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On solving a class of rank-two nonconvex programs by means of parametric quadratic semidefinite subproblems

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Abstract

The aim of this paper is to propose a solution algorithm for a particular class of rank-two nonconvex programs having a polyhedral feasible region. The considered problems cover both multiplicative, fractional and d.c. quadratic/linear programs. The algorithm is based on the so called "optimal level solutions" method. The subproblems obtained by means of this parametrical approach are quadratic semidefinite ones. The use of underestimation functions is studied with the aim of improving the efficiency of the algorithm. The results of a computational test are provided and discussed.

Key words: low-rank programs, optimal level solutions, global optimization.
AMS Math. Subj. Class. 90C05, 90C26, 90C31.
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1 Introduction

The aim of this paper is to study, from both a theoretical, an algorithmic and a computational point of view, the following class of rank-two nonconvex programs:

$$P: \begin{cases} \inf f(x) = \phi(\frac{1}{2}x^TQx + q^Tx + q_0, d^Tx + d_0) \\ x \in X = \{x \in \Re^n : Ax \leq b\} \end{cases}$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$, $q, d \in \Re^n$, $q_0, d_0 \in \Re$, $Q \in \Re^{n\times n}$ is positive semidefinite and $X \neq \emptyset$. The scalar function $\phi(y_1, y_2)$ is assumed to be continuous and strictly increasing with respect to variable y_1, and is defined for all the values in Ω where:

$$\Omega = \{(y_1, y_2) \in \Re^2 : \ y_1 = \frac{1}{2}x^TQx + q^Tx + q_0, \ y_2 = d^Tx + d_0, \ x \in X\}$$

The considered class of objective functions $f(x)$ is extremely wide and it covers both multiplicative, fractional and d.c. quadratic/linear functions. Just as an example, given any strictly increasing real function g_1, any positive function g_2 and any real function g_3, then the following functions $f(x)$ verify the assumptions of problem P by using $\phi(y_1, y_2) = g_1(y_1)g_2(y_2) + g_3(y_2)$ (see also [2, 3, 4, 5, 7]):

$$f(x) = g_1\left(\frac{1}{2}x^TQx + q^Tx + q_0\right) g_2\left(d^Tx + d_0\right) + g_3\left(d^Tx + d_0\right) \quad (1)$$

$$f(x) = g_1\left(q^Tx + q_0\right) g_2\left(d^Tx + d_0\right) + g_3\left(d^Tx + d_0\right) \quad (2)$$

Notice that function (2) uses $Q = 0$ as its quadratic part.

In other words, the aim of this paper is to provide an unifying framework for solving both the problems approached in [2, 5], involving two linear functions in the objectives, and the ones studied in [3, 4, 7], where the objective function contains a linear function and a quadratic positive definite one. In this light, notice that determining such an unique solution method is not trivial since the solution approach proposed in [3, 4, 7] is based on the nonsingularity of matrix Q while the algorithms studied in [2, 5] are based on the properties of linear programs.

The solution algorithm proposed in this paper is based on the so called "optimal level solutions" method (see [1, 2, 3, 4, 5, 6, 7, 8, 11]). It is known that this is a parametric method, which finds the optimum of the problem by determining the minima of particular subproblems. In particular, the optimal solutions of these subproblems are obtained by means of a sensitivity analysis which maintains the Karush-Kuhn-Tucker optimality conditions.

In Section 2 and Section 3 we describe how the optimal level solutions method can be applied to problem P. In Section 4 a solution algorithm is proposed and fully described. Finally, in Section 5 the results of a computational test are provided and discussed.

2 A parametric approach

In this section we show how problem P can be approached by means of the so called *optimal level solutions method* (see for all [1, 8]). With this aim, let $\xi \in \Re$ be a real parameter and let us define the corresponding parametrical subset of X:

$$X_\xi = \{x \in \Re^n : \ Ax \leq b, \ d^Tx + d_0 = \xi\}$$

In the same way, given the real parameters $\xi_1, \xi_2 \in \Re$, $\xi_1 \leq \xi_2$, the following further subset of X can be defined:

$$X_{[\xi_1,\xi_2]} = \{x \in \Re^n : \ Ax \leq b, \ \xi_1 \leq d^T x + d_0 \leq \xi_2\}$$

The parameter $\xi \in \Re$ is said to be a *feasible level* if the set X_ξ is nonempty; in this light, the set:

$$\Lambda = \left\{\xi \in \Re : \ \xi = d^T x + d_0, \ x \in X\right\}$$

represents the set of all feasible levels. The convexity of the polyhedron X implies that the set Λ is a closed convex interval. In this light, the following further notations can be introduced:

$$\xi_{min} = \inf \Lambda = d_0 + \inf_{x \in X} d^T x \quad \text{and} \quad \xi_{max} = \sup \Lambda = d_0 + \sup_{x \in X} d^T x$$

Clearly, if X is a compact set then ξ_{min} and ξ_{max} are finite and the set Λ of feasible levels is compact too.

