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Best gain-loss ratio is a poor performance measurement

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Abstract

Gain-loss ratio is known to enjoy very good properties from a normative point of view. As a confirmation, we prove a dual representation of the best market gain-loss in the presence of a random endowment for general probability spaces. The dual representation is then the key to show best gain-loss is an acceptability index.

However, gain-loss ratio was designed for and works best only for finite Ω. In most general Ω, and continuous time models best gain-loss is either infinite or fails to be attained. In addition, it displays an odd behaviour in a risk neutral situation due to scale invariance, which does not seem desirable in a portfolio optimization context. Such weaknesses definitely prove best gain-loss is a *poor* performance measurement.

Key words: Gain-loss ratio, acceptability indexes, incomplete markets, martingales, quasi concave optimization, duality methods, market modified risk measures.
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1 Introduction

Gain-loss ratio was introduced by Bernardo and Ledoit [3] to provide a novel asset pricing criterion and an alternative to the traditional Sharpe Ratio in portfolio performance evaluation. In fact, the Sharpe Ratio (SR) has a number of drawbacks: it is not monotone, thus violating a basic axiom in theory of choice; moreover, it is the target of serious objection from the asset pricing theory approach based on no-good deals. A classic reference on good deals is Cochrane and Saa-Requejo [11]. In brief, a good deal is a portfolio whose SR

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is very high high and thus, informally speaking, should be regarded as a quasi-arbitrage. Ruling out good deals, or equivalently restricting SR, produces restrictions on pricing kernels. Restricted pricing kernels are in turn desirable since they provide narrower lower and upper price intervals for contingent claims in comparison to arbitrage-free price intervals.

Of course, such idea is based on the assumption that high SR is attractive, and low SR is not. This works well in a Gaussian returns context, but in general it does not since it is totally incompatible with no-arbitrage. In fact a positive gain with finite first moment but infinite variance has zero SR, but it is very attractive as it is an arbitrage.

Bernardo and Ledoit propose as performance measure the gain-loss ratio:

$$\alpha(X) = \frac{E[X^+]}{E[X^-]}, X \neq 0$$

α has an intuitive significance, it is easily computed and enjoys many properties: monotonicity across Xs; scale invariance, that is $\alpha(\lambda X) = \alpha(X)$ for all $\lambda > 0$; law invariance, as two payoffs with the same distribution have the same α; and a classic continuity property (Fatou property). If limited to portfolios with positive expectation, it is also a quasi concave map, consistent with second order stochastic dominance, as shown by Cherny and Madan in [10] and is thus an *acceptability index* in their terminology.

Let α^* denote the best gain loss from the market, i.e. from the set $\mathcal{X}$ of non trivial portfolio gains with finite first moment:

$$\alpha^* := \sup_{X \in \mathcal{X}, X \neq 0} \alpha(X).$$

In case P is already a pricing kernel, $\alpha^* = 1$ as $E[X] = E[X^+ - X^-] = 0$ for all gains. This gives a flavor of the main result in Bernardo and Ledoit, which is the equivalence between

i) $\alpha^* < +\infty$,

ii) existence of pricing kernels with state price density Z satisfying $c \leq Z \leq C$ for some constants $C, c > 0$.

That is, restrictions on the market gain-loss ratio are equivalent to the existence of special, restricted pricing kernels bounded and bounded away from 0. They also prove a duality formula for α^*,

$$\alpha^* = \min_Z \frac{\text{ess sup } Z}{\text{ess inf } Z}$$

where Z varies over all the pricing kernels as in item ii) above. Though stated for a general probability space and in a biperiodal market model, Bernardo and Ledoit's derivation is correct only if Ω is finite. In fact, what they actually show is

$$\alpha^* = \max_{X \in \mathcal{X}, X \neq 0} \alpha(X) = \min_Z \frac{\text{ess sup } Z}{\text{ess inf } Z},$$

i.e. that the market best ratio is always attained. This is true only if Ω is finite.

Against this background, the present paper develops an analysis of the gain-loss ratio for general probability spaces. The rest of the paper is organized as follows. In Section 2 we show the above equivalence i) $\iff$ ii) in the presence of a continuous time market for general Ω. The duality technique employed here extends also Pinar's treatment [17, 18]. The assumptions made on the market model are quite general, as we do not require the process of the underlyings S to be neither a continuous diffusion, nor locally bounded.

The duality formula for α^* is correctly reformulated as $\sup \cdots = \min \cdots$ and a simple counterexample where the supremum α^*, though finite, is not attained is provided in the Examples Section 2.4.

In Section 2.3 pros and cons of gain-loss are analyzed. While in discrete time models there is a full characterization of models with finite best gain-loss ratio, in continuous time the situation is hopeless. In most commonly used models $\alpha^* = +\infty$ as any pricing kernel is unbounded as shown in detail in Example 2.7. Finally, in Section 2.5 a dual representation is provided for the best gain loss $\alpha^*(B)$ in the presence of random endowment B. The results there also highlight gain-loss drawbacks. In particular, the scale invariance property does not seem desirable at all in the presence of random endowment. If P is already a pricing kernel, it implies that either the best gain-loss is infinite if B has positive expectation or it is optimal to take infinite risk in order to off-set the negative expectation of B, which is economically unreasonable.

Finally, the dual representation of $\alpha^*(B)$ is the key to the result in Section 3. There, after some preliminaries on risk measures and acceptability indexes, the market best gain-loss ratio $\alpha^*(B)$ is described as an acceptability index on the space $L^1(P)$ of integrable random variables as a function of the random endowment B.

2 The market best gain-loss α^* and its dual representation

2.1 The market model

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a continuous time stochastic basis satisfying the usual assumptions. S is an $\mathbb{R}^d$-valued semimartingale on this basis and models the (discounted) time evolution of d underlyings up to the finite horizon T. A strategy ξ is predictable, S integrable process and the stochastic integral $\xi \cdot S$ is the corresponding gain process. Now, some integrability condition must be imposed on S in order to ensure the presence of strategies ξ with well defined gain-loss ratio. In some cases in fact it may happen that every non null terminal gain $K = \xi \cdot S_T$ verifies $E[K^+] = E[K^-] = +\infty$, see the Examples Section for a simple one period model of such extreme situation.

