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On the maximal domains of pseudoconvexity of a quadratic fractional function

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Abstract

In this paper we will characterize the maximal domains of pseudoconvexity of the ratio between a quadratic function and an affine one. Furthermore, motivated by the fact that in optimization problems the decision variables are often required to be nonnegative, we will specialize the obtained results in order to achieve conditions which guarantee that the nonnegative orthant is contained in the maximal domains of pseudoconvexity of the function.

KeyWords Generalized convexity, generalized fractional functions, quadratic functions.
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1 Introduction

As it is well known, generalized convexity plays an important role both in mathematics and in economics. Starting from the pioneer work of Arrow-Enthoven [2], several classes of generalized convex functions have been introduced and studied. Among them, the one

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of pseudoconvex functions occupies a leading position in Optimization since for this kind of functions a critical point, as well as a local minimum, is a global minimum.
Despite of the relevance of pseudoconvexity, it is not easy in general to find classes of functions which are pseudoconvex.
Pioneer works in this direction are given by Martos [10] and by Schaible [1] related to linear fractional functions, to generalized Cobb-Douglas functions and to quadratic functions.
In this paper we will focus our attention to the study of pseudoconvexity of a function f which is the ratio between a quadratic function and an affine one. Starting from the results given in [3], in Section 2 we will determine the maximal domains of pseudoconvexity of f which will allow us, in Sections 3-4, to characterize the pseudoconvexity of f on $\Re^n_+$. Finally, in Section 5 we specialize the obtained results to the three-dimensional space.

2 Maximal domains of pseudoconvexity

Consider the following quadratic fractional function:

$$f(x) = \frac{\frac{1}{2}\, x^T Qx + q^T x + q_0}{d^T x + d_0}, x \in D = \{x \in \Re^n : \ d^T x + d_0 > 0\} \tag{2.1}$$

where Q is an $n \times n$ symmetric matrix, $q, \ d \in \Re^n, \ d \neq 0, \ q_0, \ d_0 \in \Re, \ d_0 \neq 0$.
In [3], the pseudoconvexity of f has been characterized on the half-space D, nevertheless, f may be pseudoconvex on an open convex set S contained properly in D but not pseudoconvex on the whole half-space. From this point of view, the results given in [3] appear only as a sufficient but not necessary condition for the pseudoconvexity on S.
In order to find the maximal domains of pseudoconvexity of function f, we will suggest a different approach from the one proposed in [3]; such an approach is based on the characterization of the maximal domains of pseudoconvexity of a quadratic form (see [1, 4]) and on the following result.

Theorem 2.1 *Let Q be an $n \times n$ symmetric matrix with $\nu_-(Q) = 1$, and let* $A = \{y \in \Re^n : \ w \in \Re^n, \ w^T Qy = 0 \ \Rightarrow w^T Qw \geq 0\}, B = \{y \in \Re^n : \ y^T Qy \leq 0\}$.
Then, $A = B$.

Proof Consider the set $Z = \{z \in \Re^n \setminus \{0\} : \exists w \text{ such that } z^T w = 0,\ w^T Q w < 0\}$. The following property holds (see Lemma 6.2.3 in [4]):

$$\forall y \text{ such that } y^T Q y > 0, \text{then } Qy \in Z. \tag{2.2}$$

Note that A and B are closed sets. Firstly, we prove that $A \subseteq B$.
Let $y \in A$; if $y \notin B$, i.e., $y^T Q y > 0$, then, from (2.2), $Qy \in Z$, so that there exists w such that $w^T Q y = 0,\ w^T Q w < 0$ and this is a contradiction.
For the converse statement, taking into account that B verifies the property $cl(intB) = B$ since it is the union of two convex cones C_1, C_2 with $intC_1 \cap intC_2 = \emptyset$ (see [4]), it is sufficient to prove that $intB \subseteq A$.
Let $y \in intB$ and assume that $y \notin A$, then $y^T Q y < 0$ and there exists $w \in \Re^n \setminus \{0\}$ such that $w^T Q y = 0,\ w^T Q w < 0$. It follows that Q has at least two negative eigenvalues (see Lemma 6.2.1 in [4] and [8]), and this is a contradiction; consequently $intB \subseteq A$ and the proof is complete.

□

Regarding to the gradient $\nabla f(x)$ and to the Hessian matrix $\nabla^2 f(x)$, we have:

$$\nabla f(x) = \frac{1}{d^T x + d_0}[Qx + q - f(x)d], \tag{2.3}$$

$$(d^T x + d_0)\nabla^2 f(x) = Q + \frac{1}{d^T x + d_0}\left[2f(x)dd^T - (Qx+q)d^T - d(Qx+q)^T\right]. \tag{2.4}$$

The following theorem states a necessary and sufficient condition for the pseudoconvexity of f on an arbitrary open convex set contained in D.

Theorem 2.2 *Consider the quadratic fractional function $f(x)$, $x \in D$, where the matrix Q is not positive semidefinite. Then, f is pseudoconvex on an open convex set $S \subseteq D$ if and only if the following conditions hold:*
i) $\nu_-(Q) = 1$;
ii) there exist s and $\bar{x}$ such that $Qs = -q$ and $Q\bar{x} = d$;
iii) for every $x \in S$, with $x - s - f(x)\bar{x} \notin KerQ$, we have

$$d^T \bar{x} f^2(x) + 2(d^T s + d_0)f(x) + s^T Q s - 2q_0 \leq 0 \tag{2.5}$$

where $KerQ = \{x \in \Re^n : Qx = 0\}$.

Proof In [3] it has been proved that $i)$ and $ii)$ are necessary conditions for the pseudoconvexity of f. Consequently, it remains to prove, assuming the validity of $i)$ and $ii)$, that f is pseudoconvex on S if and only if $iii)$ holds.

We recall (see for instance [4]) that f is pseudoconvex on S if and only if the following conditions hold:

a) $x \in S,\ u \in \Re^n,\ u^T \nabla f(x) = 0 \ \Rightarrow u^T \nabla^2 f(x) u \geq 0$;

b) If $x_0 \in S$ is a critical point for f, then x_0 is a local minimum point for f on S.

Taking into account of (2.3) and of (2.4), it is easy to verify that b) is equivalent to

$$x \in S,\ u \in \Re^n,\ u^T \nabla f(x) = 0 \ \Rightarrow u^T Q u \geq 0. \tag{2.6}$$

Moreover, if x_0 is a critical point, then $u^T Q u \geq 0,\ \forall u \in \Re^n$; consequently, since Q is not positive semidefinite, then f is pseudoconvex on S if and only if (2.6) holds and there do not exist critical points in S.

From $ii)$ we have $\nabla f(x) = 0$ if and only if $Q(x - s - f(x)\bar{x}) = 0$. Consequently, the non-existence of critical points in S is equivalent to state that $x - s - f(x)\bar{x} \notin KerQ,\ \forall x \in S$. Setting $z = x - s - f(x)\bar{x}$, condition (2.6) can be re-written as

$$u^T Q z = 0 \ \Rightarrow u^T Q u \geq 0 \tag{2.7}$$

From Theorem 2.1, (2.7) is equivalent to $z^T Q z \leq 0$, consequently, (2.6) holds if and only if

$$(x - s - f(x)\bar{x})^T Q (x - s - f(x)\bar{x}) \leq 0,\ \forall x \in S \tag{2.8}$$

Since $x^T Q x = 2f(x)(d^T x + d_0) - 2q^T x - 2q_0$, (2.8) is equivalent to (2.5) and the proof is complete.