Adding to problem P the constraint $d^T x + d_0 = \xi$, we obtain the following subproblem:

$$\begin{cases} \inf \ \phi(\frac{1}{2}x^T Q x + q^T x + q_0, \xi) \\ \qquad\qquad x \in X_\xi \end{cases}$$

Taking into account that function $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1, the optimal solution of the previous problem can be determined by means of the following convex quadratic program:

$$P_\xi : \begin{cases} \inf \ f_\xi(x) = \frac{1}{2}x^T Q x + q^T x \\ \qquad\qquad x \in X_\xi \end{cases}$$

An optimal solution of problem P_ξ, if it exists, is called an *optimal level solution.* Given a feasible level $\xi \in \Lambda$, the set of optimal solutions of P_ξ is denoted with $S_\xi \subset X_\xi$, while the set of all the optimal level solutions is denoted with $S = \cup_{\xi \in \Lambda} S_\xi \subset X$. Obviously, an optimal solution of problem P is also an optimal level solution and, in particular, it is the optimal level solution with the smallest value. The idea of the optimal level solutions approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached.

Notice that being P_ξ, $\xi \in \Lambda$, a convex quadratic program then:

$$S_\xi \text{ is empty if and only if } \inf_{x \in X_\xi} f_\xi(x) = -\infty \tag{3}$$

In this light, it is worth studying whether or not it is possible that there exist $\xi_1, \xi_2 \in \Lambda$, $\xi_1 \neq \xi_2$, such that S_{ξ_1} is empty and S_{ξ_2} is nonempty. With this very aim, let us first denote the recession cone of the feasible region $X = \{x \in \Re^n : Ax \leq b\}$ as follows:

$$rec(X) = \{u \in \Re^n : Au \leq 0, \ u \neq 0\}$$

Notice that if $rec(X) \neq \emptyset$ then $A(x + tu) \leq b \ \forall x \in X, \forall u \in rec(X), \forall t \geq 0$.

Theorem 2.1 *Consider problem P. The following conditions are equivalent:*

i) $\exists \xi \in \Lambda$ *such that* X_ξ *is compact;*

ii) X_ξ *is compact* $\forall \xi \in \Lambda$*;*

iii) $\nexists u \in rec(X)$ *such that* $d^T u = 0$.

Proof i)⇒iii) Let X_ξ be compact with $\xi \in \Lambda$ and assume by contradiction that $\exists u \in rec(X)$ such that $d^T u = 0$. Let now be $x_0 \in X_\xi$ and consider the feasible half-line $x_0 + tu$, $t \geq 0$. Being $x_0 \in X_\xi$ it is $d^T x_0 + d_0 = \xi$; hence $d^T(x_0 + tu) + d_0 = \xi \ \forall t \geq 0$, and this contradicts the compactness of X_ξ.

iii)⇒ii) Assume by contradiction that $\exists \xi' \in \Lambda$ such that $X_{\xi'}$ is not compact. As a consequence, there exists an half-line such that $x' + tu \in X_{\xi'}$ $\forall t \geq 0$, with $u \in rec(X)$. This implies that $d^T(x' + tu) + d_0 = \xi' \ \forall t \geq 0$ and hence $d^T u = 0$, which is a contradiction.

ii)⇒i) This is trivial. □

Theorem 2.1 shows that there are no $\xi_1, \xi_2 \in \Lambda$, $\xi_1 \neq \xi_2$, such that X_{ξ_1} is compact and X_{ξ_2} is not compact. As a consequence, the following property holds:

$$\text{if } \exists \xi \in \Lambda \text{ such that } X_\xi \text{ is compact then } S_\xi \neq \emptyset \ \forall \xi \in \Lambda.$$

For the sake of completeness, the next theorem characterizes the existence of optimal level solutions in the case the sets X_ξ are not compact.

Theorem 2.2 *Consider problem P and assume that there exists* $\xi \in \Lambda$ *such that* X_ξ *is not compact. The following conditions are equivalent:*

i) $\exists \xi \in \Lambda$ *such that* $S_\xi \neq \emptyset$*;*

ii) $S_\xi \neq \emptyset \ \forall \xi \in \Lambda$*;*

iii) $q^T u \geq 0$ *for all* $u \in rec(X)$ *such that* $d^T u = 0$ *and* $Qu = 0$.

Proof *i)⇒iii)* Let $\xi \in \Lambda$ be such that $S_\xi \neq \emptyset$ and assume by contradiction that there exists $u \in rec(X)$ such that $d^T u = 0$, $Qu = 0$ and $q^T u < 0$. Let now be $x_0 \in X_\xi$ and consider the feasible half-line $x_0 + tu$, $t \geq 0$. Being $d^T u = 0$ the half-line $x_0 + tu$, $t \geq 0$, belongs to X_ξ. From $Qu = 0$ it yields:

$$\frac{1}{2}(x_0 + tu)^T Q(x_0 + tu) + q^T(x_0 + tu) = \frac{1}{2}x_0^T Q x_0 + q^T x_0 + tq^T u$$

so that $q^T u < 0$ implies:

$$\lim_{t \to +\infty} \frac{1}{2}(x_0 + tu)^T Q(x_0 + tu) + q^T(x_0 + tu) = -\infty$$

As a consequence, $S_\xi = \emptyset$ which is a contradiction.

iii)⇒ii) Assume by contradiction that $\exists \xi' \in \Lambda$ such that $S_{\xi'} = \emptyset$. This means that there exists an half-line $x_0 + tu$, $t \geq 0$, belonging to $X_{\xi'}$ such that:

$$\lim_{t \to +\infty} \frac{1}{2}(x_0 + tu)^T Q(x_0 + tu) + q^T(x_0 + tu) = -\infty \tag{4}$$

