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Endogenous restricted participation in general financial equilibrium: existence results

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Abstract

We consider an incomplete market model with numeraire assets. Each household faces an individual constraint on its participation in the asset market. In related literature, the constraint is described by a function whose sole argument is the asset portfolio. On the contrary, in our analysis the constraint depends not only on the asset portfolio, but also on asset and good prices - hence the reference to *endogenous* (in contrast to *exogenous*) in the title. We also analyze the case in which some household is excluded from the trade of some asset. Existence results are provided using a homotopy argument.

JEL classification: D50; D52.

Keywords: General equilibrium; Restricted participation; Financial markets; Existence of equilibria.

1 Introduction

This paper provides existence results for a pure exchange general equilibrium model with incomplete markets and endogenous restricted participation. We consider the same model presented in Carosi, Gori and Villanacci (2009)[1]. Each household has only partial access, in a personalized manner to the available set of assets. In related literature, the constraint is described by a function whose sole argument is the asset portfolio. On the contrary, in our analysis the constraint depends not only on the asset portfolio, but also on asset and good prices - hence the reference to *endogenous* (in contrast to *exogenous*) in the title. Each economies is described by endowments of commodities, utility functions, asset yield matrices, and restriction functions.

Assumptions on restriction functions are a natural generalizations to the endogenous setting of those used in Cass, Siconolfi and Villanacci (2001) in the exogenous case. It turns out that, in both mentioned frameworks, those assumptions imply that the admissible portfolio set has nonempty interior - see Lemma 5 below. Therefore, they do not allow to cover the case in which some household is excluded from the trade of some asset. Our proposed technique of proof can easily accomodate that economically meaningful situation.

The paper is organized as follows. In Section 2, we present the setup of the model. In Section 3, we show existence of equilibria in the case of restriction functions satisfying the proposed, standard set of assumptions; an homotopy argument is used. Finally, in Section 4, we show existence in the case of some household asset demand for some asset being forced to be zero.

2 The model

Our model is the by now very standard two-period, pure exchange economy with uncertainty and both commodities and assets. Spot commodity markets open in the first and second periods, and

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[1]See that paper for a discussion on restricted participation models.

there are $C \geq 2$ types of commodities traded at each spot, denoted by $c \in \mathcal{C} = \{1, 2, ..., C\}$. Asset markets open in just the first period, and there are $A \geq 1$ (inside) assets traded, denoted by $a \in \mathcal{A} = \{1, 2, ..., A\}$. We will also denote spots by $s \in \mathcal{S} = \{0, 1, ..., S\}$, $S \geq 1$, where $s = 0$ corresponds to the first period, today, and $s \geq 1$ the possible states of the world in the second period, tomorrow. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} = \{1, 2, ..., H\}$.

The time line for this model is as follows: today, households exchange commodities and assets, and consumption takes place. Then, tomorrow, uncertainty is resolved, households make good on their liabilities, and households again exchange and then consume commodities.

$x_h^c(s)$ is the consumption of commodity c in state s by household h, with parallel notation for the endowment of commodities, $e_h^c(s)$. Both consumption $x_h = (x_h^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ and endowment $e_h = (e_h^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ are elements of $\mathbb{R}^G_{++}$, where $G = (S+1)C$ is the total number of goods. Household h's preferences are represented by a utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$. As in most of the literature on smooth economies we will adopt throughout

Assumption u. For all $h \in \mathcal{H}$,

u1. $u_h \in C^2(\mathbb{R}^G_{++})$;

u2. u_h is differentiably strictly increasing, that is, $Du_h(x_h) \gg 0$;

u3. u_h is differentiably strictly quasi-concave, i.e.,

$$\Delta x \neq 0 \text{ and } Du_h(x_h)\,\Delta x = 0 \quad \Rightarrow \quad \Delta x^T D^2 u_h(x_h)\,\Delta x < 0;$$

u4. u_h has upper contour sets closed in the standard topology of $\mathbb{R}^G$, that is, for any $\underline{x} \in \mathbb{R}^G_{++}$, $\left\{x \in \mathbb{R}^G_{++} : u_h(x) \geq u_h(\underline{x})\right\}$ is closed in the topology of $\mathbb{R}^G$.

The set of utility functions satisfying Assumption u is denoted by $\overline{\mathcal{U}}$. Define also $\mathcal{U} = \overline{\mathcal{U}}^H$.
We will also use the following standard notation: $p^c(s)$ is the price of commodity c at spot s and $p = (p^c(s), c \in \mathcal{C}, s \in \mathcal{S})$ is the corresponding commodity price vector; q^a is the price of asset a and $q = (q^a, a \in \mathcal{A})$ is the corresponding asset price vector; $y^a(s)$ is the yield in state s of asset a in *units of the numeraire commodity*, which, for specificity, we designate to be C, and

$$Y = \begin{bmatrix} y^1(1) & \dots & y^a(1) & \dots & y^A(1) \\ \vdots & & \vdots & & \vdots \\ y^1(s) & \dots & y^a(s) & \dots & y^A(s) \\ \vdots & & \vdots & & \vdots \\ y^1(S) & \dots & y^a(S) & \dots & y^A(S) \end{bmatrix}$$

is the corresponding yield matrix; $y(s) = (y^a(s), a \in \mathcal{A})$ is the vector of asset yields in state s; z_h^a is the quantity of asset a held by household h, $z_h = (z_h^a, a \in \mathcal{A})$ is the corresponding asset portfolio and $z = (z_h, h \in \mathcal{H}) \in \mathbb{R}^{AH}$.

Concerning the financial side of the economy, and consistently with our restricted participation framework, we assume that

- there exists a given set of assets which, in number and kind, may even be sufficient for complete markets,
- each household h has only partial access, in a personalized manner to the available set of assets.

In other words, while there may be just a “few” or “many” assets, the market imperfection we consider is not incompleteness of numbers of assets, but rather restrictions on households' opportunities for transacting in assets.
It greatly simplifies our analysis (but, for the reason just mentioned, is not without loss of generality) to assume that

Assumption Y. rank $Y = A \leq S$.

Let $\mathcal{Y}$ be the set of $S \times A$ matrices satisfying the above assumption.

There are $J \geq 1$ potential participation constraints for each household. Let $\mathcal{J} = \{1, ..., J\}$ with generic element j. Then, the restriction function for household h is

$$\begin{aligned} r_h &: \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^J \\ (z_h, p, q) &\mapsto r_h(z_h, p, q) = \left(r_h^j(z_h, p, q), j \in \mathcal{J} \right) \end{aligned}$$

For each nonempty subset $\mathcal{J}_h \subseteq \mathcal{J}$, denote its cardinality by J_h, and let

$$\begin{aligned} r_h^{\mathcal{J}_h} &: \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^{J_h} \\ (z_h, p, q) &\mapsto \left(r_h^j(z_h, p, q), j \in \mathcal{J}_h \right) \end{aligned}$$

We now introduce assumptions on restriction functions.

Assumption r.

r1. For all $h \in \mathcal{H}$, r_h is $C^2(\mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A; \mathbb{R}^J)$;

r2. For all $h \in \mathcal{H}$, $j \in \mathcal{J}$, $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, r_h^j is quasi-concave in z_h;

r3. For all $h \in \mathcal{H}$, $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $r_h(0, p, q) \geq 0$;

r4. For all $h \in \mathcal{H}$, $(z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A$, $\mathcal{J}_h \subseteq \mathcal{J}$ such that $\mathcal{J}_h \neq \varnothing$,

$$r_h^{\mathcal{J}_h}(z_h, p, q) = 0 \quad \Rightarrow \quad \text{rank } D_{z_h} r_h^{\mathcal{J}_h}(z_h, p, q) = J_h;$$

r5. For all $a \in \mathcal{A}$, there exists $h \in \mathcal{H}$ such that, for every $(z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^A$,

$$D_{z_h^a} r_h(z_h, p, q) = 0.$$

Let $\mathcal{R}$ be the set of restriction functions satisfying Assumptions r1-r5 above, with generic element $r = (r_h)_{h=1}^H$. An economy is $E = (e, u, Y, r) \in \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{R} = \mathcal{E}$.
For given $(p, q, E) \in \mathbb{R}^G_{++} \times \mathbb{R}^A \times \mathcal{E}$, household $h \in \mathcal{H}$ maximization problem is as follows.

Problem (Ph)

$$\max_{(x_h, z_h)} \ u_h(x_h) \quad s.t.$$

$$\begin{aligned} & p(0) x_h(0) + q z_h \leq p(0) e_h(0) \\ & p(s) x_h(s) - p^C(s) y(s) z_h \leq p(s) e_h(s), \quad s \in \{1, ..., S\} \\ & r_h(z_h, p, q) \geq 0 \end{aligned} \tag{1}$$

Observe that normalizations of spot by spot prices are not possible because of the dependence of the restriction functions on (p, q). In fact nominal changes of prices may in general affect the constraint set of some household maximization problems. Therefore the appropriate definition of equilibrium is as follows.