The following is thus the integrability assumption on S which holds throughout the paper.

Assumption 2.1 *Let $S_T^* = \sup_{t\leq T}|S_t|$ denote the maximal functional at T. Then $S_T^* \in L^1(P)$.*

Note S_T^* coincides with the running maximum at the terminal date T if S is nonnegative. Such assumption is verified in many models used in practice:

- if time is discrete, with finite horizon, or equivalently: S is a pure jump process with jumps occurring only at fixed dates $t_1, \ldots, t_n$, the assumption is equivalent to $S_{t_i} \in L^1(P)$ for all t_i;
- if S is a Lévy process, the assumption is equivalent to the integrability of S_T only (or of S_t at any fixed $0 < t \leq T$). This is a particular case of a more general result on moments of Lévy process, see the whole [23, Section 5.25] (specifically Theorem 5.25.18).

So, at least in normal market conditions such assumption is quite reasonable. From a strict mathematical perspective it ensures that gains processes are true (and not local) martingales under pricing kernels. The set of admissible strategies and related terminal wealths are the linear spaces defined as

$$\Xi = \{\xi \mid \xi \text{ is predictable and bounded}\}, \qquad \mathcal{K} = \{K \mid K = \xi \cdot S_T \text{ for some } \xi \in \Xi\}.$$

So, $\mathcal{K}$ is the set of claims which are replicable at zero cost via a bounded strategy, that is the set of terminal gains. Note that $\xi = \mathbb{1}_A \mathbb{1}_{]s,t]}$ and its opposite $-\xi$ are in Ξ for all $A \in \mathcal{F}_s$ and for all $0 \leq s < t \leq T$, so that $K = \mathbb{1}_A(S_t - S_s)$ and $-K$ are in $\mathcal{K}$.

The best gain-loss in the above market is then

$$\alpha^* := \sup_{K\in\mathcal{K}, K\neq 0} \alpha(K).$$

Such value is always greater or equal to 1, and it is equal to 1 if and only if P is already a martingale measure for S. These facts can be easily proved, using the linearity of $\mathcal{K}$ and the above observation: $\pm\mathbb{1}_A\mathbb{1}_{]s,t]} \in \Xi$.

2.2 No λ gain-loss, its dual characterization and the duality formula for α^*

The market best gain-loss α^* is the value of a non standard optimization. In fact, the gain-loss ratio α is not concave, and not even quasi concave on $L^1(P)$. However, when restricted to variables with positive expectation it becomes quasi-concave, as shown in detail by [10]. So, the optimization can be restricted to gains with positive expectations and can be thus seen as quasi concave problem.

To characterize α^* and to link it to a no-arbitrage type result, we rely on auxiliary maximization problems with piecewise linear utility U_λ:

$$U_\lambda(x) = x^+ - \lambda x^-.$$

The convex conjugate of U_λ, $V_\lambda(y) = \sup_x(U_\lambda(x) - xy)$ is simply the functional indicator of the interval $[1, \lambda]$:

$$V_\lambda(y) = \begin{cases} 0 & \text{if } 1 \leq y \leq \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

By mere definition of the conjugate, the Fenchel inequality holds:

$$U_\lambda(x) - xy \leq V_\lambda(y) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

The next definition is understood as follows. The market is gain-loss free at a certain level λ if not only there is no gain with $\alpha \geq \lambda$, but also λ cannot be approximated arbitrarily well with gains in $\mathcal{K}$.

DEFINITION 2.2 *For a given $\lambda \in (1, +\infty]$, the market is λ gain-loss free if $\alpha^* < \lambda$.*

Theorem 2.4 below, first shown by Bernardo and Ledoit in a two periods setup, states exactly the equivalence between absence of λ gain-losses and existence of a martingale measure with a density satisfying precise bounds.

Some notation first. Let $\mathcal{C} = \{X \in L^1 \mid X \leq K \text{ for some } K \in \mathcal{K}\}$ denote the set (convex cone) of claims which are super replicable at zero cost, and consider its polar set $\mathcal{C}^0 = \{Z \in L^\infty \mid E[ZX] \leq 0 \text{ for all } X \in \mathcal{C}\}$. As $\mathcal{C} \supseteq -L^1_+$, $\mathcal{C}^0 \subseteq L^\infty_+$. $\mathcal{C}^0$ is a convex cone and thus not empty as $0 \in \mathcal{C}^0$.

However, $\mathcal{C}^0$ may be trivially $\{0\}$, i.e. its basis $\mathcal{C}^0_1 = \{Z \in \mathcal{C}^0 \mid E[Z] = 1\}$ may be empty. This may happen in common models as the Black Scholes model, see Remark 2.3 and Example 2.7 for a discussion and more details. The basis $\mathcal{C}^0_1$ however is important for gain-loss analysis. The following Lemma in fact proves that $\mathcal{C}^0_1$ is the set of bounded martingale probability densities, which in turn appear in the characterization of the market best gain-loss in Theorem 2.4.