Being $x_0 + tu \in X_{\xi'}$ $\forall t \geq 0$ it yields $u \in rec(X)$ and $d^T u = 0$. Notice also that:

$$\frac{1}{2}(x_0 + tu)^T Q(x_0 + tu) + q^T(x_0 + tu) = \frac{1}{2}t^2 u^T Q u + t(x_0^T Q u + q^T u) + \frac{1}{2}x_0^T Q x_0 + q^T x_0$$

Hence, the positive semidefiniteness of Q implies that (4) holds only if $u^T Q u = 0$. Due to the positive semidefiniteness of Q, it is $u^T Q u = 0$ if and only if $Qu = 0$. As a consequence, it results:

$$\begin{aligned} \lim_{t \to +\infty} \frac{1}{2}(x_0 + tu)^T Q(x_0 + tu) + q^T(x_0 + tu) &= \frac{1}{2}x_0^T Q x_0 + q^T x_0 + tq^T u \\ &= -\infty \end{aligned}$$

and hence $q^T u < 0$, which is a contradiction.

ii)⇒i) This is trivial. □

Theorems 2.1 and 2.2 point out that either $S_\xi \neq \emptyset$ $\forall \xi \in \Lambda$ or $S_\xi = \emptyset$ $\forall \xi \in \Lambda$. As a consequence, taking into account of (3), if $\exists \xi \in \Lambda$ such that $S_\xi = \emptyset$ then problem P reduces to:

$$P \equiv \inf_{\xi \in \Lambda} \phi(-\infty, \xi)$$

For the sake of completeness, let us finally provide the following sufficient conditions for the existence of optimal level solutions.

Corollary 2.1 *Consider problem P. If one of the following conditions holds then $S_\xi \neq \emptyset \ \forall \xi \in \Lambda$:*

i) Q is a nonsingular positive definite matrix;

ii) there exist $\alpha \in \Re^n$ and $\beta \in \Re$ such that $q = Q\alpha + \beta d$.

Proof If X_ξ is compact $\forall \xi \in \Lambda$ the result is trivial. Assume now that there exists $\xi \in \Lambda$ such that X_ξ is not compact. If Q is nonsingular then P_ξ is a strictly convex quadratic problem for all $\xi \in \Lambda$ and hence its minimum exists. If there exist $\alpha \in \Re^n$ and $\beta \in \Re$ such that $q = Q\alpha + \beta d$ then $d^T u = 0$ and $Qu = 0$ imply $q^T u = 0$, so that the result follows from *iii)* of Theorem 2.2. □

3 Moving the parameter

Let x' be the optimal level solution corresponding to level $\xi' \in \Lambda$, that is the optimal solution of:

$$P_{\xi'} : \begin{cases} \inf \frac{1}{2} x^T Q x + q^T x \\ Ax \leq b \\ d^T x = \xi' - d_0 \end{cases}$$

Let:

$$\begin{aligned} B &= \{i \ : \ A_i x' = b_i, \ i = 1, \ldots, m\} \\ N &= \{1, \ldots, m\} \setminus B \end{aligned}$$

where A_i is the i-th row of A, and let A_B, A_N, b_B and b_N be the corresponding submatrices of A and b, respectively. The following theorem shows the existence of segments of optimal level solutions.

Theorem 3.1 *Consider problem P and let x' and x'' be optimal level solutions corresponding to the feasible levels ξ' and ξ'', $\xi' < \xi''$, respectively. If x' and x'' share the same set of binding constraints B and the columns of $\begin{bmatrix} A_B^T & d \end{bmatrix}$ are linearly independent then the points of the segment $x'(\theta) = x' + \theta \Delta_x$, $\Delta_x = \frac{x'' - x'}{\xi'' - \xi'}$, $\theta \in [0, \xi'' - \xi']$, are optimal level solutions corresponding to the feasible levels $\xi' + \theta$, respectively.*

Proof The points in the segment $x'(\theta)$, $\theta \in [0, \xi'' - \xi']$, are trivially feasible due to the convexity of X. First, it is worth noticing that all the points in such a segment shares the same set of binding constraints B. This follows since:

$$\begin{array}{lll} A_B x' = b_B \ , \ A_B x'' = b_B & \Rightarrow A_B x'(\theta) = b_B \ \forall \theta \in (0, \xi'' - \xi') \\ A_N x' < b_N \ , \ A_N x'' < b_N & \Rightarrow A_N x'(\theta) < b_N \ \forall \theta \in (0, \xi'' - \xi') \end{array}$$

Being x' and x'' optimal level solutions and being the columns of $\left[A_B^T \ d\right]$ linearly independent then the following Karush-Kuhn-Tucker conditions are both necessary and sufficient:

$$Qx' + q = A_B^T \mu'_B + d\lambda' \quad \text{with} \quad \mu'_B \leqq 0, \ \mu'_N = 0 \tag{5}$$

$$Qx'' + q = A_B^T \mu''_B + d\lambda'' \quad \text{with} \quad \mu''_B \leqq 0, \ \mu''_N = 0 \tag{6}$$

To prove the level optimality of $x'(\theta)$, $\theta \in (0, \xi'' - \xi')$, we just have to verify the corresponding Karush-Kuhn-Tucker conditions. With this aim, let us introduce the following notations:

$$\Delta_x = \frac{x'' - x'}{\xi'' - \xi'} \quad , \quad \Delta_{\mu_B} = \frac{\mu''_B - \mu'_B}{\xi'' - \xi'} \quad , \quad \Delta_\lambda = \frac{\lambda'' - \lambda'}{\xi'' - \xi'}$$