Definition 1 $\left((x_h, z_h)_{h \in \mathcal{H}}, p, q\right) \in \left(\mathbb{R}^G_{++} \times \mathbb{R}^A\right)^H \times \mathbb{R}^G_{++} \times \mathbb{R}^A = \Theta$ *is an equilibrium for the economy* $E \in \mathcal{E}$ *if for each* h, (x_h, z_h) *solves Problem* (Ph) *at* (p, q, E) *and* (x, z) *solves market clearing conditions at* e

$$\begin{aligned} \sum_{h=1}^H (x_h - e_h) = 0 \\ \sum_{h=1}^H z_h = 0 \end{aligned} \tag{2}$$

In the following, for every $E \in \mathcal{E}$, we denote by $\Theta(E) \subseteq \Theta$ the set of equilibria for the economy E and by $\Theta_n(E)$ the set of *normalized equilibria*, that is,

$$\Theta_n(E) = \left\{ \left((x_h, z_h)_{h \in \mathcal{H}}, p, q\right) \in \Theta(E) : \forall s \in \mathcal{S}, p^C(s) = 1 \right\}.$$

3 Existence of normalized equilibria

In this section we prove the following existence theorem.

Theorem 2 *For every $E \in \mathcal{E}$, $\Theta_n(E) \neq \varnothing$.*

We are going to prove such a result via the system of equations of Kuhn-Tucker conditions associated with households maximizations problems, and market clearing conditions.

First of all note that in the considered model, $S+1$ Walras' laws do hold. If we define $x_h^{\backslash}(s) = (x_h^c(s), c \neq C)$, $x_h^{\backslash} = \left(x_h^{\backslash}(s), s \in \mathcal{S}\right)$ and similarly $e_h^{\backslash}(s) = (e_h^c(s), c \neq C)$, $e_h^{\backslash} = \left(e_h^{\backslash}(s), s \in \mathcal{S}\right)$, then we can write the significant market clearing conditions at e as

$$\begin{array}{l} \sum_{h=1}^{H}\left(x_h^{\backslash} - e_h^{\backslash}\right) = 0, \\ \sum_{h=1}^{H} z_h = 0. \end{array} \tag{3}$$

Define also

$$p^{\backslash}(s) = \left(p^c(s) : c \neq C\right), \quad p^{\backslash} = \left(p^{\backslash}(s), s \in \mathcal{S}\right), \quad \overline{p}(s) = \left(p^{\backslash}(s), 1\right), \quad \overline{p} = \left(\overline{p}(s), s \in \mathcal{S}\right),$$

and

$$\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{AH} \times \mathbb{R}_{++}^{(S+1)H} \times \mathbb{R}^{JH} \times \mathbb{R}_{++}^{G-(S+1)} \times \mathbb{R}^{A},$$

with generic element

$$\xi = \left(x, z, \lambda, \mu, p^{\backslash}, q\right).$$

Let us consider now $E \in \mathcal{E}$. It is immediate to prove that if

$$\left((x_h, z_h)_{h \in \mathcal{H}}, \overline{p}, q\right) \in \Theta_n(E)$$

then there exists $(\lambda_h, \mu_h)_{h \in \mathcal{H}} = (\lambda, \mu) \in \mathbb{R}_{++}^{(S+1)H} \times \mathbb{R}^{JH}$ such that

$$\xi = \left((x_h, z_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^{\backslash}, q\right)$$

solves the system $F_E(\xi) = 0$ where

$$F_E : \Xi \to \mathbb{R}^{\dim \Xi},$$

$$F_E\left(x, z, \lambda, \mu, p^{\backslash}, q\right) =$$

$$\left[\begin{array}{ll}
\underset{h \in \mathcal{H}, s \in \mathcal{S}}{(h.1.s)} & D_{x_h(s)} u_h(x_h) - \lambda_h(s)\,\overline{p}(s) \\
\underset{h \in \mathcal{H}}{(h.2.0)} & -\overline{p}(0)\left(x_h(0) - e_h(0)\right) - q z_h \\
\underset{h \in \mathcal{H}, s \in \mathcal{S}\backslash\{0\}}{(h.2.s)} & -\overline{p}(s)\left(x_h(s) - e_h(s)\right) + y(s)\, z_h \\
\underset{h \in \mathcal{H}, a \in \mathcal{A}}{(h.3.a)} & -\lambda_h(0)\, q^a + \sum_{s=1}^{S} \lambda_h(s)\, y^a(s) + \sum_{j=1}^{J} \mu_h^j D_{z_h^a} r_h^j(z_h, \overline{p}, q) \\
\underset{h \in \mathcal{H}, j \in \mathcal{J}}{(h.4.j)} & \min\left\{\mu_h^j, r_h^j(z_h, \overline{p}, q)\right\} \\
(M.x) & \sum_{h=1}^{H}\left(x_h^{\backslash} - e_h^{\backslash}\right) \\
(M.z) & \sum_{h=1}^{H} z_h
\end{array}\right] \tag{4}$$

while if

$$\xi = \left((x_h, z_h, \lambda_h, \mu_h)_{h \in \mathcal{H}}, p^{\backslash}, q\right)$$

solves the system $F_E(\xi) = 0$, then

$$\left((x_h, z_h)_{h \in \mathcal{H}}, \overline{p}, q\right) \in \Theta_n(E).$$

The above discussion implies that Theorem 2 is a consequence of the following result.

Theorem 3 *For every $E \in \mathcal{E}$, there exists $\xi \in \Xi$ such that $F_E(\xi) = 0$.*

We are going to prove Theorem 3 applying the following well known result[2].

Theorem 4 *Let M and N be two C^2 boundaryless manifolds of the same dimension, $y \in N$ and $F, G : M \to N$ be continuous functions. Assume that G is C^1 in an open neighborhood U of $G^{-1}(y)$, y is a regular value for G restricted to U, $\#G^{-1}(y)$ is odd and there exists a continuous homotopy $H : M \times [0,1] \to N$ from F to G such that $H^{-1}(y)$ is compact. Then $F^{-1}(y) \neq \varnothing$.*

We need also the following proposition.

Proposition 5 *For every $h \in \mathcal{H}$ there exists a continuous function $\widetilde{z}_h : \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^A$ such that, for every $(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $r_h(\widetilde{z}_h(p,q), p, q) \gg 0$.*

Proof. Let us fix $h \in \mathcal{H}$ and define the correspondence $\varphi_h : \mathbb{R}^G_{++} \times \mathbb{R}^A \rightrightarrows \mathbb{R}^A$ as

$$\varphi_h(p,q) = \left\{ z_h \in \mathbb{R}^A : r_h(z_h, p, q) \gg 0 \right\} = \bigcap_{j=1}^{J} \left\{ z_h \in \mathbb{R}^A : r_h^j(z_h, p, q) > 0 \right\}.$$

By Assumption r2, for every $j \in \mathcal{J}$, the set $\left\{ z_h \in \mathbb{R}^A : r_h^j(z_h, p, q) > 0 \right\}$ is convex and then φ_h is convex valued. We claim that, for every $(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $\varphi_h(p,q) \neq \varnothing$. In fact fix $(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$. We know that $r_h(0,p,q) \geq 0$ and then there exist a partition $\left\{ \mathcal{J}_h^0, \mathcal{J}_h^1 \right\}$ of $\mathcal{J}$ such that

$$\begin{array}{ll} r_h^j(0,p,q) = 0, & \forall j \in \mathcal{J}_h^0, \\ r_h^j(0,p,q) > 0, & \forall j \in \mathcal{J}_h^1. \end{array}$$

By continuity of $r_h^{\mathcal{J}_h^1}$ there exists an open neighborhood $V(0) \subseteq \mathbb{R}^A$ of 0 such that for every $z_h \in V(0)$, $r_h^{\mathcal{J}_h^1}(z_h, p, q) \gg 0$. Moreover by Assumption r4 we know that $\mathrm{rank} D_{z_h} r_h^{\mathcal{J}_h^0}(0,p,q) = J_h^0$ and then we can find $z_h^* \in V(0)$ such that $r_h^{\mathcal{J}_h^0}(z_h^*, p, q) \gg 0$. Then $r_h(z_h^*, p, q) \gg 0$ and $\varphi_h(p,q) \neq \varnothing$. Finally by Assumption r1 we have that, for every $z_h \in \mathbb{R}^A$,

$$\varphi_h^{-1}(z_h) = \{(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A : z_h \in \varphi_h(p,q)\} = \{(p,q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A : r_h(z_h, p, q) \gg 0\}$$

is open in $\mathbb{R}^G_{++} \times \mathbb{R}^A$. Then the desired result follows from Proposition 1.5.1, page 29, in Florenzano (2003). ■