LEMMA 2.3 *$Z \in \mathcal{C}^0_1$ if and only if it is a bounded martingale density.*

Proof. If $Z \in \mathcal{C}^0_1$, it is bounded nonnegative and integrates to 1, so it is a probability density of a $Q \ll P$. Moreover, $\pm\mathbb{1}_A(S_t - S_s) \in \mathcal{C}$, for all $A \in \mathcal{F}_s, s < t$, so that $E[Z\mathbb{1}_A(S_t - S_s)] = 0$, which precisely means $E_Q[S_t \mid \mathcal{F}_s] = S_s$. Conversely, if Q is a martingale probability for S, with bounded density Z, then

$$S^*_T \in L^1(P) \subseteq L^1(Q).$$

As S_T^* is Q-integrable and ξ is bounded, the integral $\xi \cdot S$ has maximal functional $(\xi \cdot S)_T^* \in L^1(Q)$ and is thus a martingale of class $\mathcal{H}^1(Q)$, see [19, Chapter IV, Sect 4]). Now, if $K \in \mathcal{C}$ by definition it can be super replicated at zero cost: $K \leq \xi \cdot S_T$ for some ξ, whence

$$E[ZK] = E_Q[K] \leq E_Q[\xi \cdot S_T] = 0.$$

The above inequality implies $Z \in \mathcal{C}^0$. □

THEOREM 2.4 *The following conditions are equivalent:*

a) the market is λ gain-loss free,

b) there exists an (equivalent) martingale probability Q such that

$$\frac{\text{ess sup} \frac{dQ}{dP}}{\text{ess inf} \frac{dQ}{dP}} < \lambda. \tag{2}$$

In case any of the two conditions above holds, the market best gain-loss α^* admits a dual representation as

$$\alpha^* = \min_{Q \in \mathcal{M}_\infty} \frac{\text{ess sup} \frac{dQ}{dP}}{\text{ess inf} \frac{dQ}{dP}} \tag{3}$$

in which $\mathcal{M}_\infty$ is the set of equivalent martingale probabilities Q with densities $Z \in \mathcal{C}_1^0$ which are (bounded and) bounded away from 0, i.e. $\{Z \in \mathcal{C}_1^0 \mid Z > c \text{ for some } c > 0\}$.

Proof. The equivalence will be proved via the auxiliary utility maximization problem

$$u_\mu := \sup_{K \in \mathcal{K}} E[U_\mu(K)].$$

The reason is clear: as $E[U_\mu(K)] = E[K^+ - \mu K^-]$, $u_\mu \leq 0$ (and in fact, $u_\mu = 0$ as $0 \in \mathcal{K}$) is equivalent to $\alpha^* \leq \mu$. Also, Fenchel pointwise inequality (1) gives, for any random variable Y

$$U_\mu(K) - KY \leq V_\mu(Y).$$

b) $\Rightarrow$ a) If there exists a Q with the stated properties, its density Z belongs to $\mathcal{C}_1^0$ by Lemma 2.3. Set $Y = \frac{Z}{\text{ess inf} Z} \in \mathcal{C}^0$. As $1 \leq Y \leq \frac{\text{ess sup} Z}{\text{ess inf} Z} := \mu < \lambda$, $V_\mu(Y) = 0$ and thus for all K Fenchel inequality simply reads as $U_\mu(K) - KY \leq 0$. Taking expectations, $E[U_\mu(K)] \leq 0$, which is in turn equivalent to $u_\mu = 0$ and to $\alpha^* \leq \mu < \lambda$.

a) $\Rightarrow$ b) Set $\mu = \alpha^*$. Then $u_\mu = 0$. Existence of a Q is now a standard duality instance. Note U_μ is monotone, so $u_\mu = \sup_{K \in \mathcal{C}} E[U_\mu(K)]$. Also, the monotone concave functional $E[U_\mu(\cdot)]$ is finite and thus continuous on L^1 by the Extended Namioka Theorem (see [5], [16]). Therefore Fenchel Duality theorem applies (see e.g. [7, Theorem I.11] or [4] for a survey of duality techniques in the utility maximization problem) and gives the formula

$$u_\mu = \min_{Y \in \mathcal{C}^0} E[V_\mu(Y)].$$

In particular the infimum in the dual is attained by a $Y^* \in \mathcal{C}^0$. Therefore $1 \leq Y^* \leq \mu = \alpha^* < \lambda$ and its scaling $Z^* = Y^*/E[Y^*]$ is a martingale density with the property required in (2).

Suppose now any of the two conditions above holds true. Then, the proof of the arrow b) $\Rightarrow a)$ actually shows

$$\alpha^* = \sup_{K \in \mathcal{K}, K \neq 0} \frac{E[K^+]}{E[K^-]} \leq \inf_{Q \in \mathcal{M}_\infty} \frac{\text{ess sup } Z}{\text{ess inf } Z}, \tag{4}$$

and the proof of the arrow a) $\Rightarrow b)$ shows that the infimum is attained by Z^* and there is no duality gap. □

The next Corollary immediately follows.

COROLLARY 2.5 *$\alpha^* < +\infty$ iff $\mathcal{M}_\infty \neq \emptyset$.*

2.3 Pros and cons of gain-loss ratio

The requirement of gain-loss free market can thus be seen as a result à-la Fundamental Theorem of Asset Pricing also in general probability spaces. A comprehensive survey of No-Arbitrage concepts and result is the reference book [12]. If compared to those theorems, the above proof looks surprisingly easy. Of course, there is a (twofold) reason. First, there is an integrability condition on S; secondly, and most importantly, the assumption of λ gain-loss free market is much stronger than absence of arbitrage (or absence of free lunch with vanishing risk).

The stronger requirement of absence of λ gain-loss arbitrage allows a straightforward reformulation in terms of a standard utility maximization problem. This reformulation as such is not possible for the general FTAP case. The reader is however referred to [21] for a proof of the FTAP in discrete time based on a technique which relies in part on the ideas of utility maximization.

In discrete time trading there is a full characterization of the models which have finite best gain-loss ratio. On one side, the Dalang-Morton-Willinger Theorem ensures that under No Arbitrage condition there always exists a bounded pricing kernel. Such kernel is not necessarily bounded away from 0. On the other side, the characterization of arbitrage free markets which admit pricing kernels satisfying prescribed lower bounds is provided by [22].

Concerning practical application, in continuous time α^* is very likely to be infinite, see Example 2.7 for an illustration in the Black-Scholes model. And even if it is finite, the supremum may not be attained. This is not due to our specific assumptions, i.e. restriction to bounded strategies in Ξ. In general the market best gain-loss is intrinsically not attained, due to the nature of the functional considered. As it is scale invariant, maximizing sequences can be selected without loss of generality of unitary L^1-norm. But bounded sets in L^1 are not necessarily (weakly) compact, so maximizing sequences may fail to converge, as shown in Example 2.8 in a one period market.

