.9346	0.0211	0.0212
VRS2	0.5401	0.8271	0.3497	0.4673	0.0188	0.0188	1.0000	1.0000	0.8847	1.0000	0.0911	0.0911	0.6559	0.9277	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0188	0.0213	1.0000	1.0000	1.0000	1.0000	0.1185	0.1279	1.0000	1.0000	0.0211	0.0222
VRSEV	1.0000	1.0000	0.6476	0.6267	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
20 scen																
VRS1	0.6694	0.9202	0.5238	0.5718	0.0197	0.0198	1.0000	1.0000	0.9475	1.0000	0.0925	0.0930	0.7845	0.9673	0.0211	0.0211
VRS2	0.5663	0.8360	0.3243	0.4491	0.0188	0.0188	1.0000	1.0000	0.9318	1.0000	0.0911	0.0911	0.6200	0.9501	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0314	0.0300	1.0000	1.0000	1.0000	1.0000	0.1185	0.1279	1.0000	1.0000	0.0211	0.0222
VRSEV	1.0000	1.0000	0.5797	0.5779	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
50 scen																
VRS1	0.7194	0.9083	0.4920	0.5546	0.0194	0.0196	1.0000	1.0000	0.9551	1.0000	0.0953	0.0958	0.8421	0.9869	0.0213	0.0215
VRS2	0.5840	0.8529	0.3145	0.4426	0.0188	0.0188	1.0000	1.0000	0.9285	1.0000	0.0911	0.0911	0.7194	0.9727	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0314	0.0300	1.0000	1.0000	1.0000	1.0000	0.1613	0.1468	1.0000	1.0000	0.0291	0.0317
VRSEV	1.0000	1.0000	0.5481	0.5482	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
100 scen																
VRS1	0.7443	0.9144	0.4878	0.5603	0.0192	0.0195	1.0000	1.0000	0.9574	1.0000	0.0976	0.0972	0.8592	0.9865	0.0214	0.0216
VRS2	0.6074	0.8675	0.3206	0.4450	0.0188	0.0188	1.0000	1.0000	0.9238	1.0000	0.0911	0.0911	0.7244	0.9692	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0314	0.0361	1.0000	1.0000	1.0000	1.0000	0.2587	0.2716	1.0000	1.0000	0.0374	0.0317
VRSEV	1.0000	1.0000	0.5438	0.5414	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
200 scen																
VRS1	0.7444	0.9112	0.4932	0.5712	0.0193	0.0194	1.0000	1.0000	0.9456	0.9973	0.0967	0.0964	0.8565	0.9821	0.0214	0.0215
VRS2	0.6021	0.8642	0.3166	0.4417	0.0188	0.0188	1.0000	1.0000	0.9037	0.9972	0.0911	0.0911	0.7154	0.9639	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0314	0.0388	1.0000	1.0000	1.0000	1.0000	0.2587	0.2981	1.0000	1.0000	0.0374	0.0321
VRSEV	1.0000	1.0000	0.5421	0.5401	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211
500 scen																
VRS1	0.7515	0.9053	0.4885	0.5659	0.0193	0.0194	1.0000	1.0000	0.9428	0.9951	0.0968	0.0970	0.8534	0.9859	0.0214	0.0215
VRS2	0.6024	0.8632	0.3136	0.4407	0.0188	0.0188	1.0000	1.0000	0.8895	0.9944	0.0911	0.0911	0.7136	0.9687	0.0211	0.0211
VRSqEV	1.0000	1.0000	1.0000	1.0000	0.0716	0.0538	1.0000	1.0000	1.0000	1.0000	0.3684	0.3809	1.0000	1.0000	0.0399	0.0444
VRSEV	1.0000	1.0000	0.5392	0.5389	0.0188	0.0188	1.0000	1.0000	1.0000	1.0000	0.0911	0.0911	1.0000	1.0000	0.0211	0.0211