Proof of Theorem 3. Let $E = (e, u, Y, r) \in \mathcal{E}$ be fixed. Then it is well known that there exists a Pareto optimal allocation x^* for u such that $\sum_{h=1}^{H} x_h^* = \sum_{h=1}^{H} e_h$. Moreover, there exists $(\chi^*, \gamma^*) \in \mathbb{R}^H_{++} \times \mathbb{R}^G_{++}$ such that (x^*, χ^*, γ^*) is the unique solution to the following system

$$\left\{ \begin{array}{ll} \chi_1^* - 1 & = 0 \\ \chi_h^* D u_h(x_h) - \gamma^* & = 0 \\ (u_h(x_h) - u_h(x_h^*))_{h \neq 1} & = 0 \\ -\sum_{h=1}^{H} (x_h - x_h^*) & = 0 \end{array} \right. \tag{5}$$

Set

$$F(\xi) = F_E(\xi), \quad \forall\, \xi \in \Xi,$$

and consider the system in the unknowns $\xi = (x, \lambda, z, \mu, p^\setminus, q) \in \Xi$ and $\tau \in [0,1]$, given by

[2] See Theorem 57, p. 199, in Villanacci et al. (2002).

$$\begin{cases}
\underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h(x_h) - \lambda_h(s)\,\overline{p}(s) & = 0 \\
\underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\left(x_h(0) - \left((1-\tau)\,e_h(0) + \tau x_h^*(0)\right)\right) - qz_h & = 0 \\
\underset{h\in\mathcal{H},s\in S\backslash\{0\}}{(h.2.s)} & -\overline{p}(s)\left(x_h(s) - \left((1-\tau)\,e_h(s) + \tau x_h^*(s)\right)\right) + y(s)\,z_h & = 0 \\
\underset{h\in\mathcal{H},a\in\mathcal{A}}{(h.3.a)} & -\lambda_h(0)\,q^a + \sum_{s=1}^{S}\lambda_h(s)\,y^a(s) + \sum_{j=1}^{J}\mu_h^j(1-\tau)\,D_{z_h^a}r_h^j\left((1-\tau)\,z_h + \tau\widetilde{z}_h(\overline{p},q),\overline{p},q\right) & = 0 \\
\underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \min\left\{\mu_h^j, r_h^j\left((1-\tau)\,z_h + \tau\widetilde{z}_h(\overline{p},q),\overline{p},q\right)\right\} & = 0 \\
(M.x) & \sum_{h=1}^{H}\left(x_h^{\backslash} - \left((1-\tau)\,e_h^{\backslash} + \tau x_h^{*\backslash}\right)\right) & = 0 \\
(M.z) & \sum_{h=1}^{H} z_h & = 0
\end{cases} \tag{6}$$

where, for every $h \in \mathcal{H}$, the function $\widetilde{z}_h$ is defined in Proposition 5.

Define now

$$H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}$$

$$(\xi,\tau) \mapsto \text{ left hand side of system (6)},$$

and

$$G : \Xi \to \mathbb{R}^{\dim \Xi}, \;\; \xi \mapsto H(\xi, 1)\,.$$

Observe that

$$H(\xi, 0) = F(\xi)\,.$$

Let us now verify that Theorem 4 can be applied. F and G are defined in the same open subset of $\mathbb{R}^{\dim \Xi}$, take values in $\mathbb{R}^{\dim \Xi}$ (and those sets are C^2 boundaryless manifolds of the same dimension) and are continuous.

Of course H is a continuous homotopy from F to G. Moreover, Lemmas 6, 7 and 8 prove the following needed results.

- $G^{-1}(0) = \{\xi^*\}$;
- G is C^1 in a neighborhood of ξ^* and $\operatorname{rank} D_\xi G(\xi^*) = \dim \Xi$;
- $H^{-1}(0)$ is compact.

From Theorem 4, it then follows, as desired, that $F^{-1}(0) \neq \varnothing$. ■

Lemma 6 *$G^{-1}(0) = \{\xi^*\} = \left\{(x^*, z^*, \lambda^*, \mu^*, p^{\backslash *}, q^*)\right\} \in \Xi$, where*

$$x_h^* = x_h^*, \quad \lambda_h^* = \left(\frac{\gamma^{*C}(s)}{\chi_h^*}, s \in \mathcal{S}\right), \quad z_h^* = 0, \quad \mu_h^* = 0, \quad \forall h \in \mathcal{H},$$

$$p^{*\backslash} = \left(\frac{\gamma^{*c}(s)}{\gamma^{*C}(s)}, s \in \mathcal{S}, c \neq C\right), \quad q^* = \sum_{s=1}^{S}\left(\frac{\gamma^{*C}(s)}{\gamma^{*C}(0)}\right) y(s)\,.$$

Proof. $G^{-1}(0)$ is the set of solutions of system (6) at $\tau = 1$, that is, the set of solutions of the system

$$\begin{cases} \underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h(x_h) - \lambda_h(s)\,\overline{p}(s) = 0 \\ \underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\left(x_h(0) - x_h^*(0)\right) - qz_h = 0 \\ \underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}(s)\left(x_h(s) - x_h^*(s)\right) + y(s)\,z_h = 0 \\ \underset{h\in\mathcal{H},a\in\mathcal{A}}{(h.3.a)} & -\lambda_h(0)\,q^a + \sum_{s=1}^{S}\lambda_h(s)\,y^a(s) = 0 \\ \underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \mu_h^j = 0 \\ (M.x) & \sum_{h=1}^{H}\left(x_h^{\setminus} - x_h^{*\setminus}\right) = 0 \\ (M.z) & \sum_{h=1}^{H} z_h = 0 \end{cases} \tag{7}$$

Using the definition of ξ^*, it is easy to check that $\xi^* \in G^{-1}(0)$. Define now $\widehat{\xi} = \left(\widehat{x}, \widehat{\lambda}, \widehat{z}, \widehat{\mu}, \widehat{p}^{\setminus}, \widehat{q}\right)$, assume $\widehat{\xi} \in G^{-1}(0)$ and prove $\widehat{\xi} = \xi^*$.
Claim 1. $\widehat{\mu} = \mu^*$. Obvious.
Claim 2. $\widehat{x} = x^*$. Suppose by contradiction $\widehat{x} \neq x^*$. Consider $\widetilde{x} = \frac{1}{2}(\widehat{x} + x^*)$. Of course it is

$$\sum_{h=1}^{H}\widetilde{x}_h = \frac{1}{2}\left(\sum_{h=1}^{H}\widehat{x}_h + \sum_{h=1}^{H}x_h^*\right) = \sum_{h=1}^{H}x_h^*. \tag{8}$$

Since (x_h^*, z_h^*) is feasible for the maximization problem of the h-th household, whose first order conditions are given by equations $(h.1.s)$, $(h.2.0)$, $(h.2.s)$ and $(h.3.a)$ in (7), it is $u_h(\widehat{x}_h) \geq u_h(x_h^*)$. Moreover, from Assumption u3, we have

$$u_h(\widetilde{x}_h) > \min\{u_h(x_h^*), u_h(\widehat{x}_h)\} = u_h(x_h^*) \quad \forall h \in \mathcal{H}. \tag{9}$$

But (8) and (9) contradict the Pareto optimality of x^*.
Claim 3. $\widehat{\lambda} = \lambda^*$. Since $\widehat{x} = x^*$, from $(h.1.s)$ in (7) we have in particular

$$\widehat{\lambda}_h(s) = D_{x_h^C(s)}u_h(\widehat{x}_h) = D_{x_h^C(s)}u_h(x_h^*) = \lambda_h^*(s), \quad \forall h \in \mathcal{H}, s \in \mathcal{S}.$$

Claim 4. $\widehat{p}^{\setminus} = p^{\setminus *}$. ¿From $(h.1.s)$ in (7) and Claims 2 and 3,

$$\widehat{\overline{p}}(s) = \frac{D_{x_h(s)}u_h(\widehat{x}_h)}{\widehat{\lambda}_h(s)} = \frac{D_{x_h(s)}u_h(x_h^*)}{\lambda_h^*(s)} = \overline{p}^*(s) \quad \forall s \in \mathcal{S}.$$

Claim 5. $\widehat{z} = z^*$. From $(h.2.s)$ $s \geq 1$, in (7) and Claim 2, we have in particular,

$$Y\widehat{z}_h = 0, \quad \forall h \in \mathcal{H}.$$

From Assumption Y, we get that for any $h \in \mathcal{H}$, $\widehat{z}_h = 0 = z_h^*$.
Claim 6. $\widehat{q} = q^*$. From $(h.3.a)$ in (7) and Claims 1 and 3 we get

$$\widehat{q} = \sum_{s=1}^{S}\frac{\widehat{\lambda}_h(s)}{\widehat{\lambda}_h(0)}y(s) = \sum_{s=1}^{S}\frac{\lambda_h^*(s)}{\lambda_h^*(0)}y(s) = q^*,$$

and the proof is completed. ■

Lemma 7 *G is C^1 in a neighborhood of ξ^* and* $\operatorname{rank} D_\xi G(\xi^*) = \dim \Xi$.