Of course, an enlargement of strategies would certainly help in capturing optimizers in some specific model. But given the intrinsic problems of gain-loss optimization, in the end we choose to work with bounded strategies, as they have a clear financial meaning and allow for a plain mathematical treatment.

2.4 Examples

Example 2.6. *A model where no gain has well-defined gain-loss ratio.* Suppose S consists of only of one jump which occurs at time T. So, $S_t = 0$ up to time $T-$, while S_T has the distribution of the jump size. If the filtration is the natural one, then a strategy is simply a real constant $\xi = c$ and terminal wealths K are of the form $K = cS_T$. Suppose the jump has a symmetric distribution with infinite first moment. Although this is an arbitrage free model, if $c \neq 0$ both $E[K^+]$ and $E[K^-]$ are infinite and gain loss ratio criterion makes no sense.

Example 2.7. *Gain-loss ratio is infinite in a Black-Scholes world.* In the Black-Scholes market model, the density of the unique pricing kernel is

$$Z = (Z_T =) \exp(-\pi W_T - \frac{\pi^2 T}{2})$$

in which W_T is Brownian motion at terminal date T and $\pi = \frac{\mu - r}{\sigma}$ is the market price of risk. Such density is both unbounded and not bounded away from 0, so $\mathcal{C}^0$ is trivial and its basis empty. Therefore, though there is no arbitrage when $\mu \neq r$ the Black Scholes market is not gain-loss free, i.e. $\alpha^* = +\infty$.

Not surprisingly, the idea behind the construction of explicit arbitrarily large gain-loss ratios is playing with sets where the density Z is either very small or very large. The former sets have a low cost if compared to the physical probability of happening, while the latter in turn happen with small probability but have a (comparatively) high cost. We give examples of both. Without loss of generality, suppose $r = 0$ and fix $1 > \epsilon > 0$. Let $A_\epsilon := \{Z < \epsilon\}$, p_ϵ its probability and $X_\epsilon = \mathbb{1}_{A_\epsilon}$, while $B_\epsilon := \{Z > \frac{1}{\epsilon}\}$, q_ϵ its probability and $Y_\epsilon = \mathbb{1}_{B_\epsilon}$. Some calculations show that X_ϵ and Y_ϵ are cash-or-nothing digital options on $S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$, either of call type with very large strike or of put type with very small strike when ϵ goes to zero.

1. Let $c_\epsilon = E[ZX_\epsilon]$ be the cost of X_ϵ, which is much smaller than p_ϵ as $c_\epsilon < \epsilon p_\epsilon < 1$. Since the market is complete $K_\epsilon := X_\epsilon - c_\epsilon$ is a gain. Its gain-loss ratio is then

$$\frac{E[K_\epsilon^+]}{E[K_\epsilon^-]} = \frac{(1 - c_\epsilon)p_\epsilon}{c_\epsilon(1 - p_\epsilon)} > \frac{1 - c_\epsilon}{\epsilon} > \frac{1}{\epsilon} - p_\epsilon$$

which tends to $+\infty$ as $\epsilon \downarrow 0$.

2. Let $b_\epsilon = E[ZY_\epsilon]$ be the cost of Y_ϵ. Then, $1 > b_\epsilon > \frac{q_\epsilon}{\epsilon}$. As before, $C_\epsilon := Y_\epsilon - b_\epsilon$ and its opposite K_ϵ are gains. The gain-loss ratio of K_ϵ is then

$$\frac{E[K_\epsilon^+]}{E[K_\epsilon^-]} = \frac{b_\epsilon(1-q_\epsilon)}{(1-b_\epsilon)q_\epsilon} > \frac{1-q_\epsilon}{\epsilon}$$

which also tends to $+\infty$ as $\epsilon \downarrow 0$.

The two items together show better why in a gain-loss free market there must be a pricing kernel bounded above *and* bounded away from 0. The strategies that lead to the digital gains $X_\epsilon - c_\epsilon$ and $Y_\epsilon - b_\epsilon$ are not bounded. However stochastic integration theory, see e.g. the book [15, Chapter 3], ensures they can be approximated arbitrarily well by simple bounded strategies with L^2 convergence of the terminal gains, so the approximating strategies are in Ξ and their gain-loss ratio blows up.

Example 2.8. *The market best gain-loss may not be attained.* Here the model can be seen as a countable collection of one-step binomial trees, with initial uncertainty on the particular tree we are in. The odds and underlying are such that the best gain-loss in a given one-step binomial tree is less than the best gain-loss of the following binomial tree. Therefore, an optimal strategy fails to exist.

Let $(\Omega, \mathcal{F}_1, P)$ be the natural, non null, numbers $\Omega = \mathbb{N}_0$ with sigma algebra $\mathcal{F}_1 = \mathcal{P}(\mathbb{N}_0)$ and let $p(k) = \frac{1}{2^k}$. Take as initial $\mathcal{F}_0 = \sigma(\{2n-1, 2n\} \mid n \geq 1)$ the sigma algebra generated by the minimal "bricks" of two elements $\{1,2\}, \{3,4\}, \{5,6\}$.... Let λ^* be a fixed real number bigger than $\sqrt{2}$; $S_0 = 0$; in the n-th "brick" $\{2n-1, 2n\}$, let $S_1(2n-1)$ be $\lambda^* n$ and $S_1(2n) = -(n+1)$, so that conditional on this brick, we have a one period binomial model.