Our aim is now to verify that the following values verifies the Karush-Kuhn-Tucker conditions for all $\theta \in [0, \xi'' - \xi']$:

$$\begin{aligned} x'(\theta) &= x' + \theta \Delta_x \\ \mu'_B(\theta) &= \mu'_B + \theta \Delta_{\mu_B} \\ \mu'_N(\theta) &= 0 \\ \lambda'(\theta) &= \lambda' + \theta \Delta_\lambda \end{aligned}$$

By means of simple calculations, from (5) and (6) we get

$$Q\Delta_x = A_B^T \Delta_{\mu_B} + d\Delta_\lambda$$

which yields, for all $\theta \in [0, \xi'' - \xi']$:

$$Qx'(\theta) + q = A_B^T \mu'_B(\theta) + d\lambda'(\theta)$$

so that just the nonpositivity of $\mu'_B(\theta)$ is left to be verified. With this aim, just notice that $\mu'_B(0) = \mu'_B \leqq 0$ and $\mu'_B(\xi'' - \xi') = \mu''_B \leqq 0$ imply $\mu'_B(\theta) = \mu'_B + \theta \Delta_{\mu_B} \leqq 0$ for all $\theta \in [0, \xi'' - \xi']$. □

Theorem 3.1 shows that two optimal level solutions x' and x'', with $\xi' = d^T x' + d_0$ and $\xi'' = d^T x'' + d_0$, $\xi' \neq \xi''$, sharing the same set B of binding constraints are extrema of a segment of optimal level solutions. It is now worth determining the biggest segment (or halfline) of optimal level solutions containing x' and x''. The points $x'(\theta) = x' + \theta\Delta_x$, $\Delta_x = \frac{x''-x'}{\xi''-\xi'}$, are feasible whenever $Ax'(\theta) \le b$, that is for all the values $\theta \in [F_L, F_R]$ where:

$$F_L = \begin{cases} -\infty & \text{if } A\Delta_x \geqq 0 \\ \max\limits_{A_i\Delta_x<0} \left\{ \frac{b_i - A_i x'}{A_i\Delta_x} \right\} & \text{otherwise} \end{cases} \quad (7)$$

$$F_R = \begin{cases} +\infty & \text{if } A\Delta_x \leqq 0 \\ \min\limits_{A_i\Delta_x>0} \left\{ \frac{b_i - A_i x'}{A_i\Delta_x} \right\} & \text{otherwise} \end{cases} \quad (8)$$

The points $x'(\theta)$, $\theta \in [F_L, F_R]$, results to be optimal level solutions if $\mu'(\theta) = \mu' + \theta\Delta_\mu \leqq 0$, $\Delta_\mu = \frac{\mu''-\mu'}{\xi''-\xi'}$, that is for all the values $\theta \in [O_L, O_R]$ where:

$$O_L = \begin{cases} -\infty & \text{if } \Delta_\mu \geqq 0 \\ \max\limits_{\Delta_{\mu_i}<0} \left\{ \frac{-\mu'_i}{\Delta_{\mu_i}} \right\} & \text{otherwise} \end{cases} \quad (9)$$

$$O_R = \begin{cases} +\infty & \text{if } \Delta_\mu \leqq 0 \\ \min\limits_{\Delta_{\mu_i}>0} \left\{ \frac{-\mu'_i}{\Delta_{\mu_i}} \right\} & \text{otherwise} \end{cases} \quad (10)$$

We can then say that the optimality is guaranteed for all the values of $\theta \in [O_L, O_R]$ while the feasibility is guaranteed for all the values $\theta \in [F_L, F_R]$. As a consequence, the biggest segment (or halfline) of optimal level solutions containing x' and x'' is given by $x'(\theta)$ with:

$$\theta \in [\theta_L, \theta_R] \text{ where } \theta_L = \max\{O_L, F_L\} \text{ and } \theta_R = \min\{O_R, F_R\}$$

Remark 3.1 It is worth noticing that in the particular case $Q = 0$, that is the case where the quadratic function $\frac{1}{2}x^TQx + q^Tx + q_0$ reduces to a linear one, we obtain:

$$\begin{aligned} \mu' = \mu'' \quad &, \quad \lambda' = \lambda'' \\ \Delta_\mu = 0 \quad &, \quad \Delta_\lambda = 0 \\ O_L = -\infty \quad &, \quad O_R = +\infty \\ \theta_L = F_L \quad &, \quad \theta_R = F_R \end{aligned}$$

The previously described behaviour suggests how to algorithmically solve the problem. Starting from an optimal level solution a segment (or half-line) of optimal level solutions can be scanned; whenever the feasibility or the optimality is lost, we just have to change the considered subset of the binding constraints B and iteratively continue the visit of optimal level solutions.

During this visit, the objective function $f(x)$ can be evaluated over the set of optimal level solution thus obtaining the global optimum.

In [4] various results are given for determining the values of the feasibility and optimality parameters. In this paper we aim to propose a simplified approach for numerically determining them.