Proof. It is immediate to prove that in a suitable small neighborhood of ξ^* we have is

$$G(\xi) = \begin{bmatrix} \underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h(x_h) - \lambda_h(s)\,\overline{p}(s) \\ \underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\,(x_h(0) - x_h^*(0)) - qz_h \\ \underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}(s)\,(x_h(s) - x_h^*(s)) + y(s)\,z_h \\ \underset{h\in\mathcal{H},a\in\mathcal{A}}{(h.3.a)} & -\lambda_h(0)\,q^a + \sum_{s=1}^{S}\lambda_h(s)\,y^a(s) \\ \underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \mu_h^j \\ (M.x) & \sum_{h=1}^{H}\left(x_h^{\setminus} - x_h^{*\setminus}\right) \\ (M.z) & \sum_{h=1}^{H} z_h \end{bmatrix}$$

and this is a C^1 function. The computation of $D_\xi G(\xi^*)$ is described below

	x_h	λ_h	z_h	μ_h	$p^{\setminus}$	q
$(h.1)$	$D^2u_h(x_h^*)$	$-\Phi(\overline{p}^*)^T$			$-\Lambda_h^*$	
$(h.2)$	$-\Phi(\overline{p}^*)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}$			
$(h.3)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}^T$				$-\lambda_h^*(0)\,I$
$(h.4)$				I		
$(M.x)$	$\widehat{I}$					
$(M.z)$			I			

where

$$\Phi(\overline{p}) = \begin{bmatrix} p^1(0) & \dots & p^{C-1}(0) & 1 & & & \\ & & & \ddots & & & \\ & & & & p^1(S) & \dots & p^{C-1}(S) & 1 \end{bmatrix}_{(S+1)\times G} \tag{10}$$

$$\widehat{I} = \begin{bmatrix} I_{(C-1)\times(C-1)}0 & & \\ & \ddots & \\ & & I_{(C-1)\times(C-1)}0 \end{bmatrix}_{[G-(S+1)]\times G} \tag{11}$$

and

$$\Lambda_h^* = \begin{bmatrix} \begin{matrix}\lambda_h^*(0)\,I_{C-1} \\ 0\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}\lambda_h^*(S)\,I_{C-1} \\ 0\end{matrix} \end{bmatrix} = \frac{1}{\chi_h^*}\begin{bmatrix} \begin{matrix}\gamma^{*C}(0)\,I_{C-1} \\ 0\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}\gamma^{*C}(S)\,I_{C-1} \\ 0\end{matrix} \end{bmatrix} = \frac{1}{\chi_h^*}\Gamma^*, \tag{12}$$

where Λ_h^* and Γ^* have G rows and $G-(S+1)$ columns and Γ^* does not depend on h.

Then the above matrix has full rank if and only if the following does:

	x_h	λ_h	z_h	$p^{\setminus}$	q
$(h.1)$	$D^2u_h(x_h^*)$	$-\Phi(\overline{p}^*)^T$		$-\Lambda_h^*$	
$(h.2)$	$-\Phi(\overline{p}^*)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}$		
$(h.3)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}^T$			$-\lambda_h^*(0)\,I$
$(M.x)$	$\widehat{I}$				
$(M.z)$			I		

Using a standard argument exploiting Assumption u3, the desired result follows[3]. ■

Lemma 8 $H^{-1}(0)$ *is compact.*

Proof. We want to show that any sequence $\left(\xi^{[k]},\tau^{[k]}\right)_{k\in\mathbb{N}}$ included in $H^{-1}(0)$ admits a convergent subsequence inside that set.
Since $\left\{\tau^{[k]}\right\}_{k\in\mathbb{N}}\subseteq[0,1]$, up to a subsequence, we can assume $\tau^{[k]}\to\widehat{\tau}\in[0,1]$. Consequently, defined

$$\begin{array}{ll} e_h^{[k]}=\left(1-\tau^{[k]}\right)e_h+\tau^{[k]}x_h^*, & \widehat{e}_h=\left(1-\widehat{\tau}\right)e_h+\widehat{\tau}x_h^*, \\ e^{[k]}=\left(1-\tau^{[k]}\right)e+\tau^{[k]}x^*, & \widehat{e}=\left(1-\widehat{\tau}\right)e+\widehat{\tau}x^*, \end{array}$$

we have $e_h^{[k]},\widehat{e}_h\in\mathbb{R}^G_{++}$, $e^{[k]},\widehat{e}\in\mathbb{R}^{GH}_{++}$ and

$$e_h^{[k]}\to\widehat{e}_h,\quad e^{[k]}\to\widehat{e}\quad\text{as }k\to\infty.$$

Claim 1. $\left(x^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{x}\in\mathbb{R}^{GH}_{++}$.
It suffices to show that $\left(x^{[k]}\right)_{k\in\mathbb{N}}$ is contained in a compact subset of $\mathbb{R}^{GH}$ included in $\mathbb{R}^{GH}_{++}$. Since $\left\{x^{[k]}\right\}_{k\in\mathbb{N}}\subseteq\mathbb{R}^{GH}_{++}$ is bounded from below from zero. Observing that Walras' law holds also for the homotopy system, we have

$$\sum_{h=1}^{H}\left(x_h^{[k]}-\left(\left(1-\tau^{[k]}\right)e_h+\tau^{[k]}x_h^*\right)\right)=0$$

and then for any $h'\in\mathcal{H}$

$$x_{h'}^{[k]}=\sum_{h=1}^{H}\left(\left(1-\tau^{[k]}\right)e_h^{[k]}+\tau^{[k]}x_h^*\right)-\sum_{h\neq h'}x_h^{[k]}.$$

Therefore, since $\left(e_h^{[k]}\right)_{k\in\mathbb{N}}$ converges and $\left(x^{[k]}\right)_{k\in\mathbb{N}}$ is bounded from below, it is bounded from above as well. We are left with showing the closedness. Remember that equations $(h.1.s)$, $(h.2.0)$, $(h.2.s)$, $(h.3.a)$ and $(h.4.j)$ in (6) say that for all $k\in\mathbb{N}$, $\left(x_h^{[k]},z_h^{[k]}\right)$ solves the problem

$$\begin{aligned} &\max_{(x_h,z_h)}\ u_h(x_h)\quad s.t. \\ &-\overline{p}^{[k]}(0)\left(x_h(0)-(1-\tau^{[k]})e_h(0)-\tau^{[k]}x_h^*(0)\right)-q^{[k]}z_h=0, \\ &-\overline{p}^{[k]}(s)\left(x_h(s)-(1-\tau^{[k]})e_h(s)-\tau^{[k]}x_h^*(s)\right)+y(s)z_h=0,\quad s\in\{1,...,S\} \\ &r_h\left((1-\tau^{[k]})z_h+\tau^{[k]}\widetilde{z}_h(\overline{p}^{[k]},q^{[k]}),\overline{p}^{[k]},q^{[k]}\right)\geq0. \end{aligned}\tag{13}$$

From Assumption u3 and the properties of $\widetilde{z}_h$ it is simple to prove that, for all $k\in\mathbb{N}$, we have that the vector $\left(e_h^{[k]},0\right)$ belongs to the constraint set described in (13), and therefore by definition of $x_h^{[k]}$, we have that

$$u_h\left(x_h^{[k]}\right)\geq u_h\left(e_h^{[k]}\right)\geq\min_{e_h\in\left\{e_h^{[k]}\right\}_{k\in\mathbb{N}}\cup\{\widehat{e}_h\}}u_h(e_h)=\underline{u}_h,\quad\forall k\in\mathbb{N},h\in\mathcal{H},$$

where the minimum is well defined because $\{e_h^{[k]}\}_{k\in\mathbb{N}}\cup\{\widehat{e}_h\}$ is a compact set. Therefore, $\left\{x^{[k]}\right\}_{k\in\mathbb{N}}\subseteq\left\{x_h\in\mathbb{R}^G_{++}:u_h(x_h)\geq\underline{u}_h\right\}$, which is a closed set in $\mathbb{R}^G$ and contained in $\mathbb{R}^G_{++}$ from Assumption u4. Therefore, $\left\{x^{[k]}\right\}_{k\in\mathbb{N}}$ is contained in a closed set, too. We can assume then, up to a subsequence that $x^{[k]}\to\hat{x}\in\mathbb{R}^{GH}_{++}$ as $k\to\infty$.

Claim 2. $\left(\lambda^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{\lambda}\in\mathbb{R}^{(S+1)H}_{++}$.
Since $p^{[k]C}(s)=1$, from $(h.1.s)$ in (6), we have

$$D_{x_h^C(s)}u_h\left(x_h^{[k]}\right)=\lambda_h^{[k]}(s),\quad\forall h\in\mathcal{H},s\in\mathcal{S}.$$

[3]See Villanacci et al. (2002), Lemma 18, p. 319.

Letting $k \to \infty$ in both sides, we get

$$\lim_{k\to\infty} \lambda_h^{[k]}(s) = \lim_{k\to\infty} D_{x_h^C(s)} u_h \left(x_h^{[k]}\right) = D_{x_h^C(s)} u_h \left(\widehat{x}_h\right) = \widehat{\lambda}_h(s) > 0$$

where the last strict inequality comes from Assumption u2.

Claim 3. $\left(p^{\backslash[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{p}^{\backslash} \in \mathbb{R}_{++}^{G-(S+1)}$.