□

Taking into account of ii) of the previous Theorem, it is easy to verify that the condition $x - s - f(x)\bar{x} \notin KerQ,\ \forall x \in S$, holds if and only if $\nabla f(x) \neq 0,\ \forall x \in S$. Consequently, in order to determine the maximal domains of pseudoconvexity of f, we must study the inequality (2.5) and find the critical points of f.

Obviously, the inequality (2.5) is strictly related to the roots of the following second order equation

$$d^T\bar{x}\alpha^2 + 2(d^Ts + d_0)\alpha + s^TQs - 2q_0 = 0 \tag{2.9}$$

The next theorem points out that there is a relationship between the solutions of the inequality (2.5) and the sign of the quadratic function

$$Q(x,\alpha) = (x - (s + \alpha\bar{x}))^TQ(x - (s + \alpha\bar{x})) \tag{2.10}$$

where α is a root of (2.9).

Theorem 2.3 *Consider function f with $\nu_-(Q) = 1$. Assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$, and the existence of a root α of (2.9).*
Then, $\{x \in D: \ f(x) \lesseqgtr \alpha\} = \{x \in D: \ Q(x,\alpha) \lesseqgtr 0\}$.

Proof We have
$f(x) - \alpha = \dfrac{x^TQx + 2(q - \alpha d)^Tx + 2(q_0 - \alpha d_0)}{2(d^Tx + d_0)} = \dfrac{x^TQx - 2(s + \alpha\bar{x})^TQx + 2(q_0 - \alpha d_0)}{2(d^Tx + d_0)}$.
On the other hand, since α is a root of (2.9), we obtain
$2(q_0 - \alpha d_0) = d^T\bar{x}\alpha^2 + 2\alpha d^Ts + s^TQs = \alpha^2\bar{x}^TQ\bar{x} + 2\alpha s^TQ\bar{x} + s^TQs = (s + \alpha\bar{x})^TQ(s + \alpha\bar{x})$.
Consequently, $f(x) - \alpha = \dfrac{(x - (s + \alpha\bar{x}))^TQ(x - (s + \alpha\bar{x}))}{2(d^Tx + d_0)}$ and the thesis follows. □

With respect to the critical points of the function f, setting $D^* = \{x \in D: \ d^Tx + d_0 \neq 0\}$, we have the following result.

Theorem 2.4 *Consider function f with $\nu_-(Q) = 1$ and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$. Then, f has some critical points in D^* if and only if the second-order equation (2.9) has a solution. Furthermore, the set of critical points of f corresponding to a root α of (2.9) is $H(\alpha) = \{x \in D^*: \ x = s + \alpha\bar{x} + w, \ w \in KerQ\}$.*

Proof If $x^* \in D^*$ be a critical point of f, from (2.3) we have $x^* - (s + f(x^*)\bar{x} = w \in KerQ$; furthermore, $0 = w^TQw = [x^* - (s + f(x^*)\bar{x})]^TQ[x^* - (s + f(x^*)\bar{x})]$, i.e., $\alpha^* = f(x^*)$ is a root of (2.9). Consequently, $x^* \in H(\alpha^*)$.
Conversely, let α^* be a root of (2.9) and let $x^* = s + \alpha^*\bar{x} + w^* \in H(\alpha^*)$. We have $Q(x^*, \alpha^*) = 0$ so that, from Theorem 2.3, $f(x^*) = \alpha^*$. It results
$Q[x^* - (s + f(x^*)\bar{x})] = Q[s + \alpha^*\bar{x} + w - (s + f(x^*)\bar{x})] = Q[\alpha^*\bar{x} - f(x^*)\bar{x})] = (\alpha^* - f(x^*))d = 0.$

Consequently, x^* is a critical point of f.

The proof is complete. $\square$

As a direct consequence of Theorem 2.4, we get the following results.

Corollary 2.1 *Consider function f with $\nu_-(Q) = 1$ and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$. Then, the following conditions hold:*

i) f does not have critical points in D^* if and only if the second-order equation (2.9) does not have solutions;

ii) the set of all critical points of f is $H = H(\alpha_1) \cup H(\alpha_2)$, where α_1, α_2, are the roots of (2.9) not necessarily distinct from each other.

Remark 2.1 *It is important to note that a critical point of f, if one exists, belongs to the boundary of the cone $\{x \in \Re^n : \; Q(x, \alpha) \leq 0\}$ or to the boundary of the cone $\{x \in \Re^n : \; Q(x, \alpha) \geq 0\}$. Consequently, f does not have critical points on every open convex set S contained in the interior of one of the two cones. It follows that in iii) of Theorem 2.2, condition $x - s - f(x)\bar{x} \notin KerQ$, $x \in S$, can be omitted.*

Now we are able to state the main result related to the pseudoconvexity of f on an open convex set contained in D.

With this aim, we introduce the following notations:

- $\Delta = (d^Ts + d_0)^2 - d^T\bar{x}(s^TQs - 2q_0)$ is the discriminant of (2.9);
- α_1, α_2, with $\alpha_1 < \alpha_2$, denote the real roots of (2.9), when $\Delta > 0$;
- $\bar{\alpha} = \dfrac{2q_0 - s^TQs}{2(d^Ts + d_0)}$ is the real root of (2.9), when $\Delta = 0$ or when $d^T\bar{x} = 0$, $d^Ts + d_0 \neq 0$;
- $S^-(\alpha) = \{x \in D : Q(x, \alpha) \leq 0\}$; $S^+(\alpha) = \{x \in D : Q(x, \alpha) \geq 0\}$.

Theorem 2.5 *Consider function f with $\nu_-(Q) = 1$ and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$.*

i) If $d^T\bar{x} = 0$, and $d^Ts + d_0 > 0$, then f is pseudoconvex on every open convex set $S \subset S^-(\bar{\alpha})$;

ii) If $d^T\bar{x} = 0$, and $d^Ts + d_0 < 0$, then f is pseudoconvex on every open convex set $S \subset S^+(\bar{\alpha})$;

iii) If $d^T\bar{x} = 0$, $d^Ts + d_0 = 0$, and $s^TQs - 2q_0 \leq 0$, then f is pseudoconvex on every open convex set $S \subseteq D$;

iv) If $d^T\bar{x} < 0$, and $\Delta \leq 0$, then f is pseudoconvex on every open convex set $S \subseteq D$;

v) If $d^T\bar{x} < 0$, and $\Delta > 0$, then f is pseudoconvex on every open convex set $S \subset S^-(\alpha_1)$ or $S \subset S^+(\alpha_2)$;

vi) If $d^T\bar{x} > 0$, and $\Delta > 0$, then f is pseudoconvex on every open convex set $S \subset S^-(\alpha_1) \cap S^+(\alpha_2)$.