Let x' be the optimal level solution corresponding to the level ξ' and let x'' be the optimal level solution corresponding to the level $\xi'' = \xi' + \delta$, with $\delta > 0$ small enough to guarantee that x' and x'' belong to the same segment of optimal level solutions and share the same set of binding constraints B. Hence, it is:

$$\Delta_x = \frac{x'' - x'}{\delta}$$

The values μ'_B, λ' and μ''_B, λ'', can then be computed by solving the following linear systems, respectively:

$$\begin{bmatrix} A_B \\ d^T \end{bmatrix}^T \begin{bmatrix} \mu_B \\ \lambda \end{bmatrix} = Qx' + q \quad , \quad \begin{bmatrix} A_B \\ d^T \end{bmatrix}^T \begin{bmatrix} \mu_B \\ \lambda \end{bmatrix} = Qx'' + q \qquad (11)$$

so that

$$\Delta_{\mu_B} = \frac{\mu''_B - \mu'_B}{\delta} \quad , \quad \Delta_\lambda = \frac{\lambda'' - \lambda'}{\delta}$$

The described approach is summarized in procedure "*Parameters*()". This procedure determines a segment (or halfline) of optimal level solutions of the kind $x' + \theta\Delta_x$, $\theta \in (\theta_L, \theta_R)$.

Procedure **Parameters**(*inputs*: ξ'; *outputs*: Δ_x, F_L, F_R, O_L, O_R, θ_L, θ_R)
Fix $\delta > 0$ small enough and let $\xi'' = \xi' + \delta$;
Determine the values of x', x'' and let $\Delta_x = \frac{x''-x'}{\delta}$;
let $B = \{i \, : \, A_i x' = b_i, \; A_i x'' = b_i, \; i = 1, \ldots, m\}$ and $N = \{1, \ldots, m\} \backslash B$;
let $\tilde{B} \subseteq B$ such that the columns of $\left[A^T_{\tilde{B}} \; d\right]$ are linearly independent
and $\operatorname{rank}\left(\left[A^T_{\tilde{B}} \; d\right]\right) = \operatorname{rank}\left(\left[A^T_B \; d\right]\right)$;
let $\mu'_{\tilde{B}}$ and $\mu''_{\tilde{B}}$ be the solutions of the linear systems (11);
let $\mu'_i = \mu''_i = 0$ for all $i \notin \tilde{B}$ and let $\Delta_\mu = \frac{\mu''-\mu'}{\delta}$;
let F_L, F_R, O_L, O_R as described in (7), (8), (9), (10);
set $\theta_L := \max\{O_L, F_L\}$ and $\theta_R := \min\{O_R, F_R\}$;
end proc.

Notice also that in the case $Q = 0$ some of the outputs of procedure "*Parameters*()" can be computed as described in Remark 3.1.

Remark 3.2 It is worth comparing the method described in procedure "*Parameters*()" with the ones proposed in [3, 4, 7]. The aim of these methods is to determine $x'(\theta) = x' + \theta\Delta_x$ and $\mu'(\theta) = \mu' + \theta\Delta_\mu$. In procedure "*Parameters*()" both problems $P_{\xi'}$ and $P_{\xi''}$ are numerically solved, μ' and μ'' are obtained by means of systems (11), and hence $\Delta_x = \frac{x''-x'}{\delta}$ and $\Delta_\mu = \frac{\mu''-\mu'}{\delta}$. In the case Q is nonsingular just $P_{\xi'}$ has to be numerically solved, and all of the required parameters can be computed by means of the inverse of the following matrix (see [3]):

$$S = \begin{bmatrix} Q & -A_{\bar{B}}^T & -d \\ A_{\bar{B}} & 0 & 0 \\ d^T & 0 & 0 \end{bmatrix}$$

which results to be nonsingular due to the nonsingularity of Q.

4 Solution algorithm

In order to find a global minimum (assuming that one exists) it would be necessary to solve problems P_ξ for all the feasible levels. In this section we will show that this can be done by means of a finite number of iterations, using the results of the previous section.

The method scans all the feasible levels starting from a certain feasible level ξ'; the levels $\xi > \xi'$ are visited in increasing order, while the levels $\xi < \xi'$ are visited in decreasing order. For the sake of convenience, the described algorithm visits the feasible levels only in increasing order, hence the levels $\xi < \xi'$ can be analyzed by solving the following problem which is equivalent to P:

$$P \equiv \tilde{P} : \begin{cases} \inf \ f(x) = \tilde{\phi}(\frac{1}{2}x^TQx + q^Tx + q_0, \tilde{d}^Tx + \tilde{d}_0) \\ x \in X \end{cases}$$

where $\tilde{\phi}(y_1, y_2) = \phi(y_1, -y_2)$, $\tilde{d} = -d$ and $\tilde{d}_0 = -d_0$. In this light, $\tilde{\phi}(y_1, y_2)$ is strictly increasing with respect to variable y_1. Notice also that the decreaseness of the feasible levels of P corresponds to the increaseness of the feasible levels of $\tilde{P}$.

Procedure "*Main*()" initialize the algorithm by determining the set of feasible levels and a starting incumbent solution, then it uses procedure "*Visit*()" to obtain the global optimal solution (if it exists). Notice that in procedure "*Main*()" there is also one more optional subprocedure, namely "*ImproveStartingValues*()", which is aimed to improve the starting incumbent optimal solution. The usefulness of this optional procedure will be pointed out later.