Again from $(h.1.s)$ in (6), $D_{x_h^{\backslash}(s)} u_h \left(x_h^{[k]}\right) - \lambda_h^{[k]}(s) p^{\backslash}(s) = 0$. Therefore, we get

$$\lim_{k\to\infty} p^{\backslash[k]}(s) = \lim_{k\to\infty} \frac{D_{x_h^{\backslash}(s)} u_h \left(x_h^{[k]}\right)}{\lambda_h^{[k]}(s)} = \frac{D_{x_h^{\backslash}(s)} u_h \left(\widehat{x}_h\right)}{\widehat{\lambda}_h(s)} = \widehat{p}^{\backslash} > 0$$

where again the last strict inequality comes from Assumption u2.

Claim 4. $\left(z^{[k]}\right)_{k=1}^{\infty}$ *admits a subsequence converging to* $\widehat{z} \in \mathbb{R}^{AH}$.

From equation $(h.2.s)$ $s \geq 1$ in (6), and using the fact that Y has full rank A and the previous claims, we also get that $\left(z^{[k]}\right)_{k=1}^{\infty}$ converges.

Claim 5. $\left(q^{[k]}\right)_{k=1}^{\infty}$ *admits a subsequence converging to* $\widehat{q} \in \mathbb{R}^{A}$.

From the previous claims we know that, up to a subsequence, $\left(z^{[k]}, p^{\backslash[k]}\right)$ converges to $\left(\widehat{z}, \widehat{p}^{\backslash}\right)$. Let us fix now $a \in \mathcal{A}$ and prove $q^{a[k]} \to \widehat{q}^{a}$. From Assumption r5 there exists $h \in \mathcal{H}$ such that

$$(z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}_{++}^G \times \mathbb{R}^A \quad \Rightarrow \quad D_{z_h^a} r_h(z_h, p, q) = 0$$

and therefore for every k we get

$$D_{z_h^a} r_h \left((1-\tau^{[k]}) z_h^{[k]} + \tau^{[k]} \widetilde{z}_h(\overline{p}^{[k]}, q^{[k]}), \overline{p}^{[k]}, q^{[k]}\right) = 0$$

Then, $(h.3.a)$ in (6) can be written as

$$-\lambda_h^{[k]}(0)\, q^{a[k]} + \sum_{s=1}^{S} \lambda_h^{[k]}(s)\, y^a(s) = 0$$

and then

$$q^{a[k]} = \frac{\sum_{s=1}^{S} \lambda_h^{[k]}(s) y(s)}{\lambda_h^{[k]}(0)} \to \frac{\sum_{s=1}^{S} \widehat{\lambda}_h(s) y(s)}{\widehat{\lambda}_h(0)} = \widehat{q}^{a}.$$

Claim 6. $\left(\mu^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{\mu} \in \mathbb{R}^{JH}$.

Fix $h \in \mathcal{H}$ and let $\left\{\mathcal{J}_h^0, \mathcal{J}_h^1\right\}$ be a partition of the set of indexes $\mathcal{J}$ such that

$$\mathcal{J}_h^0 = \left\{ j \in \mathcal{J} : r_h^j \left((1-\widehat{\tau})\, \widehat{z}_h + \widehat{\tau} \widetilde{z}_h(\overline{\widehat{p}}, \widehat{q}), \overline{\widehat{p}}, \widehat{q} \right) = 0 \right\}$$

and

$$\mathcal{J}_h^1 = \left\{ j \in \mathcal{J} : r_h^j \left((1-\widehat{\tau})\, \widehat{z}_h + \widehat{\tau} \widetilde{z}_h(\overline{\widehat{p}}, \widehat{q}), \overline{\widehat{p}}, \widehat{q} \right) > 0 \right\}$$

with cardinality J_h^0 and J_h^1, respectively. If $j \in \mathcal{J}_h^1$, if k is large enough it is

$$r_h^j \left(\left(1-\tau^{[k]}\right) z_h^{[k]} + \tau^{[k]} \widetilde{z}_h(\overline{p}^{[k]}, q^{[k]}), \overline{p}^{[k]}, q^{[k]} \right) > 0$$

which implies $\mu_h^{j,[k]} = 0$. Then $\mu_h^{j,[k]} \to 0$ as $k \to \infty$, for all $j \in \mathcal{J}_h^1$.

From Assumption r4,

$$\text{rank} \left(D_{z_h} r_h^{\mathcal{J}_h^0} \left((1-\widehat{\tau})\, \widehat{z}_h + \widehat{\tau} \widetilde{z}_h(\overline{\widehat{p}}, \widehat{q}), \overline{\widehat{p}}, \widehat{q} \right) \right) = J_h^0.$$

Then $\left|\det M\left(\widehat{z}_h, \widehat{p}^{\setminus}, \widehat{q}, \widehat{\tau}\right)\right| > 0$, where $M\left(\widehat{z}_h, \widehat{p}^{\setminus}, \widehat{q}, \widehat{\tau}\right)$ is a well chosen square J_h^0-dimensional sub-matrix of

$$D_{z_h} r_h^{\mathcal{J}_h^0}\left(\left(1-\widehat{\tau}\right)\widehat{z}_h + \widehat{\tau}\widetilde{z}_h(\overline{\widehat{p}}, \widehat{q}), \overline{\widehat{p}}, \widehat{q}\right).$$

Let $M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right)$ be the square sub-matrix of

$$D_{z_h} r_h^{\mathcal{J}_h^0}\left(\left(1-\tau^{[k]}\right) z_h^{[k]} + \tau^{[k]}\widetilde{z}_h(\overline{p}^{[k]}, q^{[k]}), \overline{p}^{[k]}, q^{[k]}\right) \tag{14}$$

whose columns are the same than the ones of $M\left(\widehat{z}_h, \widehat{p}^{\setminus}, \widehat{q}, \widehat{\tau}\right)$. Of course

$$M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \to M\left(\widehat{z}_h, \widehat{p}^{\setminus}, \widehat{q}, \widehat{\tau}\right);$$

if k is large enough we have also $\left|\det M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right)\right| > 0$ and then

$$M^{-1}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \to M^{-1}\left(\widehat{z}_h, \widehat{p}^{\setminus}, \widehat{q}, \widehat{\tau}\right).$$

Making the needed permutations of rows and columns of (14) in order to have $M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right)$ in the top-left corner we get

$$\begin{array}{cc} & \begin{array}{cc} \quad J_h^0 \quad & \quad\quad A - J_h^0 \end{array} \\ \begin{array}{c} J_h^0 \\ J - J_h^0 \end{array} & \begin{pmatrix} M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) & M_{12}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \\ M_{21}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) & M_{22}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \end{pmatrix} \end{array}$$

Performing analogous permutation of the components of μ_h and $\eta = \left(-\lambda_h \left[\begin{smallmatrix} -q \\ Y \end{smallmatrix}\right]\right)$ in order to have the equalities in equations $(h.3.a)$, of (6) still satisfied we get:

$$\left[\mu_h^{0,[k]} \quad \mu_h^{1,[k]}\right]\left(1-\tau^{[k]}\right)\left[\begin{array}{cc} M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) & M_{12}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \\ M_{21}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) & M_{22}\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right) \end{array}\right] = \left[\begin{array}{cc} \eta^{I,[k]} & \eta^{II,[k]} \end{array}\right]$$

where $\mu_h^{0,[k]} \in \mathbb{R}^{J_h^0}$, $\mu_h^{1,[k]} \in \mathbb{R}^{J-J_h^0}$, $\eta^{I,[k]} \in \mathbb{R}^{J_h^0}$, $\eta^{II,[k]} \in \mathbb{R}^{A-J_h^0}$.

Since $\mu_h^{1,[k]} = 0$, we have in particular

$$\mu_h^{0,[k]}\left(1-\tau^{[k]}\right)\left[M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right)\right] = \eta^{I,[k]}$$

and then

$$\mu_h^{0,[k]} = \eta^{I,[k]}\left[M\left(z_h^{[k]}, p^{\setminus[k]}, q^{[k]}, \tau^{[k]}\right)\right]^{-1}$$

which implies $\mu_h^{0,[k]} \to \widehat{\mu}_h^0$ as $k \to \infty$ because the left hand side does converge. ■

4 Existence when some households cannot trade some assets

Let us consider an economy $E \in \mathcal{E}$. For every $h \in \mathcal{H}$ let us define the set $\mathcal{A}_h \subseteq \mathcal{A}$ such that, for every $a \in \mathcal{A}_h$,

$$D_{z_h^a} r_h(z_h, p, q) = 0, \quad \forall\,(z_h, p, q) \in \mathbb{R}^A \times \mathbb{R}_{++}^G \times \mathbb{R}^A.$$

Of course it may be $\mathcal{A}_h = \varnothing$ but, because of Assumption r5, surely we have

$$\bigcup_{h\in\mathcal{H}} \mathcal{A}_h = \mathcal{A}.$$

Let us define then the set $\mathfrak{B}(E) \subseteq \left(2^{\mathcal{A}}\right)^H$ as follows.[4] Given $\mathcal{B} = (\mathcal{B}_h)_{h\in\mathcal{H}} \in \left(2^{\mathcal{A}}\right)^H$ we have $\mathcal{B} \in \mathfrak{B}(E)$ if

[4] We denote by $2^{\mathcal{A}}$ the power set of $\mathcal{A}$.