It is straightforward to see that $S_1 \in L^1(P)$, so Assumption 2.1 is satisfied; there is no arbitrage and, moreover, $\mathcal{M}_\infty \neq 0$. Here a strategy ξ can be identified with the sequence $(\xi_n)_n$ of its values on $\{2n-1, 2n\}, n \in \mathbb{N}_0$. For the purpose of this example only, let us work with all terminal wealths $\widetilde{\mathcal{K}} \ni K = \xi \cdot S_1 \in L^1$, i.e.

$$\sum_{n \geq 1} |\xi_n| (\frac{\lambda^* n}{2^{2n-1}} + \frac{n+1}{2^{2n}}) < +\infty \text{ or, in short } \sum_{n \geq 1} |\xi_n| \frac{n}{2^{2n}} < +\infty$$

That is, we do not impose boundedness on ξ. The results are

a- for every non null $K \in \widetilde{\mathcal{K}}$, $\frac{E[K^+]}{E[K^-]} < 2\lambda^*$

b- but $\alpha^* = \sup_{K \in \widetilde{\mathcal{K}}, K \neq 0} \frac{E[K^+]}{E[K^-]} = 2\lambda^*$.

As $\widetilde{K}$ is the largest conceivable set of gains in gain-loss maximization, this shows that the best gain-loss is intrinsically not attained. Namely, it is not a matter of strategy restrictions (boundedness or similar).

An intuitive explanation on why a) and b) above hold is: since λ^* is big enough, being long on S gives a better gain-loss ratio than being short and from such gains the ratio is strictly smaller than $2\lambda^*$; as in each one-step binomial tree the best gain-loss is $2\lambda^* \frac{n}{n+1}$, which increases to its limit $2\lambda^*$ when n tends to infinity, the optimum is not attained.

Now, the computations. Fix $K \neq 0$, i.e. $\xi_n \neq 0$ for some n. Since

$$E[K^+] = \lambda^* \sum_{\xi_n > 0} \frac{\xi_n n}{2^{2n-1}} + \sum_{\xi_n < 0} (-\xi_n) \frac{(n+1)}{2^{2n}},$$

call α the sum of the first series and set $\beta = \sum_{\xi_n<0}(-\xi_n)\frac{n}{2^{2n}}$. Then, $E[K^+] \leq \lambda^* \alpha + 2\beta$, while

$$E[K^-] = \lambda^* \sum_{\xi_n<0} (-\xi_n) \frac{n}{2^{2n-1}} + \sum_{\xi_n>0} \frac{\xi_n(n+1)}{2^{2n}} = 2\lambda^*\beta + \frac{\alpha}{2} + \sum_{\xi_n>0} \frac{\xi_n}{2^{2n}}.$$

Since $(\lambda^*)^2 > 2$,

$$\frac{E[K^+]}{E[K^-]} \leq \frac{\lambda^*\alpha + 2\beta}{2\lambda^*\beta + \frac{\alpha}{2} + \sum_{\xi_n>0} \frac{\xi_n}{2^{2n}}} \leq \lambda^* \frac{(\alpha + \lambda^*\beta)}{\frac{\alpha}{2} + \frac{\lambda^*\beta}{2} + c} < 2\lambda^*$$

where $c := \frac{3}{2}\lambda^*\beta + \sum_{\xi_n>0} \frac{\xi_n}{2^{2n}} > 0$ as $\xi_n \neq 0$ for some n. This proves item a) above.

To show item b), define the strategy ξ^n as follows:

$$\xi^n(\omega) = \begin{cases} 1 & \text{if } \omega \in \{2n-1, 2n\} \\ 0 & \text{otherwise.} \end{cases}$$

This is a position of $+1$ in S if we are in the $n-th$ binomial tree, corresponding to the brick $\{2n-1, 2n\}$, and do nothing otherwise. Then $\xi \in \Xi$ as it is bounded, and $k^n = \xi^n \cdot S_1$ has gain loss ratio $2\lambda^* \frac{n}{n+1}$, which concludes the proof.

From an analytic point of view, the maximizing sequence $k^n = \xi^n \cdot S_1$ converge in L^1 to 0, but in 0 α is not defined. Given the scale invariance of gain-loss, we can without loss of generality take as maximizing sequence the normalized version:

$$\frac{k_n}{E[|k_n|]}$$

which is not uniformly integrable and thus has no limit. Note further that the ratio $S_1(2n-1)/S_1(2n)$ remains bounded and bounded away from zero when n tends to infinity. This is the key to recover existence of a $Q \in \mathcal{M}_\infty$. A simple modification of this example, with e.g. $S_1(2n-1) = 1$ and $S_1(2n) = -2^{-n}$ as in [12, Remark 6.5.2], leads to an arbitrage free market model with no Q bounded away from zero.

2.5 Best gain loss with a random endowment B and its dual representation

Suppose the investor at time T has a random endowment $B \in L^1$. She can optimize over the market in order to reduce exposure. The best gain-loss in the presence of B will then be:

$$\alpha^*(B) := \sup_{K \in \mathcal{K}} \alpha(B + K).$$

Note $\alpha^*(0) = \alpha^*$ and therefore the results in this Section are the generalization of the case $B = 0$. As shown below, if the world is not risk neutral possessing a claim whatsoever does not reduce the best gain-loss ratio $\alpha^*(0)$.

LEMMA 2.9 *Suppose P is not a martingale measure. Then $\alpha^*(B) \geq \alpha^*(0) > 1$.*

Proof. Since P is not a martingale measure, there is a gain K with non null expectation. Then, either K or $-K$ have positive expectation. Say $\alpha(K) > 1$. For any $t > 0$, the scale invariance property of α gives

$$\alpha(B + tK) = \alpha(\frac{B}{t} + K).$$

An application of dominated convergence gives, when $t \uparrow +\infty$, $\alpha\left(\frac{B}{t} + K\right) \to \alpha(K) > 1$. The conclusion follows by taking the supremum over Ks. □

Remark 2.10. The proof of the previous Lemma shows in fact the scale invariance property is not so desirable in the presence of a random endowment. In fact, it rather "kills" the contribution of the endowment!

Define for a shorthand $\mathcal{Q}_b := \{Q \text{ probab. } \mid \exists y > 0, E[V_b(y\frac{dQ}{dP})] < +\infty\}$ to be the set of probabilities with finite V_b entropy.

THEOREM 2.11 *Fix a random endowment $B \in L^1$ which is not replicable, i.e. $B \notin \mathcal{K}$.*

a) Case P is not a martingale measure or $E[B] > 0$. Then, $\alpha^*(B) > 1$ and the following conditions are equivalent:

i) $\alpha^*(B) < +\infty$

ii) $E_Q[B] = 0$ for some $Q \in \mathcal{M}_\infty$.