Procedure "*Visit*()" scans iteratively the given set of feasible levels obtaining the best solution. Notice that "*Visit*()" uses a subprocedure "*MinRestriction*()" which determines the minimum of the continuous single valued

Procedure **Main**(*inputs*: P; *outputs*: Opt, $OptVal$)

Compute the values $\xi_{min} := d_0 + \inf_{x \in X} d^T x$ and $\xi_{max} := d_0 + \sup_{x \in X} d^T x$;

Let $\xi' \in (\xi_{min}, \xi_{max})$;

if $\inf_{x \in X_{\xi'}} f_{\xi'}(x) = -\infty$

then

$OptVal := \inf_{\xi \in \Lambda} \phi(-\infty, \xi)$ and $Opt := \arg\min_{\xi \in \Lambda} \phi(-\infty, \xi)$;

else

Let $x' := \arg \min_{x \in X_{\xi'}} f_{\xi'}(x)$, $\bar{x} := x'$ and let $UB := f(\bar{x})$;

Optional : $[x', \xi', \bar{x}, UB] := ImproveStartingValues()$;

$[\overline{x}, UB] := Visit(P, \xi', \xi_{max}, x', \overline{x}, UB)$;

$[\overline{x}, UB] := Visit(\tilde{P}, -\xi', -\xi_{min}, x', \overline{x}, UB)$;

$Opt := \overline{x}$ and $OptVal := UB$;

end if;

end proc.

Procedure **ImproveStartingValues**(*outputs*: x', ξ', $\bar{x}$, UB)

if $|\xi_{min} \cdot \xi_{max}| < +\infty$ *then* $\xi_1 := \xi_{min}$ and $\xi_2 := \xi_{max}$

else let $\xi_{big} >> 0$, $\xi_1 := \max\{\xi_{min}; -\xi_{big}\}$ and $\xi_2 := \min\{\xi_{max}; \xi_{big}\}$

end if;

Let $x_1 := \arg \min_{x \in X_{\xi_1}} f_{\xi_1}(x)$ and $x_2 := \arg \min_{x \in X_{\xi_2}} f_{\xi_2}(x)$;

if $f(x_1) < UB$ *then*

$x' := x_1$, $\xi' := \xi_1$, $\bar{x} := x'$ and $UB := f(\bar{x})$

end if;

if $f(x_2) < UB$ *then*

$x' := x_2$, $\xi' := \xi_2$, $\bar{x} := x'$ and $UB := f(\bar{x})$

end if;

end proc.

Procedure **Visit**(*inputs*: P, ξ', ξ_{max}, x', $\bar{x}$, UB; *outputs*: Opt, $OptVal$)
  while $\xi' < \xi_{max}$
    $[\Delta_x, F_L, F_R, O_L, O_R, \theta_L, \theta_R] := Parameters(\xi')$;
    let $z(\theta) = f(x' + \theta\Delta_x)$;
    set $[\overline{\theta}, z_{inf}] := MinRestriction(z(\theta), [\theta_L, \theta_R])$;
    if $z_{inf} = -\infty$ *then* $\overline{x} := [\,]$; $UB := -\infty$; $\xi' := \xi_{max}$ *else*
      if $z_{inf} < UB$ *then*
        $UB := z_{inf}$;
        if $\overline{\theta} = +\infty$ *then* $\overline{x} := [\,]$ *else* $\overline{x} := x' + \overline{\theta}\Delta_x$ *end if*;
      end if;
      set $\Delta_\xi := \theta_R$;
      # *if* $\xi' + \Delta_\xi < \xi_{max}$ *then* $\Delta_\xi := Skip(\xi', F_R, O_R)$ *end if*;
      # *if* $\xi' + \Delta_\xi < \xi_{max}$ *then* $\Delta_\xi := Jump(\xi', \Delta_\xi)$ *end if*;
      set $\xi' := \xi' + \Delta_\xi$;
    end if;
  end while;
  $Opt := \overline{x}$; $OptVal := UB$;
end proc.

function $z(\theta)$ in a closed interval. Observe that procedure "*MinRestriction*()" can be implemented numerically, and eventually improved for specific functions $f(x)$ (see [3, 4, 11]).

The optional subprocedures "*Skip*()" and "*Jump*()" evaluate the opportunity of avoiding the explicit visit of some feasible levels by means of so-called underestimation functions. A function $\psi(\xi)$ is said to be an underestimation function if it verifies the following property for all the feasible levels ξ:

$$\min_{x\in X_\xi} f(x) \geq \psi(\xi)$$

Given $\xi_a \in [\xi', \xi_{max}]$ it can be easily seen that:

$$\psi(\xi) \geq UB \;\forall \xi \in [\xi', \xi_a] \quad \Rightarrow \quad \min_{x\in X_{[\xi',\xi_{max}]}} f(x) = \min_{x\in X_{[\xi_a,\xi_{max}]}} f(x)$$

In other words, whenever $\psi(\xi) \geq UB \;\forall \xi \in [\xi', \xi_a]$ the feasible levels $[\xi', \xi_a]$ can be skipped (implicitly visited) since they do not allow to improve the incumbent optimal solution. In this light, given a set $Y \supseteq X$ and defining $Y_\xi = \{y \in Y : \; d^T y + d_0 = \xi\}$, a straightforward underestimation function is given by:

$$\psi(\xi) = \min_{y\in Y_\xi} f(y)$$

Whenever $F_R < O_R$ the function $z(\theta) = \psi(\xi'+\theta) = \min_{y\in Y_{\xi'+\theta}} f(y)$, where $Y = \{y \in \Re^n : \; A_{\tilde{B}} y \underline{\leq} b_{\tilde{B}}\} \supseteq X$, represents a tight underestimation for the

Procedure **Skip**(*inputs*: ξ', F_R, O_R; *outputs*: Δ_ξ)
if $F_R < O_R$ *then*
set $\tilde{\theta} := \min\{O_R; \xi_{max} - \xi'\}$;
let $\mathcal{L} = \{\theta \in [F_R, \tilde{\theta}] : z(\theta) < UB\}$;
if $\mathcal{L} = \emptyset$ *then* $\Delta_\xi := \tilde{\theta}$
else if $\inf\{\mathcal{L}\} > \Delta_\xi$ *then* $\Delta_\xi := \inf\{\mathcal{L}\}$ *end if*;
end if;
end if;
end proc.