1. for every $h \in \mathcal{H}$, $\mathcal{B}_h \subseteq \mathcal{A}_h$,
2. for every $a \in \mathcal{A}$, there exists $h \in \mathcal{H}$ such that $a \in \mathcal{A}_h \setminus \mathcal{B}_h$.

Moreover, for every $h \in \mathcal{H}$, let $B_h = \#(\mathcal{B}_h)$ and $B = \sum_{h\in\mathcal{H}} B_h$.

For given $(p, q, E) \in \mathbb{R}^G_{++} \times \mathbb{R}^A \times \mathcal{E}$ and $\mathcal{B} \in \mathfrak{B}(E)$, household $h \in \mathcal{H}$ maximization problem is now as follows.

Problem (Ph2)

$$\max_{(x_h,z_h)} \; u_h(x_h) \quad s.t.$$

$$\begin{aligned} & p\,(0)\,x_h\,(0) + qz_h \leq p\,(0)\,e_h\,(0) \\ & p\,(s)\,x_h\,(s) - p^C\,(s)\,y\,(s)\,z_h \leq p\,(s)\,e_h\,(s)\,, \quad s \in \{1,...,S\} \\ & r_h(z_h, p, q) \geq 0 \\ & z_h^a = 0, \quad a \in \mathcal{B}_h \end{aligned} \tag{15}$$

Moreover the definition of equilibrium is as follows.

Definition 9 $\left((x_h, z_h)_{h\in\mathcal{H}}, p, q\right) \in \left(\mathbb{R}^G_{++} \times \mathbb{R}^A\right)^H \times \mathbb{R}^G_{++} \times \mathbb{R}^A = \Theta$ *is an equilibrium for the economy* $E \in \mathcal{E}$ *and for* $\mathcal{B} \in \mathfrak{B}(E)$ *if for each* h*,* (x_h, z_h) *solves Problem* $(Ph2)$ *at* (p, q, E) *and* (x, z) *solves market clearing conditions at* e

$$\begin{aligned} & \sum_{h=1}^{H} (x_h - e_h) = 0 \\ & \sum_{h=1}^{H} z_h = 0 \end{aligned} \tag{16}$$

In the following, for every $E \in \mathcal{E}$ and $\mathcal{B} \in \mathfrak{B}(E)$ we denote by $\widetilde{\Theta}(E, \mathcal{B}) \subseteq \Theta$ the set of equilibria for the economy E and for $\mathcal{B} \in \mathfrak{B}(E)$ by $\widetilde{\Theta}_n(E, \mathcal{B})$ the set of *normalized equilibria*, that is,

$$\widetilde{\Theta}_n(E, \mathcal{B}) = \left\{ \left((x_h, z_h)_{h\in\mathcal{H}}, p, q\right) \in \widetilde{\Theta}(E, \mathcal{B}) : \forall s \in \mathcal{S}, p^C(s) = 1 \right\}.$$

The following existence theorem hold.

Theorem 10 *For every* $E \in \mathcal{E}$ *and for every* $\mathcal{B} \in \mathfrak{B}(E)$, $\widetilde{\Theta}_n(E, \mathcal{B}) \neq \varnothing$.

The proof of Theorem 10 follows the same lines that the one of Theorem 2. In fact for this new kind of equilibria $S + 1$ Walras' laws hold too and then the significant market clearing conditions are given by (3).

Consider then $E \in \mathcal{E}$ and $\mathcal{B} \in \mathfrak{B}(E)$ as fixed and define

$$\widetilde{\Xi} = \mathbb{R}^{GH}_{++} \times \mathbb{R}^{AH} \times \mathbb{R}^{(S+1)H}_{++} \times \mathbb{R}^{JH} \times \mathbb{R}^B \times \mathbb{R}^{G-(S+1)}_{++} \times \mathbb{R}^A$$

with generic element

$$\widetilde{\xi} = \left((x_h, z_h, \lambda_h, \mu_h, \beta_h)_{h=1}^H, p^\setminus, q\right) = \left(x, z, \lambda, \mu, \beta, p^\setminus, q\right),$$

where, for every $h \in \mathcal{H}$, $\beta_h \in \mathbb{R}^{B_h}$.

It is immediate to prove that if

$$\left((x_h, z_h)_{h=1}^H, \overline{p}, q\right) \in \widetilde{\Theta}_n(E, \mathcal{B})$$

then there exists $(\lambda, \mu, \beta) \in \mathbb{R}^{(S+1)H}_{++} \times \mathbb{R}^{JH} \times \mathbb{R}^B$ such that

$$\widetilde{\xi} = \left((x_h, z_h, \lambda_h, \mu_h, \beta_h)_{h=1}^H, p^\setminus, q\right)$$

solves the system $\widetilde{F}_{(E,\mathcal{B})}(\widetilde{\xi})=0$ where

$$\widetilde{F}_{(E,\mathcal{B})}:\widetilde{\Xi}\to\mathbb{R}^{\dim\Xi},$$

$$\widetilde{F}_{(E,\mathcal{B})}\left(x,z,\lambda,\mu,\beta,p^{\backslash},q\right)=$$

$$\begin{bmatrix}
\underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h\left(x_h\right)-\lambda_h\left(s\right)\overline{p}\left(s\right)\\
\underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}\left(0\right)\left(x_h\left(0\right)-e_h\left(0\right)\right)-qz_h\\
\underset{h\in\mathcal{H},s\in\mathcal{S}\backslash\{0\}}{(h.2.s)} & -\overline{p}\left(s\right)\left(x_h\left(s\right)-e_h\left(s\right)\right)+y\left(s\right)z_h\\
\underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.3.a1)} & -\lambda_h\left(0\right)q^a+\sum_{s=1}^{S}\lambda_h\left(s\right)y^a(s)+\sum_{j=1}^{J}\mu_h^jD_{z_h^a}r_h^j\left(z_h,\overline{p},q\right)+\beta_h^a\\
\underset{h\in\mathcal{H},a\in\mathcal{A}\backslash\mathcal{B}_h}{(h.3.a2)} & -\lambda_h\left(0\right)q^a+\sum_{s=1}^{S}\lambda_h\left(s\right)y^a(s)+\sum_{j=1}^{J}\mu_h^jD_{z_h^a}r_h^j\left(z_h,\overline{p},q\right)\\
\underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \min\left\{\mu_h^j,r_h^j\left(z_h,\overline{p},q\right)\right\}\\
\underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.5.a)} & z_h^a=0\\
(M.x) & \sum_{h=1}^{H}\left(x_h^{\backslash}-e_h^{\backslash}\right)\\
(M.z) & \sum_{h=1}^{H}z_h
\end{bmatrix} \tag{17}$$

while if

$$\widetilde{\xi}=\left(\left(x_h,z_h,\lambda_h,\mu_h,\beta_h\right)_{h=1}^{H},p^{\backslash},q\right)$$

solves the system $\widetilde{F}_{(E,\mathcal{B})}(\widetilde{\xi})=0$, then

$$\left(\left(x_h,z_h\right)_{h=1}^{H},\overline{p},q\right)\in\widetilde{\Theta}_n(E,\mathcal{B}).$$

Then Theorem 10 is a consequence of the following result.

Theorem 11 *For every $E\in\mathcal{E}$ and $\mathcal{B}\in\mathfrak{B}(E)$, there exists $\widetilde{\xi}\in\widetilde{\Xi}$ such that $\widetilde{F}_{(E,\mathcal{B})}(\widetilde{\xi})=0$.*

Proof of Theorem 11. Let $E=(e,u,Y,r)\in\mathcal{E}$ and $\mathcal{B}\in\mathfrak{B}(E)$ be fixed. Then it is well known that there exists a Pareto optimal allocation x^* for u such that $\sum_{h=1}^{H}x_h^*=\sum_{h=1}^{H}e_h$. Moreover, there exists $(\chi^*,\gamma^*)\in\mathbb{R}_{++}^H\times\mathbb{R}_{++}^G$ such that (x^*,χ^*,γ^*) is the unique solution to the following system

$$\begin{cases}
\chi_1^*-1 & =0\\
\chi_h^*Du_h\left(x_h\right)-\gamma^* & =0\\
\left(u_h\left(x_h\right)-u_h\left(x_h^*\right)\right)_{h\neq1} & =0\\
-\sum_{h=1}^{H}x_h+\sum_{h=1}^{H}x_h^* & =0
\end{cases} \tag{18}$$