In any other case f is not pseudoconvex on $S \subseteq D$ whatever the open convex set S be.

Proof Referring to Theorem 2.2 we must find the set of $x \in S$ verifying (2.5).
Note that when $d^T\bar{x} = 0$, (2.5) reduces to the inequality $2(d^Ts+d_0)f(x)+s^TQs-2q_0 \leq 0$ so that $f(x) \leq \bar{\alpha} = \dfrac{2q_0 - s^TQs}{2(d^Ts + d_0)}$ or $f(x) \geq \bar{\alpha}$ according to the sign of $d^Ts + d_0$. From Theorem 2.3, *i*) and *ii*) follow.
If $d^T\bar{x} = 0, and d^Ts + d_0 = 0$, then (2.5) reduces to the inequality $s^TQs - 2q_0 \leq 0$, so that *iii*) follows.
Taking into account Theorem 2.3, conditions *iv*), *v*), *vi*), can be deduced directly by studying the sign of the second order inequality (2.5). □

Let us note that f is pseudoconvex on the whole domain D if and only if iii) or iv) of Theorem 2.5 holds, so that by means of the suggested approach we have extended the results given in [3].
In order to clarify how Theorem 2.5 works, in the following example we will find the maximal domains of pseudoconvexity of a quadratic fractional function.

Example 2.1 *Consider the function $f(x_1, x_2) = \dfrac{x_1^2 - x_2^2}{x_2 + 4}$, defined on the half-space $D = \{(x_1, x_2) \in \Re^2 : x_2 + 4 > 0\}$.*
It is easy to verify that the critical point $\hat{x} = (0,0)^T \in D$ is not a minimum point for f, so that the function is not pseudoconvex on D.
On the other hand, we have $\bar{x} = (0, -\frac{1}{2})^T$, $s = (0,0)^T$, $d^T\bar{x} = -\frac{1}{2}$, $d^Ts + d_0 = 4$,

$\alpha_1 = 0$, $\alpha_2 = 16$, *so that, from v) of Theorem 2.5, f is pseudoconvex on every open convex set S contained in* $S^-(0) = \{(x_1, x_2) \in \Re^2 : \; x_1^2 - x_2^2 \le 0, \; x_2 > -4\}$ *or in* $S^+(16) = \{(x_1, x_2) \in \Re^2 : \; x_1^2 - (x_2 + 8)^2 \ge 0, \; x_2 > -4\}$.

At last, we point out that, by means of Theorem 2.5, it is possible to get the results given in [4] related to the characterization of the maximal domains of pseudoconvexity of a function which is the sum of a linear and a linear fractional function.

3 Pseudoconvexity on the nonnegative orthant

Motivated by the fact that in many real problems the decision variables are required to be nonnegative, in this section we will analyze the pseudoconvexity of function f on the nonnegative orthant $\Re^n_+$.

In order to guarantee that $\Re^n_+$ is contained in the domain D of f, from now on we will assume $d \in \Re^n_+ \setminus \{0\}$, and $d_0 > 0$.

Note that the results established in the previous section are related to the pseudoconvexity on an open convex set; with respect to a closed convex set X, we recall that a function is pseudoconvex on X if and only if it is pseudoconvex on the interior of X and there do not exist critical points on the boundary of X or they are minimum points.

With respect to the pseudoconvexity of f on $int\Re^n_+$, taking into account Theorem 2.5, we must find conditions which ensure that $int\Re^n_+ \subset S^-(\alpha)$, or $int\Re^n_+ \subset S^+(\alpha)$. In order to characterize the first inclusion, we will use some results (see [1, 4, 10]) related to the pseudoconvexity of a quadratic function which are summarized in the following theorem, where $A \le 0$ means that all the elements of matrix A are nonpositive.

Theorem 3.1 *Consider the quadratic function* $A(x) = \frac{1}{2}x^T Ax + a^T x$, *where* A *is a* $m \times n$ *matrix with with* $\nu_-(A) = 1$, *and assume the existence of* $v \in \Re^n$ *such that* $Av = -a$. *Then, the following conditions hold:*

i) the function $A(x)$ *is pseudoconvex on every open convex set contained in the cone* $\{x \in \Re^n : (x - v)^T A(x - v) \le 0\}$;

ii) $A(x)$ *is pseudoconvex on* $int\Re^n_+$ *if and only if* $A \le 0, a \in \Re^n_-, a^T v \ge 0$.

The following theorem holds.

Theorem 3.2 *Consider function f with $\nu_-(Q) = 1$, and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$. Then, $int\Re^n_+ \subset S^-(\alpha)$ if and only if*

$$Q \leq 0, \; q - \alpha d \in \Re^n_-, \; q_0 - \alpha d_0 \leq 0 \tag{3.11}$$

where α is a root of (2.9).

Proof Assume $int\Re^n_+ \subset S^-(\alpha)$ and consider the function $g(x) = \frac{1}{2}x^TQx+(q-\alpha d)^Tx$. Since $\nu_-(Q) = 1$, and $Q(s+\alpha\bar{x}) = -(q-\alpha d)$, taking into account Theorem 3.1, $g(x)$ is pseudoconvex on $int\Re^n_+$, so that $Q \leq 0$, $q-\alpha d \in \Re^n_-$, $(q-\alpha d)^T(s+\alpha\bar{x}) \geq 0$. On the other hand, since α is a root of (2.9), we have $(q-\alpha d)^T(s+\alpha\bar{x}) = -s^TQs-2d^Ts\alpha-d^T\bar{x}\alpha^2 = 2(d_0\alpha-q_0)$. Consequently, $(q-\alpha d)^T(s+\alpha\bar{x}) \geq 0$ if and only if $q_0 - d_0\alpha \leq 0$ and (3.11) holds. Viceversa, since $S^-(\alpha) = \{x \in D : \frac{1}{2}x^TQx + (q-\alpha d)^Tx + q_0 - \alpha d_0 \leq 0\}$, the validity of (3.11) implies $int\Re^n_+ \subset S^-(\alpha)$.
The proof is complete. □

Note that the approach used for characterizing the inclusion $int\Re^n_+ \subset S^-(\alpha)$ (based on the pseudoconvexity of a suitable quadratic function) cannot be applied in finding conditions which ensure $int\Re^n_+ \subset S^+(\alpha)$. Furthermore, the symmetric conditions of (3.11), i.e., $Q \geq 0$, $q - \alpha d \in \Re^n_+$, $q_0 - \alpha d_0 \geq 0$ is only sufficient for the validity of $int\Re^n_+ \subset S^+(\alpha)$. All this points out that the characterization of the last inclusion in terms of the data Q, $q - \alpha d$, $q_0 - \alpha d_0$, is not easy to carry on.
Obviously, we have $int\Re^n_+ \subset S^+(\alpha)$ if and only if there exists the minimum of $Q(x, \alpha)$ on $\Re^n_+$ and the minimum value is nonnegative. This remark allows us to state the following fundamental theorem related to the characterization of the pseudoconvexity of f on the nonnegative orthant.
A necessary and sufficient condition for the existence of the minimum of $Q(x, \alpha)$ will be stated in Section 4.