If any of the two conditions i), ii) is satisfied, α^* has the dual representation

$$\alpha^*(B) = \min_{Q \in \mathcal{M}_\infty, E_Q[B]=0} \frac{\text{ess sup } Z}{\text{ess inf } Z}. \tag{5}$$

b) Case P is a martingale measure and $E[B] \leq 0$. Then, $\alpha^*(B) = 1$.

Proof. a) In this case $\alpha^*(B) > 1$. In fact, if P is not a martingale measure $\alpha^*(B) > 1$ follows from Lemma 2.9. Else if $E[B] > 0$ then trivially $\alpha^*(B) \geq \alpha(B) > 1$. The rest of the proof is similar to that of Theorem 2.4:

i)⇒ ii) Set $b = \alpha^*(B)$. Then $b \in (1, +\infty)$ and in particular $(B + K)^-$ is always non null. So,

$$0 = \alpha^*(B) - b = \sup_{K \in \mathcal{K}} \frac{E[U_b(B + K)]}{E[(B + K)^-]}.$$

The denominator is positive, whence

$$0 = \alpha^*(B) - b \leq \frac{\sup_K E[U_b(B+K)]}{\inf_K E[(B+K)^-]} \leq 0$$

which means $\sup_K E[U_b(B+K)] = 0$. As in the proof of Theorem 2.4 an application of Fenchel duality gives:

$$0 = \sup_K E[U_b(B+K)] = \min_{Q \in \mathcal{C}_1^0, y \geq 0} \{yE[\frac{dQ}{dP}B] + E[V_b(y\frac{dQ}{dP})]\}.$$

Given the structure of V_b, $E_Q[B] \geq 0$ for all $Q \in \mathcal{Q}_b \cap \mathcal{C}_1^0$ i.e. for all the (equivalent) martingale measures with finite V_b entropy. Moreover, any couple of minimizers y^*, Q^* satisfies $y^* > 0$ and $dQ^* = Z^* dP \in \mathcal{Q}_b \cap \mathcal{C}_1^0 \subseteq \mathcal{M}_\infty$, which is then not empty. Then, $E[V_b(y^*\frac{dQ^*}{dP})] + y^* E_{Q^*}[B] = 0$ gives $E_{Q^*}[B] = 0$ and ii) follows.

ii) $\Rightarrow$ i) If there exists a $dQ = ZdP$ with the stated properties, Z belongs to $\mathcal{C}_1^0$ by Lemma 2.3. Set $y = \frac{1}{\text{ess inf } Z}$ and $\mu = \frac{\text{ess sup } Z}{\text{ess inf } Z}$ so that $1 \leq yZ \leq \mu$. Fenchel inequality gives $U_\mu(B+K) - (K+B)yZ \leq V_\mu(yZ) = 0$. Taking expectations, $E[U_\mu(K)] \leq 0$ for all K, which gives $u_\mu \leq 0$ and $\alpha^*(B) \leq \mu$.

The duality formula (5) has been implicitly proved in the above lines.

b) Case P is a martingale measure and $m := E[B] \leq 0$. Then, $E[B+K] = m$ for all K. Note $(B+K)^-$ is always non null. Otherwise $B \geq -K = K'$ and since B is not replicable, such inequality would imply $E[B] > E[K'] = 0$ which is absurd. So,

$$\alpha^*(B) = \sup_{K \in \mathcal{K}} \alpha(B+K) = 1 + \sup_K \frac{m}{E[(B+K)^-]}.$$

The functional to be optimized on the right is always 1 if $m = 0$ and otherwise always less than 1, which is easily seen to be the supremum. In fact, just pick any $K \neq 0$ and define $K_n = nK$. Then, $E[|K_n|] \uparrow +\infty$, which here is equivalent to $E[K_n^+] \uparrow +\infty$ *and* $E[K_n^-] \uparrow \infty$, so that $E[(B+K_n)^-] \to +\infty$.

□

Remark 2.12. The results just found constitute the basis for a strong objection against gain-loss ratio. From case a) above, best gain-loss with a claim is finite only if the claim belongs to $\cup_{Q \in \mathcal{M}_\infty} \{B \in L^1(Q) \mid E_Q[B] = 0\}$. Moreover, if P is already a pricing kernel a) and b) together imply that either the best gain-loss is infinite if B has positive expectation or it is optimal to take infinite risk so to off-set the negative expectation of B, which does not make economic sense.

From a strict mathematical viewpoint, there is quite a difference from what happens in standard concave optimization. There, if P is a martingale measure and $B = m$ is constant, the optimal solution is simply not to invest in the market. This is due to risk aversion and mathematically it is a consequence of Jensen's inequality:

$$E[U(m+K)] \leq U(m+E[K]) = U(m).$$

On the contrary, when $m < 0$, while $\alpha(m) = 0$ the market optimized $\alpha^*(m) = 1$.

3 Best gain loss is an acceptability index

3.1 From the family of risks associated to α to the market modified family of risks

Cherny and Madan introduced the concept of acceptability index for bounded claims from an analysis of the properties a performance measure should possess. Let us recall their definition, here restated in the more general case of maps defined on L^p.