Procedure **Jump**(*inputs*: ξ', Δ_ξ; *outputs*: Δ_ξ)
let $\psi(\xi)$ be an underestimation function for $f_\xi(x)$;
let $\mathcal{L} = \{\xi \in [\xi' + \Delta_\xi, \xi_{max}] : \psi(\xi) < UB\}$;
if $\mathcal{L} = \emptyset$ *then* $\Delta_\xi := \xi_{max} - \xi'$
else if $\inf\{\mathcal{L}\} > \xi' + \Delta_\xi$ *then* $\Delta_\xi := \inf\{\mathcal{L}\} - \xi'$ *end if*;
end if;
end proc.

objective function in the interval $[F_R, O_R]$, hence some levels corresponding to $\theta \in [F_R, O_R]$ can be skipped in the case $z(\theta) \geq UB$. This opportunity is described in subprocedure "*Skip*()".

In addition to that, any other underestimation function $\psi(\xi)$ can be analogously used to further jump levels in the interval $[\xi' + \Delta_\xi, \xi_{max}]$, as it is shown in subprocedure "*Jump*()".

In this case, an useful underestimation function can be given by $\psi(\xi) = \min_{y \in Y_\xi} f(y)$ where $Y = \{y \in \Re^n : A_V y \leq b_V\} \supseteq X$, $V = \bar{B} \setminus \{k\}$, k is the index which determines O_R by vanishing $\mu_k'(\theta)$, that is to say the one such that $\frac{-\mu_k'}{\Delta_{\mu_k}} = \min_{\Delta_{\mu_i} > 0} \left\{ \frac{-\mu_i'}{\Delta_{\mu_i}} \right\}$.

Notice that, as better is the value UB of the incumbent optimal solution as more effective is the implicit visit of the feasible levels produced by the optional subprocedures "*Skip*()" and "*Jump*()". For this very reason, a "good" starting optimal level solution could improve the performance of the algorithm. The role of subprocedure "*ImproveStartingValues*()" is indeed to improve the starting optimal level solution by comparing it with the optimal level solutions corresponding to feasible levels close to ξ_{min} and ξ_{max}.

The correctness of the proposed algorithm follows since all the feasible levels are scanned (either explicitly or implicitly) and the optimal solution, if it exists, is also an optimal level solution. It remains to verify the convergence (finiteness), that is to say that the procedure stops after a finite number of steps. With this aim it is worth pointing out that:

- if $O_R < F_R$ at least one of the multipliers corresponding to the binding constraints vanishes and in the next iteration a new segment of optimal level solutions is determined and visited;
- if $F_R \leq O_R$ the whole segment is visited and in the next iteration a new segment of optimal level solutions is determined and visited.

As a consequence, at every iterative step of the proposed algorithm, the set of binding constraints changes; note also that the level is increased from ξ' to $\xi' + \theta_R > \xi'$, so that it is not possible to obtain again an already used set of binding constraints; the convergence then follows since we have a finite number of possible sets of binding constraints.

Remark 4.1 Let us point out that problems P_ξ are independent of the function ϕ. This means that problems having the same feasible region, the same Q, q and d, but different function ϕ (either multiplicative or fractional or d.c.), they share the same set of optimal level solutions. As a consequence, when procedure "*Main*()" explicitly visits all the feasible levels (that is to say that the optional subprocedures "*Skip*()" and "*Jump*()" are not used), these different problems are solved by means of the same iterations of the while cycle in procedure "*Visit*()".

5 Computational results

In this section the results of a computational experience are provided in order to point out both the correctness and the performance of the proposed algorithm. All the procedures described in the previous sections have been fully implemented with the software MatLab 7.10 R2010a on a computer having 2 Gb RAM and a dual core processor at 2.66 GHz.

The following four different objective functions have been used in the computational test:

	$\phi(y_1, y_2)$	$f(x)$
P_1	$y_1 - y_2^2$	$\left(\frac{1}{2}x^TQx + q^Tx\right) - \left(d^Tx\right)^2$
P_2	$y_1 y_2^3$	$\left(\frac{1}{2}x^TQx + q^Tx\right)\left(d^Tx\right)^3$
P_3	y_1/y_2^2	$\left(\frac{1}{2}x^TQx + q^Tx\right) / \left(d^Tx\right)^2$
P_4	$y_2^2 log(y_1)$	$\left(d^Tx\right)^2 log\left(\frac{1}{2}x^TQx + q^Tx\right)$

where in P_2 and P_3 function d^Tx is positive over the feasible region, while in P_4 function $\frac{1}{2}x^TQx + q^Tx$ is positive over the feasible region.