Theorem 3.3 *Consider function f where $\nu_-(Q) = 1$. Then, f is pseudoconvex on $\Re^n_+$ if and only if there exist s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$ and one of the following* *conditions holds:*

i) $d^T\bar{x} = 0,\ d^Ts + d_0 > 0,\ Q \leq 0,\ q - \bar{\alpha}d \in \Re^n_- \setminus \{0\},\ q_0 - \bar{\alpha}d_0 \leq 0;$

ii) $d^T\bar{x} = 0,\ d^Ts + d_0 < 0,$ *and the minimum value of* $Q(x, \bar{\alpha})$ *on* $\Re^n_+$ *is nonnegative;*

iii) $d^T\bar{x} = 0,\ d^Ts + d_0 = 0,\ s^TQs - 2q_0 \leq 0;$

iv) $d^T\bar{x} < 0,\ \Delta \leq 0;$

v) $d^T\bar{x} < 0,\ \Delta > 0,$ *and* $Q \leq 0,\ q - \alpha_1 d \in \Re^n_- \setminus \{0\},\ q_0 - \alpha_1 d_0 \leq 0,$ *or the minimum value of* $Q(x, \alpha_2)$ *on* $\Re^n_+$ *is nonnegative.*

Proof Referring to Theorem 2.5, it is sufficient to note that, when $d^T\bar{x} > 0$ and $\Delta > 0$, f is pseudoconvex on $int\Re^n_+$ if and only if $Q = 0$ and this contradicts the assumption $\nu_-(Q) = 1$. Consequently, case *vi)* of Theorem 2.5 cannot occur, so that f is pseudoconvex on $int\Re^n_+$ if and only if one of the conditions $i) - v)$ holds.
Note that in cases *iii)* and *iv)*, the function is pseudoconvex on D and, in particular, on $int\Re^n_+$. In the other cases, the thesis will be achieved by proving that there do not exist critical points belonging to the boundary of $\Re^n_+$.
Corresponding to a root α of (2.9), the critical points of f are of the kind $x^*_\alpha = s + \alpha\bar{x} + w$, $w \in KerQ$, so that

$$d^Tx^*_\alpha = d^Ts + \alpha d^T\bar{x} \tag{3.12}$$

$$Qx^*_\alpha = -(q - \alpha d) \tag{3.13}$$

If $q - \alpha d \in \Re^n_- \setminus \{0\}$ and $Q \leq 0$, then, from (3.13), we have $Qx^*_\alpha \in \Re^n_+ \setminus \{0\}$ so that $x^*_\alpha \notin \Re^n_+$.
If $d^T\bar{x} = 0$, $d^Ts + d_0 < 0$, then $d^Ts < 0$ and, from (3.12), we have $d^Tx^*_{\bar{\alpha}} = d^Ts < 0$; consequently $x^*_{\bar{\alpha}} \notin \Re^n_+$.
If $d^T\bar{x} < 0$, $\Delta > 0$, taking into account (2.9), we have $\alpha_1 + \alpha_2 = -2\dfrac{d^Ts + d_0}{d^T\bar{x}}$, and since $\alpha_1 < \alpha_2$, it results $\alpha_2 > -\dfrac{d^Ts + d_0}{d^T\bar{x}}$. On the other hand, from (3.12), we have $\alpha_2 = \dfrac{d^Tx^*_{\alpha_2} - d^Ts}{d^T\bar{x}}$ so that $\dfrac{d^Tx^*_{\alpha_2}}{d^T\bar{x}} > -\dfrac{d_0}{d^T\bar{x}}$, i.e, $d^Tx^*_{\alpha_2} < -d_0 < 0$. Thus, we have $x^*_{\alpha_2} \notin \Re^n_+$, and the proof is complete. □

When one of the two roots of equation (2.9) is equal to zero, it is possible to obtain sufficient conditions for the pseudoconvexity of the function f, expressed in terms of the data. More exactly we have the following corollary.

Corollary 3.1 *Consider the function f where $\nu_-(Q) = 1$ and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$, $s^T Qs = 2q_0$. Then, f is pseudoconvex on $\Re^n_+$ if one of the following conditions holds:*

i) $d^T\bar{x} \leq 0$, $d^T s + d_0 > 0$, $Q \leq 0$, $q \in \Re^n_- \setminus \{0\}$, $q_0 \leq 0$;

ii) $d^T\bar{x} \leq 0$, $d^T s + d_0 < 0$, $x^T Qx + 2q^T x + 2q_0 \geq 0$, $\forall x \in \Re^n_+$;

iii) $d^T\bar{x} < 0$, $\Delta \leq 0$;

Remark 3.1 *Note that if $Q \leq 0$, $q \in \Re^n_- \setminus \{0\}$, $q_0 \leq 0$, then the numerator of f, $\frac{1}{2}x^T Qx + q^T x + q_0$, is pseudoconvex on $\Re^n_+$ (see [10]), so that condition i) of Corollary 3.1 can be interpreted as a sufficient condition for the pseudoconvexity of the ratio between a pseudoconvex quadratic function and an affine one. The following example show that the ratio between a pseudoconvex function and an affine one is not in general pseudoconvex.*

Example 3.1 *Consider the function*
$$f(x_1, x_2) = \frac{\frac{1}{2}(-x_1^2 - 4x_1x_2 - 3x_2^2) - x_1 - 2x_2 - \frac{1}{2}}{x_1 + x_2 + d_0}, (x_1, x_2) \in D$$
where $D = \{(x_1, x_2) : x_1 + x_2 + d_0 > 0\}$.
The quadratic function $\frac{1}{2}(-x_1^2 - 4x_1x_2 - 3x_2^2) - x_1 + 2x_2 - \frac{1}{2}$ is pseudoconvex on $\Re^2_+$.
We have $\bar{x} = (1, -1)^T$, $s = (-1, 0)^T$, $s^T Qs - 2q_0 = 0$, $d^T\bar{x} = 0$, $d^T s + d_0 = -1 + d_0$.
If $d_0 < 1$, then f is pseudoconvex on every convex set contained in $D^* = \{(x_1, x_2) \in D : (x_1 + 3x_2 + 1)(x_1 + x_2 + 1) \leq 0\}$ but it is not pseudoconvex on $\Re^2_+ \not\subset D^*$.
Note that when $d_0 = 1$, then f is pseudoconvex on the whole half-space D, while $d_0 > 1$ implies the pseudoconvexity of f on the maximal domain $\{(x_1, x_2) \in D : -(x_1 + 3x_2 + 1)(x_1 + x_2 + 1) \leq 0\}$ containing $\Re^2_+$.

Example 3.2 *Consider the function $f(x, y) = \dfrac{-x^2 - 2xy}{x + 3y + 3}$. It results $\bar{x} = (-\frac{3}{2}, 1)$, so that $d^T\bar{x} = \frac{3}{2} > 0$. From vi) of Theorem 2.2, the function is pseudoconvex on every open convex set contained in $\{(x, y) : f(x, y) \geq \alpha_1 = -4\} \cap \{(x, y) : f(x, y) \leq \alpha_2 = 0\}$, but not on $\Re^2_+ \setminus \{0\}$ even if the function $-x^2 - 2xy$ is pseudoconvex on $\Re^2_+ \setminus \{0\}$.*

When the vector q is the null vector or when it is proportional to vector d, the characterization of pseudoconvexity assumes a simpler form since the case $int\Re^n_+ \subset S^+_\alpha$ does not occur.