DEFINITION 3.1 *An acceptability index β on L^p is a nonnegative map satisfying: quasi-concavity; (non decreasing) monotonicity; scale invariance; and the Fatou property, i.e. if $(X_n)_n$ is a dominated sequence in L^p and $X_n \to X$ a.s. then*

$$\beta(X) \geq \limsup \beta(X_n).$$

As shown by [10], an acceptability index can be characterized via an associated family of coherent risks $(\rho_x)_x$, indexed on a positive real interval and increasing in x:

$$\beta(X) = \sup\{x \mid \rho_x(X) \leq 0\}.$$

The gain loss ratio α is easily seen to possess all the properties but global quasi concavity, see [10] for the proofs where the authors define gain-loss ratio for X as $E[X]/E[X^-]$, i.e. our $\alpha(X) - 1$. Even if it is not quasi concave on L^1, gain loss ratio becomes quasi concave if restricted to the positions with nonnegative expectation. In other terms,

$$\widetilde{\alpha}(X) := \begin{cases} \alpha(X) & \text{if } E[X] > 0 \\ 1 & \text{otherwise,} \end{cases}$$

is an acceptability index. Therefore

$$\widetilde{\alpha}(X) = \sup\{\lambda \geq 1 \mid \rho_\lambda(X) \leq 0\},$$

where

$$\rho_\lambda(X) := \inf\{m \mid X + m \in \mathcal{A}_\lambda\},$$

in which the acceptance set is defined as the convex cone of claims with nonnegative U_λ-expected utility:

$$\mathcal{A}_\lambda = \{X \in L^1 \mid E[U_\lambda(X)] \geq 0\}.$$

For the sake of completeness, note that modulo a sign ρ_λ is a divergence utility in the terminology of Cherny and Kupper [9] (see also [13] for more background on this concept, specifically for functionals defined on L^p).

The dual representation of ρ_λ is given in the next Proposition. The proof follows standard duality arguments and is omitted. For the expression of the optimal Q^* and the characterization of the probability p^* see [10, Proposition 4].

PROPOSITION 3.2 *ρ_λ defines a coherent, law invariant, risk measure on L^1. The dual representation of ρ_λ is*

$$\rho_\lambda(X) = \max_{Q \in \mathcal{Q}_\lambda} E_Q[-X]$$

where $\mathcal{Q}_\lambda = \{Q \text{ probab.} \mid \exists y > 0, E[V_\lambda(y\frac{dQ}{dP})] < +\infty\}$. The maximizer is unique and it is of the form

$$\frac{dQ^*}{dP} = \frac{1 + (\lambda - 1)\mathbb{1}_{\{X \le q_{p^*}\}}}{1 + (\lambda - 1)p^*}$$

where q_{p^*} indicates the p^*-quantile of the distribution of X.

Let ρ indicate the agent's static risk measure and let $\mathcal{A}_\rho$ be the corresponding (maximal) acceptance set. She can invest in order to reduce his/her risk exposure and the resulting optimized risk measure is the so-called market modified risk measure. It was first introduced by [8] and then analyzed in detail by Barrieu-El Karoui [1, Section 3.1.3] for risk measures on L^∞:

$$\rho^M(X) := \inf\{m \mid \exists \xi \in \Xi, \text{s.t.}\, (m + \xi \cdot S_T + X) \in \mathcal{A}_\rho\}.$$

We briefly recall here that this construction is a generalization of the classical super replication price. In fact, suppose the agent perception of risk is quantified by the most severe risk measure ν, i.e. the acceptance set $\mathcal{A}_\nu$ is the convex cone of non-negative claims. Suppose the agent has sold the option B for the maturity T and has thus an exposure of $-B$ at that time. Then $\nu^M(-B) = \inf\{m \mid \exists \xi \in \Xi, \text{s.t.}\, (m + \xi \cdot S_T - B) \ge 0\}$ is simply the super replication price of B, given the market strategies ξ. Recall also that, in full generality,

$$\nu^M(X) = \sup_{Q \in \mathcal{M}} E_Q[-X],$$

that is the penalty function of ν^M is, modulo a sign, the support function of the set of martingale probabilities $\mathcal{M}$. Since our primal space is L^1 and the duality applied (L^1, L^∞), $\mathcal{M}$ here will be the set of martingale probabilities with bounded density.

We are interested in the representation of the market modified risk $\rho_\lambda^M(X) := \inf\{m \mid \exists \xi \in \Xi, \text{s.t.}\, E[U_\lambda(m + \xi \cdot S_T + X)] \ge 0\}$.

PROPOSITION 3.3 *Let the market be λ gain-loss free. Then, the market modified risk ρ_λ^M is finite on L^1 and it has the dual representation:*

$$\rho_\lambda^M(X) = \max_{Q \in \mathcal{Q}_\lambda \cap \mathcal{M}} E_Q[-X] \tag{6}$$

where $\mathcal{M}$ are the all the martingale probabilities, i.e. modulo a sign it is the support function of the set $\mathcal{Q}_\lambda \cap \mathcal{M}$.

Proof. (A sketch). Step 1: ρ_λ^M is well defined and finite over L^1. For a fixed integrable r.v. Y the function $m \to E[U_\lambda(m+Y)]$ is continuous, increasing and has infinite limits when $x \to \pm\infty$, hence there is a unique m s.t. $E[U_\lambda(m+Y)] = 0$. Therefore, the risk measure can be re-written:

$$\rho_\lambda^M(X) = \inf\{m \mid \exists \xi \in \mathcal{H} \,\text{s.t.}\, E[U_\lambda(m + \xi \cdot S_T + X)] = 0\}.$$

Now, pick a martingale measure $Q \in \mathcal{Q}_\lambda$, which exists by Theorem 2.4 and let Z be its density. Fix any couple (m, ξ) with $E[U_\lambda(m + \xi \cdot S_T + X)] = 0$. Fenchel inequality gives

$$U_\lambda(m + \xi \cdot S_T + X) - \frac{Z}{\text{ess inf } Z}(m + \xi \cdot S_T + X) \leq V_\lambda(\frac{Z}{\text{ess inf } Z}) = 0.$$

Taking expectations and rearranging

$$+\infty > m \geq E_Q[-X] > -\infty$$

Step 2: regularity. The Extended Namioka Theorem for convex monotone functionals applies, in the version stated by [16] or [5]. ρ_λ^M is automatically norm continuous on L^1 and subdifferentiable. This, together with normalization and translation invariance properties, gives a dual representation over probabilities with density in L^∞ as

$$\rho_\lambda^M(X) = \max_{Q \in \mathcal{Q}^*} E_Q[-X].$$

As all risk measures on L^1, it has the Fatou property, see [5].