Define

$$F(\widetilde{\xi})=\widetilde{F}_{(E,\mathcal{B})}(\widetilde{\xi})\quad\forall\,\widetilde{\xi}\in\widetilde{\Xi}$$

and consider the system in the unknowns $\xi = (x, \lambda, z, \mu, \beta, p^{\setminus}, q) \in \widetilde{\Xi}$ and $\tau \in [0,1]$ given by

$$
\begin{cases}
\underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h(x_h) - \lambda_h(s)\,\overline{p}(s) = 0 \\
\underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\left(x_h(0) - ((1-\tau)\,e_h(0) + \tau x_h^*(0))\right) - qz_h = 0 \\
\underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}(s)\left(x_h(s) - ((1-\tau)\,e_h(s) + \tau x_h^*(s))\right) + y(s)\,z_h = 0 \\
\underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.3.a1)} & -\lambda_h(0)\,q^a + \sum\limits_{s=1}^{S}\lambda_h(s)\,y^a(s) + \sum\limits_{j=1}^{J}\mu_h^j(1-\tau)D_{z_h^a}r_h^j\left((1-\tau)\,z_h + \tau\widetilde{z}_h(\overline{p},q),\overline{p},q\right) + \beta_h^a = 0 \\
\underset{h\in\mathcal{H},a\in\mathcal{A}\setminus\mathcal{B}_h}{(h.3.a2)} & -\lambda_h(0)\,q^a + \sum\limits_{s=1}^{S}\lambda_h(s)\,y^a(s) + \sum\limits_{j=1}^{J}\mu_h^j(1-\tau)D_{z_h^a}r_h^j\left((1-\tau)\,z_h + \tau\widetilde{z}_h(\overline{p},q),\overline{p},q\right) = 0 \\
\underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \min\left\{\mu_h^j, r_h^j\left((1-\tau)\,z_h + \tau\widetilde{z}_h(\overline{p},q),\overline{p},q\right)\right\} = 0 \\
\underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.5.a)} & z_h^a = 0 \\
(M.x) & \sum\limits_{h=1}^{H}\left(x_h^{\setminus} - e_h^{\setminus}\right) = 0 \\
(M.z) & \sum\limits_{h=1}^{H} z_h = 0
\end{cases}
\tag{19}
$$

where, for every $h \in \mathcal{H}$, the function $\widetilde{z}_h$ is defined in Proposition 5.

Define now

$$H : \widetilde{\Xi} \times [0,1] \to \mathbb{R}^{\dim \widetilde{\Xi}}$$

$$\left(\widetilde{\xi}, \tau\right) \mapsto \text{ left hand side of system (19)},$$

and

$$G : \widetilde{\Xi} \to \mathbb{R}^{\dim \widetilde{\Xi}}, \;\; \xi \mapsto H\left(\widetilde{\xi}, 1\right).$$

Observe that

$$H\left(\widetilde{\xi}, 0\right) = F\left(\widetilde{\xi}\right).$$

Let us now verify that Theorem 4 can be applied. F and G are defined in the same open subset of $\mathbb{R}^{\dim \widetilde{\Xi}}$, take values in $\mathbb{R}^{\dim \widetilde{\Xi}}$ (and those sets are C^2 boundaryless manifolds of the same dimension) and are continuous.

Of course H is a continuous homotopy from F to G. Moreover, Lemmas 12, 13 and 14 prove the following needed results.

- $G^{-1}(0) = \left\{\widetilde{\xi}^*\right\}$;
- G is C^1 in a neighborhood of $\widetilde{\xi}^*$ and $\operatorname{rank} D_{\widetilde{\xi}} G\left(\widetilde{\xi}^*\right) = \dim \widetilde{\Xi}$;
- $H^{-1}(0)$ is compact.

From Theorem 4, it then follows, as desired, that $F^{-1}(0) \neq \varnothing$. ■

Lemma 12 $G^{-1}(0) = \left\{\widetilde{\xi}^*\right\} = \left\{(x^*, z^*, \lambda^*, \mu^*, \beta^*, p^{\setminus *}, q^*)\right\} \in \widetilde{\Xi}$, *where*

$$x_h^* = x_h^*, \quad \lambda_h^* = \left(\frac{\gamma^{*C}(s)}{\chi_h^*}, s \in \mathcal{S}\right), \quad z_h^* = 0, \quad \mu_h^* = 0, \quad \beta_h^* = 0 \quad \forall h \in \mathcal{H},$$

$$p^{*\setminus} = \left(\frac{\gamma^{*c}(s)}{\gamma^{*C}(s)}, s \in \mathcal{S}, c \neq C\right), \quad q^* = \sum_{s=1}^{S}\left(\frac{\gamma^{*C}(s)}{\gamma^{*C}(0)}\right) y(s).$$

Proof. $G^{-1}(0)$ is the set of solutions of system (19) at $\tau = 1$, that is, the set of solution of the system

$$\begin{cases} \underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h(x_h) - \lambda_h(s)\,\overline{p}(s) = 0 \\ \underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}(0)\,(x_h(0) - x_h^*(0)) - qz_h = 0 \\ \underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}(s)\,(x_h(s) - x_h^*(s)) + y(s)\,z_h = 0 \\ \underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.3.a1)} & -\lambda_h(0)\,q^a + \sum\limits_{s=1}^{S}\lambda_h(s)\,y^a(s) + \beta_h^a = 0 \\ \underset{h\in\mathcal{H},a\in\mathcal{A}\setminus\mathcal{B}_h}{(h.3.a2)} & -\lambda_h(0)\,q^a + \sum\limits_{s=1}^{S}\lambda_h(s)\,y^a(s) = 0 \\ \underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \mu_h^j = 0 \\ \underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.5.a)} & z_h^a = 0 \\ (M.x) & \sum\limits_{h=1}^{H}\left(x_h^\backslash - e_h^\backslash\right) = 0 \\ (M.z) & \sum\limits_{h=1}^{H} z_h = 0 \end{cases} \tag{20}$$

Using the definition of $\widetilde{\xi}^*$, it is easy to check that $\widetilde{\xi}^* \in G^{-1}(0)$. Define now $\widehat{\xi} = \left(\widehat{x}, \widehat{\lambda}, \widehat{z}, \widehat{\mu}, \widehat{\beta}, \widehat{p}^\backslash, \widehat{q}\right)$ and assume $\widehat{\xi} \in G^{-1}(0)$: we prove $\widehat{\xi} = \widetilde{\xi}^*$.
Claim 1. $\widehat{\mu} = \mu^*$. Obvious.
Claim 2. $\widehat{x} = x^*$. Suppose by contradiction $\widehat{x} \neq x^*$. Consider $\widetilde{x} = \frac{1}{2}(\widehat{x} + x^*)$. Of course it is

$$\sum_{h=1}^{H}\widetilde{x}_h = \frac{1}{2}\left(\sum_{h=1}^{H}\widehat{x}_h + \sum_{h=1}^{H}x_h^*\right) = \sum_{h=1}^{H}x_h^*. \tag{21}$$

Since (x_h^*, z_h^*) is feasible for the maximization problem of the h-th household, whose first order conditions are given by equations $(h.1.s)$, $(h.2.0)$, $(h.2.s)$, $(h.3.a1)$, $(h.3.a2)$ and $(h.5.a)$ in (20), it is $u_h(\widehat{x}_h) \geq u_h(x_h^*)$. Moreover, from Assumption u3, we have

$$u_h(\widetilde{x}_h) > \min\{u_h(x_h^*), u_h(\widehat{x}_h)\} = u_h(x_h^*) \quad \forall h \in \mathcal{H}. \tag{22}$$

But (21) and (22) contradict the Pareto optimality of x^*.
Claim 3. $\widehat{\lambda} = \lambda^*$. Since $\widehat{x} = x^*$, from $(h.1.s)$ in (20) we have in particular

$$\widehat{\lambda}_h(s) = D_{x_h^C(s)}u_h(\widehat{x}_h) = D_{x_h^C(s)}u_h(x_h^*) = \lambda_h^*(s), \quad \forall h \in \mathcal{H}, s \in \mathcal{S}.$$

Claim 4. $\widehat{p}^\backslash = p^{\backslash *}$. From $(h.1.s)$ in (20) and Claims 2 and 3,

$$\widehat{\overline{p}}(s) = \frac{D_{x_h(s)}u_h(\widehat{x}_h)}{\widehat{\lambda}_h(s)} = \frac{D_{x_h(s)}u_h(x_h^*)}{\lambda_h^*(s)} = \overline{p}^*(s) \quad \forall s \in \mathcal{S}.$$

Claim 5. $\widehat{z} = z^*$. From $(h.2.s)$ $s \geq 1$, in (20) and Claim 2, we have in particular,

$$Y\widehat{z}_h = 0, \quad \forall h \in \mathcal{H}.$$

From Assumption Y, we get that for any $h \in \mathcal{H}$, $\widehat{z}_h = 0 = z_h^*$.
Claim 6. $\widehat{q} = q^*$. From the assumption on $\mathcal{B}$ and $(h.3.a2)$ in (20), for every $a \in \mathcal{A}$, there exists $h \in \mathcal{H}$

$$\widehat{q}^a = \sum_{s=1}^{S}\frac{\widehat{\lambda}_h(s)}{\widehat{\lambda}_h(0)}y^a(s) = \sum_{s=1}^{S}\frac{\lambda_h^*(s)}{\lambda_h^*(0)}y^a(s) = q^{a*},$$

Claim 7. $\widehat{\beta} = \beta^*$. From $(h.3.a1)$ in (20) and Claims 1 and 6 we get the claim and the proof is now completed. ■

Lemma 13 *G is C^1 in a neighborhood of $\widetilde{\xi}^*$ and* $\operatorname{rank} D_{\widetilde{\xi}}G\left(\widetilde{\xi}^*\right) = \dim\widetilde{\Xi}$.