Theorem 3.4 *Consider the function* $f(x) = \dfrac{\frac{1}{2}x^TQx + q_0}{d^Tx + d_0}$ *where* $q_0 \neq 0$, $\nu_-(Q) = 1$. *Then, f is pseudoconvex on $\Re^n_+$ if and only if there exists $\bar{x}$ such that $Q\bar{x} = d$, and one of the following conditions holds:*

i) $d^T\bar{x} = 0,\ Q \leq 0,\ q_0 > 0$;

ii) $d^T\bar{x} < 0,\ \Delta > 0,\ Q \leq 0,\ \alpha_1 > 0,\ q_0 - \alpha_1 d_0 \leq 0$;

iii) $d^T\bar{x} < 0,\ d_0^2 + 2q_0 d^T\bar{x} \leq 0$.

Proof Referring to Theorem 3.3, the assumption $q = 0$ implies the existence of $s \in \Re^n$ such that $Qs = 0$; consequently, $d^Ts = \bar{x}^TQs = 0$ and the quadratic function $Q(x,\alpha)$ becomes

$$Q(x,\alpha) = x^TQx - 2\alpha d^Tx + \alpha^2\bar{x}^TQ\bar{x} = x^TQx - 2\alpha d^Tx + \alpha^2 d^T\bar{x}$$

where α is a root of the equation

$$d^T\bar{x}\ \alpha^2 + 2d_0\alpha - 2q_0 = 0.$$

Note that $ii)$ and $iii)$ of Theorem 3.3 cannot occur since $d^Ts + d_0 = d_0 > 0$, while conditions $i)$ and $iii)$ are equivalent to conditions $i)$ and $iv)$ of Theorem 3.3, respectively.
If $d^T\bar{x} < 0$, $\Delta = d_0^2 + 2q_0 d^T\bar{x} > 0$, then $\alpha_2 > 0$, so that $Q(0,\alpha_2) = \alpha^2 d^T\bar{x} < 0$, and thus, by continuity, $Q(x,\alpha_2)$ assumes some negative values on $int\Re^n_+$. Consequently, condition $v)$ of Theorem 3.3 reduces to $ii)$, and the proof is complete. □

When $q_0 = 0$, the function $f(x) = \dfrac{\frac{1}{2}x^TQx}{d^Tx + d_0}$ is never pseudoconvex on the whole domain D and, since the origin is a critical point, f is not pseudoconvex on $\Re^n_+$. Nevertheless we have the following result.

Theorem 3.5 *Consider the function* $f(x) = \dfrac{\frac{1}{2}x^TQx}{d^Tx + d_0}$ *where* $\nu_-(Q) = 1$. *Then, f is pseudoconvex on $\Re^n_+ \setminus \{KerQ\}$ if and only if $Q \leq 0$ and there exists $\bar{x}$ such that $Q\bar{x} = d$ and $d^T\bar{x} \leq 0$.*

Proof Equation (2.9) reduces to $d^T\bar{x}\ \alpha^2 + 2d_0\alpha = 0$. When $d^T\bar{x} < 0$, we have $\alpha_1 = 0$ and $\alpha_2 = -2\dfrac{d_0}{d^T\bar{x}} > 0$. Since $Q(0,\alpha_2) = \alpha^2 d^T\bar{x} < 0$, the case $int\Re^n_+ \subset S^+(\alpha)$ does not occur. Corresponding to the root $\alpha = 0$, we have $Q(x,0) = x^TQx$ so that $Q(x,0) \leq 0\ \ \forall x \in \Re^n_+$ if and only if $Q \leq 0$.

At last, note that the set of all critical points of f coincides with $KerQ$. The proof is complete.

□

4 On the existence of the minimum of $Q(x,\alpha)$ on $\Re^n_+$

As it has been stated in the previous section, the study of the sign of the quadratic function $Q(x,\alpha)$ plays a crucial role in characterizing the pseudoconvexity of f on $\Re^n_+$. In particular, in order to guarantee the inclusion $\Re^n_+ \subset S^+_\alpha$, in Theorem 3.3 we have implicitely assumed that the infimum of the quadratic function $Q(x,\alpha)$ on $\Re^n_+$ is attained as a minimum. In this section, we will give a necessary and sufficient condition for the existence of the minimum of $Q(x,\alpha)$. Successively, when the minimum value is nonnegative, we will prove the existence of a minimum point belonging to a k-dimensional face of $\Re^n_+$, with $k \leq n-2$. This last property will allow us to characterize, in Section 5, the pseudoconvexity of f in terms of the data Q, q, q_0, in the three dimensional case.

Theorem 4.1 *Consider function f where $\nu_-(Q) = 1$ and assume the existence of s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$. Consider the quadratic function $Q(x,\alpha)$, where α is a root of (2.9).*

We have $\inf_{x\in\Re^n_+} Q(x,\alpha) = \min_{x\in\Re^n_+} Q(x,\alpha)$ if and only if the following conditions hold:

i) $x^TQx \geq 0,\ \forall x \in \Re^n_+$;

ii) $v \in \Re^n_+,\ v^TQv = 0 \Rightarrow (q-\alpha d)^Tv \geq 0$.

Proof Firstly, we prove that $i)$ and $ii)$ are necessary conditions for having $\inf_{x\in\Re^n_+} Q(x,\alpha) = \min_{x\in\Re^n_+} Q(x,\alpha)$. If $i)$ and/or $ii)$ does not hold, then there exists $v \in \Re^n_+$ such that $v^TQv < 0$ or $v^TQv = 0$ and $(q-\alpha d)^Tv < 0$. In both cases we have $Q(tv,\alpha) = t^2v^TQv + 2t(q-\alpha d)^Tv + 2(q_0 - \alpha d_0) \to -\infty$ when $t \to +\infty$, and this is a contradiction since the infimum is not attained.

It remains to prove the sufficiency of $i)$ and $ii)$. Assume, by contradiction, that the infimum $\ell = \inf_{x \in \Re^n_+} Q(x, \alpha)$ is not attained. Then, there exists a sequence $\{z_n\} \subset \Re^n_+$, such that $\| z_n \| \to +\infty$ and $Q(z_n, \alpha) \to \ell$.

For every n, consider the problem:

$$\min_{x \in Z(n)} Q(x, \alpha), \;\; Z(n) = \{x \in \Re^n_+ : \; e^T x \leq e^T z_n\}$$

where e is the vector having all its components equal to 1.

Since $Z(n)$ is a compact set, the problem has an optimal solution y_n. The sequence $\{y_n\}$ is such that $Q(y_n, \alpha) \leq Q(z_n, \alpha)$, so that $Q(y_n, \alpha) \to \ell$; furthermore, $\{y_n\}$ is unbounded, otherwise, for the continuity of the function, the infimum is attained.