Step 3: identification of $\mathcal{Q}^*$. The line is identical to [1]. ρ_λ^M is obtained by taking the so-called inf-convolution of ρ_λ and of $\delta_\mathcal{K}$, the functional indicator of the set of replicable claims:

$$\rho_\lambda^M(X) = \inf_{\xi \in \Xi} \rho_\lambda(\xi \cdot S_T + X) = \inf_{X' \in L^1}(\rho_\lambda(X - X') + \delta_\mathcal{K}(X'))$$

where the last equality holds since $\mathcal{K}$ is a vector space. Now, recalling that ρ_{worst} is a neutral element in the inf-convolution $\Box$, and that this operation is associative,

$$\rho_\lambda^M = (\rho_\lambda \Box \delta_\mathcal{K}) \Box \rho_{worst} = \rho_\lambda \Box (\delta_\mathcal{K} \Box \rho_{worst}) = \rho_\lambda \Box \nu^M,$$

that is, ρ_λ^M is the inf-convolution of ρ_λ and the market risk measure ν^M. Therefore, its minimal penalty function is the sum of the two respective minimal penalty functions, that is the sum of the two deltas, $\delta_{\mathcal{Q}_\lambda}$ and $\delta_\mathcal{M}$, which amounts to $\delta_{\mathcal{Q}_\lambda \cap \mathcal{M}}$ (the proofs in [1, Section 3] for bounded risks go through in the same way).

Finally a combination of the results in item 1 and 2 above gives the desired relation (6). □

Remark 3.4. A market modified risk is a more tolerant notion of hedging cost, since it is based on a more tolerant hedging. However, with gain-loss the minimal hedging strategy mail fail to exist as $\rho^M_\lambda(X)$ may be only an infimum and not a minimum. For the same reasons pointed out in Remark 2.3, minimizing sequences of integrals, though L^1 bounded, may fail to converge.

3.2 α^* is an acceptability index

Our final result is an extension of [10, Proposition] in the presence of a market: α^* is an acceptability index, though not a law invariant one as its value does depend at least on the joint distribution of the claim B and S.

PROPOSITION 3.5 *Suppose* $\mathcal{M}_\infty \neq \emptyset$. *With the convention* $\sup \emptyset = 1$, $\alpha^* : L^1 \to [1, +\infty]$ *admits a representation in terms of the family of market modified risks* $(\rho^M_\lambda)_{\lambda \geq 1}$*:*

$$\alpha^*(B) = \sup\{\lambda \geq 1 \mid \rho^M_\lambda(B) \leq 0\} = \sup\{\lambda \geq 1 \mid \min_{\mathcal{Q}_\lambda \cap \mathcal{M}} E_Q[B] \geq 0\}$$

and is thus an acceptability index. When $\alpha^* < +\infty$, *the supremum is attained. Moreover, the set of probabilities* $(\mathcal{Q}_\lambda \cap \mathcal{M})_{\lambda \geq 1}$ *is the set of supporting kernels of* α^*, *i.e. the maximal one.*

Proof. Call $\Lambda = \{\lambda \mid E_Q[B] \geq 0 \ \forall Q \in \mathcal{Q}_\lambda \cap \mathcal{M}\}$. We show first $\alpha^*(B) \geq \sup \Lambda$. If Λ is empty, the inequality is true by definition since as shown in Theorem 2.11 $\alpha^*(B) \geq 1 = \sup \emptyset$ always. Otherwise, select $\lambda \in \Lambda$. By Fenchel Duality Theorem once again,

$$\sup_K E[U_\lambda(B+K)] = \min_{y>0, Q \in \mathcal{M}_\infty} (y E_Q[B] + E[V_\lambda(y \frac{dQ}{dP})]) \geq 0,$$

which implies $\alpha^*(B) \geq \lambda$.

Suppose now $\sup \Lambda = \lambda^* < +\infty$ (otherwise there is nothing more to prove). In such case, Fenchel Duality Theorem applied with U_b, with any fixed $b > \lambda^*$, gives $\sup_K E[U_b(B+K)] < 0$ as there is a $Q \in \mathcal{Q}_b \cap \mathcal{M}$ with $E_Q[B] < 0$, which implies $\alpha^*(B) \leq b$, i.e. $\alpha^*(B) = \lambda^*$. That λ^* belongs to Λ follows from the proof of Theorem 2.11, arrow i) $\to$ ii). This completes the first part of proof.

The family $(\mathcal{Q}_\lambda \cap \mathcal{M})_{\lambda \geq 1}$ of probabilities satisfies a closure property, which ensures maximality exactly as in [10]. Suppose in fact

$$\alpha^*(B) = \sup\{\lambda \geq 1 \mid \min_{\mathcal{Q}_\lambda \cap \mathcal{M}} E_Q[B] \geq 0\} = \sup\{\lambda \geq 1 \mid \min_{\mathcal{P}_\lambda} E_Q[B] \geq 0\}$$

for some other larger family of probabilities $(\mathcal{P}_\lambda)_\lambda$. Suppose for some λ we have $\mathcal{Q}_\lambda \cap \mathcal{M} \subsetneq \mathcal{P}_\lambda$ and fix $Q^0 \in \mathcal{P}_\lambda \setminus \mathcal{Q}_\lambda \cap \mathcal{M}$. The set

$$\mathcal{W} := \{Y \mid Y = \frac{Z}{\text{ess inf } Z} \text{ for some } Z \in \mathcal{Q}_\lambda \cap \mathcal{M}\}$$

is closed in $L^1(P)$ and $Q^0 \notin \mathcal{W}$. An application of Hahn-Banach Theorem gives a $B \in L^\infty$ such that

$$E_{Q^0}[B] < 0, E[YB] > 0 \quad \forall Y \in \mathcal{W} \text{ or, equivalently, } E_Q[B] > 0 \; \forall Q \in \mathcal{Q}_\lambda \cap \mathcal{M}$$

which at the same time gives $\alpha^*(B) < \lambda$ and $\alpha^*(B) \geq \lambda$, a contradiction. □

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