The problems have been randomly created; in particular, matrices and vectors $Q \in \Re^{n \times n}$, $q, d \in \Re^n$, $A \in \Re^{m \times n}$, $b \in \Re^m$, $m = 3n$, have been generated with components in the interval [-10,10] by using the "rand()" MatLab function (numbers generated with uniform distribution). Matrix Q is generated with $rank(Q) = round\left(\frac{2}{3}n\right)$. Within the procedures, the linear problems and the convex quadratic problems have been solved with the "linprog()" and "quadprog()" MatLab functions, respectively.

For each amount "n" of variables a certain number of problems have been randomly generated and each of these problems have been solved for both the objective functions in P_1, P_2, P_3 and P_4. In particular, 2000 randoms problems have been generated for $n = 10, 20, 30, 40$, 1500 randoms problems for $n = 50, 60$, and finally 1000 randoms problems for $n = 70, 80, 90, 100$. The average number of iterations and the CPU times spent by the algorithm to solve the problems are given as the result of the test (see Table 1).

Notice that, in the case neither "*Skip*()" nor "*Jump*()" subprocedure is used (hence all the feasible levels are explicitly scanned), all the problems are solved in the same number of iterations (see Remark 4.1). In this case subprocedure "*ImproveStartingValues*()" is useless and hence it has not been invoked in the computational test.

Notice also that $z(\theta)$ is the tightest underestimation function in the interval $[F_R, OR]$. For this very reason, the use of subprocedure "*Jump*()" without the "*Skip*()" one is unuseful. In this light, in the computational test we considered the following three cases:

- no optional subprocedures used;
- use "*Skip*()" subprocedure only;
- use both "*Skip*()" and "*Jump*()".

The obtained results point out the effectiveness of the improvements proposed in Section 5; in particular, the performance is strongly improved, expecially for problem P_2 or for big values of n, for both the number of iterations and the spent CPU time.

6 Conclusions

The proposed algorithm allows to solve in an unified framework a large class of problems, covering both multiplicative, fractional and d.c. quadratic/linear functions. It has been shown that the use of underestimation functions greatly improves the algorithm performance. The correctness of the method guarantees that the global minimum is found even in the case of unbounded feasible regions.

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	n	Number of Iterations			CPU Times (seconds)		
		None	*Skip*	*Skip/Jump*	*None*	*Skip*	*Skip/Jump*
P_1	10	28.8515	7.2575	5.1245	1.28639	0.390385	0.347485
	20	67.443	14.424	8.635	6.07783	1.46444	1.23979
	30	105.951	23.856	12.707	17.0449	4.12004	3.28474
	40	147.536	35.3195	17.111	38.9603	9.73514	7.30851
	50	188.287	48.1387	22.2473	75.0776	19.6976	14.382
	60	229.435	61.6433	27.8787	128.983	35.2012	25.4962
	70	272.448	76.895	33.729	212.605	60.6313	42.9276
	80	315.833	92.979	40.547	327.051	96.6563	68.6803
	90	359.246	109.712	47.137	496.465	151.836	106.503
	100	402.76	126.698	53.092	709.204	223.102	153.514
P_2	10	28.8515	6.9685	4.9635	1.28634	0.376965	0.34148
	20	67.443	13.234	7.5925	6.08737	1.35519	1.13585
	30	105.951	21.888	10.702	17.0653	3.79204	2.8782
	40	147.536	32.867	14.4795	38.9963	9.06548	6.37885
	50	188.287	45.066	18.7027	75.1319	18.4278	12.4018
	60	229.435	57.7887	23.3933	129.06	32.9272	21.8418
	70	272.448	72.376	28.3	212.69	56.9236	36.6511
	80	315.833	87.618	33.92	327.12	90.8385	58.413
	90	359.246	103.206	39.276	496.583	142.514	90.262
	100	402.76	119.637	44.305	709.359	210.112	130.019
P_3	10	28.8515	9.173	7.739	1.28136	0.467775	0.48758
	20	67.443	18.3945	14.521	6.0768	1.79026	1.92403
	30	105.951	29.4515	21.7085	17.0495	4.92232	5.12722
	40	147.536	42.2685	29.1655	38.9726	11.3595	11.3205
	50	188.287	56.1647	36.7727	75.1094	22.5043	21.5167
	60	229.435	70.916	45.038	129.027	39.72	37.0744
	70	272.448	87.488	53.594	212.628	67.83	61.3302
	80	315.833	104.656	62.099	327.079	107.038	94.5416
	90	359.246	122.353	71.394	496.536	167.053	145.298
	100	402.76	140.726	80.55	709.316	244.552	209.031
P_4	10	28.8515	7.673	5.6655	1.22116	0.382665	0.363315
	20	67.443	15.6405	10.363	5.941	1.50697	1.41303
	30	105.951	25.528	15.3615	16.8527	4.25285	3.75595
	40	147.536	37.031	20.479	38.7059	9.95337	8.26987
	50	188.287	49.6467	25.6787	74.9309	19.9792	15.7627
	60	229.435	62.1427	30.6233	128.782	34.9489	26.7558
	70	272.448	76.938	36.066	212.302	59.8919	44.0342
	80	315.833	92.921	43.014	327.651	95.7944	69.966
	90	359.246	107.826	47.531	497.011	148.161	104.186
	100	402.76	125.217	54.55	710.057	218.984	152.414

Table 1: Computational Results

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