We have $\{y_n\} \subset int\Re^n_+$ or, there exists a subsequence of $\{y_n\}$, which without loss of generality we can assume the same sequence, contained in the relative interior of a k dimensional face Γ. With the convention $\Gamma = int\Re^n_+$ when $\{y_n\} \subset int\Re^n_+$, the problem

$$\min_{x \in Y(n)} Q(x, \alpha), \;\; Y(n) = \{x \in \Gamma : \; e^T x \leq e^T y_n\}$$

has the property that y_n is an optimal solution binding to the constraint $e^T x \leq e^T y_n$.

Set $q^* = q - \alpha d$. By means of the Karush-Kuhn-Tucker conditions, for every n, we have

$$Qy_n + q^* = \lambda_n e, \;\; \lambda_n \leq 0. \tag{4.14}$$

Assume the existence of $\bar{n}$ such that $\lambda_{\bar{n}} = 0$, or, equivalently, that $y_{\bar{n}}$ is a critical point. The quadratic form of the restriction of $Q(x, \alpha)$ on Γ is positive semidefinite or it has a negative eigenvalue μ. In the first case, $y_{\bar{n}}$ is a global minimum of $Q(x, \alpha)$ on Γ and, consequently, on $\Re^n_+$, and this is a contradiction. In the second case, let $w \in \Gamma$ be an eigenvalue associated to μ; the point $y_{\bar{n}}$ is a strict local maximum for the restriction $\varphi(t) = Q(y_{\bar{n}} + tw, \alpha)$ and this absurd since $y_{\bar{n}}$ is the minimum point of $Q(x, \alpha)$ on $Y(\bar{n})$.

It follows that in (4.14), $\lambda_n < 0$, $\forall n$.

From (4.14), we have $Q\dfrac{y_n}{\| y_n \|} + \dfrac{q^*}{\| y_n \|} = \dfrac{\lambda_n}{\| y_n \|}e$. Since $\lim_{n \to +\infty} Q\dfrac{y_n}{\| y_n \|} + \dfrac{q^*}{\| y_n \|} = Qv$, with $v \in \Re^n_+$, $\| v \| = 1$, then the sequence $\dfrac{\lambda_n}{\| y_n \|}$ converges to a nonpositive value β, so that $Qv = \beta e$. From $i)$ we have $v^T Qv \geq 0$, so that $v^T Qv = \beta e^T v$ holds if and only if $\beta = 0$ and thus $Qv = 0$. From (4.14), we have $v^T Q y_n + q^{*T} v = \lambda_n e^T v$, i.e., $q^{*T} v = \lambda_n e^T v$, so that the sequence $\{\lambda_n\}$ is constant; consequently $q^{*T} v < 0$ and this contradicts $ii)$. The proof is complete. □

The following theorem is related to the location of a minimum point of the quadratic function $Q(x, \alpha)$ in the case $int\Re^n_+ \subset S^+(\alpha)$.

Theorem 4.2 *Consider function f where $\nu_-(Q) = 1$. If there exist s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$ and $\inf_{x \in \Re^n_+} Q(x, \alpha) = \min_{x \in \Re^n_+} Q(x, \alpha) \geq 0$, then there exists a minimum point which belongs to a k-dimensional face of $\Re^n_+$ with $k \leq n - 2$.*

Proof Note that a minimum point of $Q(x, \alpha)$ on $\Re^n_+$ cannot belongs to $int\Re^n_+$, because of the condition $\nu_-(Q) = 1$.

Assume the existence of a minimum point x^* belonging to the relative interior of a $(n-1)$-dimensional face F of $\Re^n_+$, say $x^* = (x_1^*, ..., x_{n-1}^*, 0)^T, x_i^* > 0, i \neq n$, and consider the hyperplane Γ of equation $x_n = 0$.

We will analyze the two exaustive cases $Q(x^*, \alpha) = 0$, $Q(x^*, \alpha) > 0$.

case $Q(x^*, \alpha) = 0$.

Consider the line r of equation $x = x^* + t(\hat{x} - x^*), t \in \Re$, where $\hat{x} = s + \alpha\bar{x}$. From (2.10) we have $Q(x, \alpha) = (1 - t)^2 Q(x^*, \alpha) = 0$, $\forall t \in \Re$. We have $r \cap int\Re^n_+ = \emptyset$ because of the condition $\nu_-(Q) = 1$. Since $x^* \in r$ is an interior point of F, any point of $r \cap F$ is a minimum for $Q(x, \alpha)$ on $\Re^n_+$. Taking into account that $r \cap F$ intersects a k-dimensional face of $\Re^n_+$ with $k < n - 1$, the thesis is achieved.

case $Q(x^*, \alpha) > 0$.

Since $\Gamma \supset F$, x^* is a local minimum for the restriction g of $Q(x, \alpha)$ on Γ, so that Q is positive semidefinite on Γ, and, consequently, x^* is a global minimum for the restriction g. From (2.10) we have $Q(\hat{x} + tw, \alpha) = 0$, $\forall t \in \Re, \forall w \in W = \{w \in \Re^n : \ w \neq 0,\ w^T Q w = 0\}$. Then, $\hat{x} + tw \notin \Gamma$, $\forall t \in \Re, \forall w \in W$ and this implies $\hat{x}_n \neq 0$ and $w_n = 0$. It follows $W \subset \Gamma$, i.e., Q is positive semidefinite (not definite) on Γ. Consequently, there exists at least a line s of minimum points for g on Γ passing trough x^*, which intersects a k-dimensional face of $\Re^n_+$ with $k < n-1$. The proof is complete. $\square$

In the next section, we will see how the obtained results will allow us to characterize the inclusion $S^+(\alpha) \subset int\Re^n_+$ in the case $n = 3$, $n = 2$. Unfortunately, when $n > 3$, it is not possible, in general, to know a priori the dimension of the face on which the minimum point is located, as it is shown in the following example.

Example 4.1 *Consider the function* $f(x) = \dfrac{\frac{1}{2}x^TQx + q^Tx + q_0}{d^Tx + d_0}, x \in D$

where $Q = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$, $q = (14, -2, -4, -2)^T, q_0 = 30, d = (2, 0, 0, 0)^T,$

$d_0 = 3,\ D = \{x = (x_1, x_2, x_3, x_4) : 2x_1 + 3 > 0\}.$

We have $\nu_-(Q) = 1,\ \bar{x} = (0, 1, -2, 1)^T,\ s = (-2, -4, 12, -4)^T,\ s^TQs - 2q_0 = 0,\ d^T\bar{x} = 0,$ $d^Ts + d_0 = -1$, *so that* f *is pseudoconvex on every convex subset of* $S^*(0) = \{x \in D : Q(x, 0) = \frac{1}{2}x^TQx + q^Tx + q_0 \geq 0\}$.

Since $Q \geq 0, q_{ii} > 0, i = 1, ..., 4$, *it results* $\{x \in \Re^4_+ : x^TQx = 0\} = \{0\}$; *consequently, the quadratic function* $Q(x, 0)$ *assume its minimum value on* $\Re^4_+$.