Table 2: Variable returns to scale: 4 inputs, 5 outputs, 8 DMUs

DMU	$\sum_{s=1}^{S} p^s \sum_{r=1}^{q} u_r^s y_{rj}^s$	$\leq$	$\sum_{i=1}^{m} v_i x_{ij} - u_0 \quad \forall j$	Scen	$\sum_{r=1}^{q} u_r^s y_{rj}^s$	$\leq$	$\sum_{i=1}^{m} v_i x_{ij} - u_0 \quad \forall j \; \forall s$
1				1	16.5049	$\leq$	8.0283
1				2	0	$\leq$	8.0283
1	8.0283	$\leq$	8.0283	3	13.8874	$\leq$	8.0283
1				4	9.7491	$\leq$	8.0283
1				5	0	$\leq$	8.0283
2				1	24.0499	$\leq$	8.3635
2				2	0	$\leq$	8.3635
2	8.3635	$\leq$	8.3635	3	9.29076	$\leq$	8.3635
2				4	8.4769	$\leq$	8.3635
2				5	0	$\leq$	8.3635
3				1	3.7220	$\leq$	9.6888
3				2	0	$\leq$	9.6888
3	2.3174	$\leq$	9.6888	3	5.1008	$\leq$	9.6888
3				4	2.7644	$\leq$	9.6888
3				5	0	$\leq$	9.6888
4				1	8.1538	$\leq$	7.4071
4				2	0	$\leq$	7.4071
4	4.5331	$\leq$	7.4071	3	11.1219	$\leq$	7.4071
4				4	3.3899	$\leq$	7.4071
4				5	0	$\leq$	7.4071
5				1	17.1815	$\leq$	7.4853
5				2	0	$\leq$	7.4853
5	7.4853	$\leq$	7.4853	3	15.7576	$\leq$	7.4853
5				4	4.4875	$\leq$	7.4853
5				5	0	$\leq$	7.4853
6				1	7.5644	$\leq$	7.8420
6				2	0	$\leq$	7.8420
6	3.7388	$\leq$	7.8420	3	7.8153	$\leq$	7.8420
6				4	3.3141	$\leq$	7.8420
6				5	0	$\leq$	7.8420
7				1	24.3288	$\leq$	9.2015
7				2	0	$\leq$	9.2015
7	9.2015	$\leq$	9.2015	3	15.0977	$\leq$	9.2015
7				4	6.5811	$\leq$	9.2015
7				5	0	$\leq$	9.2015
8				1	5.7368	$\leq$	9.4354
8				2	0	$\leq$	9.4354
8	2.7788	$\leq$	9.4354	3	6.1825	$\leq$	9.4354
8				4	1.9749	$\leq$	9.4354
8				5	0	$\leq$	9.4354

Table 3: Constraints comparison

5.3 Comparing CRS1, CRS2, CRSqEV and CRSEV

The same computational tests have been done for constant return to scale models. For the sake of compactness we omit the corresponding Tables. As it is shown in Theorem 4.1, the efficiency level equal to one can be reached in extreme cases. From a computational point of view, this result is confirmed in our tests. The most efficient unit for CRS1 and CRS2, even if it does not reach the unitary value, is the same than in models CRSEV and CRSqEV. A ranking of efficiency in the scenario based models can be stated with more accuracy than the one obtained with the expected value approach. As in the case of corresponding models under the assumption of variable returns to scale, it can be observed that the models CRSqEV appears to be less selective in identifying the most efficient units.

6 Conclusions

This paper presents a scenario approach to handle uncertainty in a Data Envelopment Analysis model. Two different formulations are defined both with Variable Returns to Scale (VRS1 and VRS2) and with Constant Returns to Scale (CRS1 and CRS2). Both VRS1 and VRS2 models show a different approach to manage uncertainty in data with respect to the main literature. The main research field in DEA considers a chance-constrained approach which permits constraint violations up to specified probability limits and which generally assume normal data distribution. The proposed models remove the hypothesis of normal data distribution and uses a scenario generation approach to include data perturbations. The results of a full computational test are collected by comparing VRS1 and VRS2 formulations with two alternative formulations, a first one differing in constraints formulation (VRSqEV) and the other one considering a classical expected value model (VRSEV). It can be observed that the rank between DMUs efficiency scores is different, some efficient DMUs in the models VRSqEV and VRSEV are not efficient neither in VRS1 nor in VRS2. A simple example remarks the reason of such a different behaviour. The proposed models VRS1 and VRS2 can be a useful strategic management tool aimed to determine a restrictive efficiency score ranking. A natural future validation of these models could be their application to concrete problems discussing a more accurate scenario selection and parameter estimation.

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- ☐ n. 9 Ester Lari, Alcune osservazioni sull'equazione funzionale Ø(x,y,z)=Ø(Ø(x,y,t),t,z)
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- ☐ n. 11 Giovanni Boletto, Metodi di scomposizione del tasso di inflazione
- ☐ n. 12 Claudio Sodini, A New Algorithm for the Strictly Convex Quadratic Programming Problem
- ☐ n. 13 Laura Martein, On Generating the Set of all Efficient Points of a Bicriteria Fractional Problem
- ☐ n. 14 Laura Martein, Applicazioni della programmazione frazionaria nel campo economico-finanziario
- ☐ n. 15 Laura Martein, On the Bicriteria Maximization Problem
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- ☐ n. 30 Anna Marchi, A sequential method for a bicriteria problem arising in portfolio selectiontheory.
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- ☐ n. 51 Gilberto Ghilardi, Elementi di uno schema di campionamento areale per alcune rilevazioni ufficiali in Italia.
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- ☐ n. 53 Laura Lecchini - Odo Barsotti, Le rôle des immigrés extra- communautaires dans le marché du travail

- ☐ n. 54 Riccardo Cambini, Alcune condizioni di ottimalità relative ad un insieme stellato.
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- ☐ n. 112 Riccardo Cambini, Generalized concavity for bicriteria functions.
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- ☐ n. 130 Alessandro Valentini, Impatto delle immigrazioni sulla popolazione italiana: confronto tra scenari alternativi.
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