It is easy to verify that do not exist minimum points of $Q(x, 0)$ *belonging to the relative interior of a three-dimensional face of* $\Re^4_+$. *With respect to the two-dimensional faces, we have a unique minimum point given by* $x^{(1)} = (0, 1, 0, 1)^T$, *with* $Q(x^{(1)}, 0) = 58$; *nevertheless,* $\min_{x \in \Re^4_+} Q(x, 0) = Q(x^{(2)}, 0) = 44$, *where* $x^{(2)} = (0, 0, 4, 0)$.

5 The three-dimensional case

As pointed out in Section 3, the problem of finding necessary and sufficient conditions which ensure $S^+(\alpha) \subset int\Re^n_+$ in terms of the structure of Q is not easy to carry on. An effort in this direction is given in this section which is devoted to give a complete characterization of the above inclusion in the case where matrix Q is of order 3.

In what follows we will use the following notations:

$| A |$ denotes the determinant of a square matrix A;

Q_{ii} is the submatrix of Q obtained by deleting the i-th row and column.

Theorem 5.1 *Let $Q = (q_{ij}),\ i, j \in \{1, 2, 3\}$, be a symmetric matrix with $\nu_-(Q) = 1$. Then, $x^T Qx \geq 0,\ \forall x \in \Re^3_+$ if and only if the following conditions hold:*

i) $q_{ii} \geq 0,\ i = 1, 2, 3$;

ii) for every $q_{ij} < 0$ we have $| Q_{kk} | \geq 0,\ k \neq i, j$;

iii) if there exist two negative elements, say q_{ij}, q_{ik}, then $q_{ij}q_{ik} - q_{ii}q_{jk} \leq 0$.

Proof Assume that $x^T Qx \geq 0,\ \forall x \in \Re^3_+$. Setting $x_j = 0, \forall j \neq i$, we have $q_{ii}x_i^2 \geq 0$, so that $q_{ii} \geq 0,\ i = 1, .., 3$, and $i)$ holds.

In what follows the restriction of $x^T Qx$ obtained by setting $x_j = 0$, will be denoted with $g_j(x_i, x_k)$.

Suppose the existence of one negative element q_{ij} and consider the restriction $g_k(x_i, x_j) = q_{ii}x_i^2 + 2q_{ij}x_jx_i + q_{jj}x_j^2$. If $q_{ii} = 0$ or $q_{jj} = 0$, then $g_k(x_i, x_j) < 0$ for suitable positive values of $x_i,\ x_j$, so that $q_{ii} > 0$ and $q_{jj} > 0$. Furthermore, the discriminant $\Delta^* = q_{ij}^2 - q_{ii}q_{jj} = - | Q_{kk} |$ must be nonpositive otherwise, once again, $g_k(x_i, x_j) < 0$ for suitable positive values of $x_i,\ x_j$. Consequently, $ii)$ holds.

Assume now the existence of two negative elements q_{ij}, q_{ik}; we must prove that $q_{ij}q_{ik} - q_{ii}q_{jk} \leq 0$. By considering the restrictions $g_k(x_i, x_j),\ g_j(x_i, x_k)$, it easy to prove that $q_{ii} > 0, \forall i \in \{1, 2, 3\}$. We can express the quadratic form $x^T Qx$ as follows:

$$q_{ii}x_i^2 + 2(q_{ij}x_j + q_{ik}x_k)x_i + q_{jj}x_j^2 + 2q_{jk}x_jx_k + q_{kk}x_k^2. \tag{5.15}$$

Since $q_{ij}x_j + q_{ik}x_k < 0,\ \forall x_j,\ x_k > 0$, the discriminant Δ of the trinomial (5.15) must be nonpositive. By simple calculations we obtain

$$\Delta = - | Q_{jj} | x_j^2 - | Q_{kk} | x_k^2 + 2(q_{ij}q_{ik} - q_{ii}q_{jk})x_jx_k.$$

From $ii)$ we have $| Q_{jj} | \geq 0, | Q_{kk} | \geq 0$. If $| Q_{jj} | = | Q_{kk} | = 0$, then $\Delta \leq 0$ if and only if $q_{ij}q_{ik} - q_{ii}q_{jk} \leq 0$. Assume, without loss of generality, that $| Q_{jj} | > 0$. The discriminant of Δ is equal to $-q_{ii} \mid Q \mid x_k^2$ and, for every $x_k \neq 0$, it is positive since $\nu_-(Q) = 1$

implies $| Q |< 0$. It follows that $\Delta \leq 0, \ \forall x_j, \ x_k > 0$, if and only if $q_{ij}q_{ik} - q_{ii}q_{jk} \leq 0$. Consequently, $iii)$ is proven.

Note that we cannot have $q_{ij} < 0, q_{ik} < 0, q_{jk} < 0$, otherwise it results $q_{ij}q_{ik} - q_{ii}q_{jk} > 0$, so that Δ assumes positive values for some $x_j, \ x_k > 0$, contradicting the assumption $x^T Qx \geq 0, \ \forall x \in \Re^3_+$.

Assume now that conditions $i) - iii)$ hold.

If $q_{ij} \geq 0, \ \forall i, j$, then obviously we have $x^T Qx \geq 0, \ \forall x \in \Re^3_+$.

If there exists only one negative element, say q_{jk}, then from $ii)$ the submatrix Q_{ii} is positive semidefinite, so that from (5.15), we have $x^T Qx \geq 0, \ \forall x \in \Re^3_+$.

If there exist two negative elements, say q_{ij}, q_{ik}, from $ii)$ we have $| Q_{jj} |\geq 0, | Q_{kk} |\geq 0$, so that $q_{ii} > 0, i = 1, 2, 3$. Taking into account of $iii)$, we have that the discriminant Δ of (5.15) is nonpositive, so that $x^T Qx \geq 0, \ \forall x \in \Re^3_+$.

The proof is complete. □

Remark 5.1 *Regarding the structure of the matrix Q, from Theorem 5.1, in order to have $x^T Qx \geq 0, \ \forall x \in \Re^3_+$, the following conditions must be verified:*

- *Q cannot have three negative elements;*
- *if $q_{ii} = 0$, then $q_{ij} \geq 0, \ \forall j \neq i$;*
- *if $| Q_{ii} |< 0$, then $q_{jk} > 0, j, k \neq i$ and $j \neq k$.*

Denoting by q_i, d_i the i-th component of q, d, respectively, we have the following theorem.

Theorem 5.2 *Consider function f where $\nu_-(Q) = 1$, $d \in \Re^3_+ \setminus \{0\}$, $d_0 > 0$. If there exist s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$, then $int\Re^3_+ \subset S^+(\alpha)$ if and only if $x^T Qx > 0, \ \forall x \in int\Re^3_+$ and the following conditions hold:*

i) If there exists an index i such that $q_{ii} = 0$, then $q_i - \alpha d_i \geq 0$;

if there exists $q_{ij} < 0$ such that $q_{ij}^2 = q_{ii}q_{jj}$, then $(q_i - \alpha d_i)\sqrt{q_{jj}} + (q_j - \alpha d_j)\sqrt{q_{ii}} \geq 0$.

ii) there not exist a unique minimum point belonging to the relative interior of a two-dimensional face; furthermore, $q_0 - \alpha d_0 \geq 0$, and $2(q_0 - \alpha d_0) \geq \max_{i \in I} \frac{(q_i - \alpha d_i)^2}{q_{ii}}$, $I = \{i : q_i - \alpha d_i < 0\}$, where α is a root of (2.9).

Proof The inclusion $int\Re^3_+ \subset S^+(\alpha)$ holds if and only if $\underset{x\in\Re^3_+}{inf}\ Q(x,\alpha) = \underset{x\in\Re^3_+}{min}\ Q(x,\alpha) \geq 0$. Taking into account Theorem 4.1, firstly we prove that $i)$ is equivalent to $ii)$ of Theorem 4.1.

Assume the validity of the implication: $v \in \Re^3_+,\ v^TQv = 0 \Rightarrow (q-\alpha d)^T v \geq 0$.

If $q_{ii} = 0$, we have $(e^i)^TQe^i = q_{ii} = 0$ so that $(q-\alpha d)^Te^i = (q_i - \alpha d_i) \geq 0$, whre e^i denotes the vector having its i-th component equal to 1 and all others equal to zero.

The existence of $q_{ij} < 0$ with $q^2_{ij} = q_{ii}q_{jj}$, together with the assumption $x^TQx > 0,\ \forall x \in int\Re^3_+$, implies $q_{ii} > 0, q_{jj} > 0$. We have $q_{ii}x_i^2 + 2q_{ij}x_ix_j + q_{jj}x_j^2 = (\sqrt{q_{ii}}x_i - \sqrt{q_{jj}}x_j)^2$, so that $v = (\sqrt{q_{jj}}, \sqrt{q_{ii}}, 0)^T$ verifies the equality $v^TQv = 0$. It follows that $(q-\alpha d)^Tv = (q_i - \alpha d_i)\sqrt{q_{jj}} + (q_j - \alpha d_j)\sqrt{q_{ii}} \geq 0$.

Assume now that $i)$ holds and let $v \neq 0$ such that $v^TQv = 0$. Since $w^TQw > 0,\ \forall w \in int\Re^3_+$, than v belongs to the boundary of $\Re^n_+$, so that the following exaustive cases occur: v is of the kind $v = ke^i, k > 0$, or v is of the kind $v = (\bar{v}_i, \bar{v}_j, 0)^T, \bar{v}_i > 0, \bar{v}_j > 0$. In the first case we have $(e^i)^TQe^i = q_{ii} = 0$, so that $i)$ implies $(q_i - \alpha d_i) = (q-\alpha d)^Te^i \geq 0$, and thus $(q-\alpha d)^Tv = (q-\alpha d)^Tke^i \geq 0$.

In the second case consider the restriction g_k of the quadratic form x^TQx on $x_k = 0, k \neq i,j$, i.e., $g_k(x_i,x_j) = q_{ii}x_i^2 + 2q_{ij}x_ix_j + q_{jj}x_j^2$. Since $g_k(x_i,x_j) \geq 0, \forall(x_i,x_j) \in \Re^2_+$ and $g_k(\bar{v}_i,\bar{v}_j) = 0$, the matrix Q_{kk} is positive semidefinite and $q^2_{ij} - q_{ii}q_{jj} = 0$.

If $q_{ij} < 0$, then $g_k(\bar{v}_i,\bar{v}_j) = (\sqrt{q_{ii}}\bar{v}_i - \sqrt{q_{jj}}\bar{v}_j)^2$, so that $\bar{v}_i = k\sqrt{q_{jj}}, \bar{v}_j = k\sqrt{q_{ii}}, k > 0$; it follows $(q-\alpha d)^Tv = k\left[(q_i - \alpha d_i)\sqrt{q_{jj}} + (q_j - \alpha d_j)\sqrt{q_{ii}}\right] \geq 0$.

If $q_{ij} = 0$, then $g_k(\bar{v}_i,\bar{v}_j) = 0$ implies $q_{ii} = q_{jj} = 0$ so that $q_i - \alpha d_i \geq 0,\ q_j - \alpha d_j \geq 0$ and thus $(q-\alpha d)^Tv = (q_i - \alpha d_i)\bar{v}_i + (q_j - \alpha d_j)\bar{v}_j \geq 0$.

It remains to prove that the minimum value of $Q(x,\alpha)$ on $\Re^3_+$ is nonnegative if and only if $ii)$ holds.

If $\underset{x\in\Re^3_+}{min}\ Q(x,\alpha) \geq 0$, then Theorem 4.2 implies the existence of a minimum point x_0 which is the origin or which belongs to an edge s^k of $\Re^3_+$. If $x_0 = 0$, then $I = \emptyset$ and $q_0 - \alpha d_0 \geq 0$. If $x_0 \in s^k$, then $k \in I$ and $\underset{x\in\Re^3_+}{min}\ Q(x,\alpha) = \dfrac{-(q_k - \alpha d_k)^2 + 2(q_0 - \alpha d_0)q_{kk}}{q_{kk}} \geq 0$. Consequently, $2(q_0 - \alpha d_0) \geq \dfrac{(q_k - \alpha d_k)^2}{q_{kk}} = \max\limits_{i\in I} \dfrac{(q_i - \alpha d_i)^2}{q_{ii}}$.

Assume now that $ii)$ holds. If there exists a minimum point which belongs to a two-

dimensional face, then there exists a line r of equation $x = x_0 + tu, t \in \Re$ such that $Q(x,\alpha) = Q(x_0,\alpha)$, $\forall x \in r$ and which intersect at least one edge of $\Re^3_+$. The assumptions $q_0 - \alpha d_0 \geq 0$, and $2(q_0 - \alpha d_0) \geq \max_{i \in I} \frac{(q_i - \alpha d_i)^2}{q_{ii}}$, $I = \{i : q_i - \alpha d_i < 0\}$, imply that $\min_{x \in \Re^3_+} Q(x,\alpha) \geq 0$.
The proof is complete. □

The obtained results can be specialized to the two-dimensional case.

Theorem 5.3 *Consider function f where $Q = (q_{ij})$, $i,j = 1,2$, $\nu_-(Q) = 1$, $d \in \Re^2_+ \setminus \{0\}$, $d_0 > 0$. If there exist s and $\bar{x}$ such that $Qs = -q$, $Q\bar{x} = d$, then $int\Re^2_+ \subset S^+(\alpha)$ if and only if $q_{ii} \geq 0$, $i = 1,2$, $q_{12} > 0$, $q_i - \alpha d_i \geq 0$, $i = 1,2$, $q_0 - \alpha d_0 \geq 0$.*

Proof Applying Theorem 5.1 to the symmetric matrix of order 3 obtained by Q adding a null row and a null column we have $x^T Q x \geq 0$, $\forall x \in \Re^2_+$ if and only if $q_{ii} \geq 0$, $i = 1,2$, $q_{12} > 0$. The thesis is achieved taking into account *ii* of Theorem 5.2. □

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