Proof. It is immediate to prove that in a suitable small neighborhood of $\widetilde{\xi}^*$ we have

$$G(\widetilde{\xi}) = \left[\begin{array}{cl} \underset{h\in\mathcal{H},s\in\mathcal{S}}{(h.1.s)} & D_{x_h(s)}u_h\left(x_h\right) - \lambda_h\left(s\right)\overline{p}\left(s\right) \\ \underset{h\in\mathcal{H}}{(h.2.0)} & -\overline{p}\left(0\right)\left(x_h\left(0\right) - x_h^*(0)\right) - qz_h \\ \underset{h\in\mathcal{H},s\in S\setminus\{0\}}{(h.2.s)} & -\overline{p}\left(s\right)\left(x_h\left(s\right) - x_h^*(s)\right) + y\left(s\right)z_h \\ \underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.3.a1)} & -\lambda_h\left(0\right)q^a + \sum\limits_{s=1}^{S}\lambda_h\left(s\right)y^a(s) + \beta_h^a \\ \underset{h\in\mathcal{H},a\in\mathcal{A}\setminus\mathcal{B}_h}{(h.3.a2)} & -\lambda_h\left(0\right)q^a + \sum\limits_{s=1}^{S}\lambda_h\left(s\right)y^a(s) \\ \underset{h\in\mathcal{H},j\in\mathcal{J}}{(h.4.j)} & \mu_h^j \\ \underset{h\in\mathcal{H},a\in\mathcal{B}_h}{(h.5.a)} & z_h^a \\ (M.x) & \sum\limits_{h=1}^{H}\left(x_h^{\setminus} - e_h^{\setminus}\right) \\ (M.z) & \sum\limits_{h=1}^{H} z_h \end{array} \right]$$

and this is a C^1 function. The computation of $D_{\widetilde{\xi}}G\left(\widetilde{\xi}^*\right)$ is described below

	x_h	λ_h	z_h	μ_h	β_h	$p^{\setminus}$	q
$(h.1)$	$D^2u_h\left(x_h^*\right)$	$-\Phi\left(\overline{p}^*\right)^T$				$-\Lambda_h^*$	
$(h.2)$	$-\Phi\left(\overline{p}^*\right)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}$				
$(h.3)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}^T$			$\widetilde{I}_{\mathcal{B}_h}$		$-\lambda_h^*\left(0\right)I$
$(h.4)$				I			
$(h.5)$			$\widetilde{I}_{\mathcal{B}_h}^T$				
$(M.x)$	$\widehat{I}$						
$(M.z)$			I				

where $\Phi\left(\overline{p}\right)$, $\widehat{I}$, Λ_h^* are defined in (10), (11) and (12) respectively, and if $\mathcal{B}_h = \{b_1, \ldots, b_{B_h}\}$, with $b_1 < \ldots < b_{B_h}$, then

$$\widetilde{I}_{\mathcal{B}_h} = \left(\gamma_{i,j}\right)_{i=1,\ldots,A}^{j=1,\ldots,B_h} \quad \text{where} \quad \gamma_{i,j} = \begin{cases} 1 & \text{if } i = b_j \\ 0 & \text{if } i \neq b_j \end{cases}$$

The above matrix has full rank if and only if the following does:

$M^* =$

	x_h	λ_h	z_h	β_h	$p^{\setminus}$	q
$(h.1)$	$D^2u_h\left(x_h^*\right)$	$-\Phi\left(\overline{p}^*\right)^T$			$-\Lambda_h^*$	
$(h.2)$	$-\Phi\left(\overline{p}^*\right)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}$			
$(h.3)$		$\begin{bmatrix} -q^* \\ Y \end{bmatrix}^T$		$\widetilde{I}_{\mathcal{B}_h}$		$-\lambda_h^*\left(0\right)I$
$(h.5)$			$\widetilde{I}_{\mathcal{B}_h}^T$			
$(M.x)$	$\widehat{I}$					
$(M.z)$			I			

(23)

We are going to use a simple modification of the standard argument. Consider a vector

$$\Delta\widetilde{\xi} = \left(\left(\Delta x_h, \Delta z_h, \Delta\lambda_h, \Delta\mu_h, \Delta\beta_h\right)_{h=1}^{H}, \Delta p^{\setminus}, \Delta q\right)$$

and prove that if $M\Delta\widetilde{\xi}=0$ then $\Delta\widetilde{\xi}=0$. Let us explicitly write $M\Delta\widetilde{\xi}=0$ as follows:

$$\begin{cases} (h.1) & D^2u_h\left(x_h^*\right)\Delta x_h-\Phi\left(\overline{p}^*\right)^T\Delta\lambda_h-\Lambda_h^*\Delta p^{\setminus}=0\\ (h.2) & -\Phi\left(\overline{p}^*\right)\Delta x_h+\left[\begin{smallmatrix}-q^*\\ Y\end{smallmatrix}\right]\Delta z_h=0\\ (h.3) & \left[\begin{smallmatrix}-q^*\\ Y\end{smallmatrix}\right]^T\Delta\lambda_h+\widetilde{I}_{\mathcal{B}_h}\Delta\beta_h-\lambda_h^*\left(0\right)\Delta q=0\\ (h.5) & \widetilde{I}_{\mathcal{B}_h}^T\Delta z_h=0\\ (M.x) & \sum\limits_{h=1}^{H}\Delta x_h^{\setminus}=0\\ (M.z) & \sum\limits_{h=1}^{H}\Delta z_h=0\end{cases}\tag{24}$$

and remember that, from (20) we have in particular

$$\begin{cases} (h.1) & Du_h\left(x_h^*\right)-\lambda_h^*\Phi\left(\overline{p}^*\right)=0\\ (h.3) & \left[\begin{smallmatrix}-q^*\\ Y\end{smallmatrix}\right]^T\lambda_h^*=0\end{cases}\tag{25}$$

First we claim that if, for every $h\in\mathcal{H}$, $\Delta x_h=0$ then $\Delta\widetilde{\xi}=0$. In fact if, for every $h\in\mathcal{H}$, $\Delta x_h=0$ then, from $(h.2)$ in (24) we have

$$\begin{bmatrix}-q^*\\ Y\end{bmatrix}\Delta z_h=0,$$

and from Assumption Y we have $\Delta z_h=0$. From $(h.1)$ in (24) it follows $\Delta\lambda_h=0$ that implies $\Delta p^{\setminus}=0$ as well. From $(h.3)$ in (24) we obtain the equality

$$\widetilde{I}_{\mathcal{B}_h}\Delta\beta_h=\lambda_h^*\left(0\right)\Delta q.$$

We know that, for every $a\in\mathcal{A}$, there exists $h\in\mathcal{H}$ such that $a\in\mathcal{A}_h\setminus\mathcal{B}_h$. Then we have that, for every $a\in\mathcal{A}$, $\Delta q^a=0$ that is $\Delta q=0$. As a consequence we have that, for every $h\in\mathcal{H}$, $\Delta\beta_h=0$ too and the proof of the claim is complete.

Let us assume now there exists $h'\in\mathcal{H}$ such that $\Delta x_h'\neq 0$ and prove that this leads to a contradiction. First of all let us show that, for every $h\in\mathcal{H}$, $Du_h(x_h^*)\Delta x_h=0$. In fact multiplying $(h.1)$ in (25) by Δx_h, for every $h\in\mathcal{H}$, we obtain

$$Du_h\left(x_h^*\right)\Delta x_h=\lambda_h^*\Phi\left(\overline{p}^*\right)\Delta x_h.$$

From $(h.2)$ in (24) we have

$$Du_h\left(x_h^*\right)\Delta x_h=\lambda_h^*\begin{bmatrix}-q^*\\ Y\end{bmatrix}\Delta z_h$$

and the right hand side is zero because of $(h.3)$ in (25). The claim then follows and since $\Delta x_{h'}\neq 0$ and Assumption u3 holds, we have in particular that

$$\sum_{h=1}^{H}\chi_h^*\Delta x_hD^2u_h(x_h^*)\Delta x_h<0.\tag{26}$$

We end the proof of the lemma showing that it has also to be

$$\sum_{h=1}^{H}\chi_h^*\Delta x_hD^2u_h(x_h^*)\Delta x_h=0,\tag{27}$$

and then finding a contradiction.
From $(h.1)$ in (24) we have

$$\chi_h^*\Delta x_hD^2u_h(x_h^*)\Delta x_h=\chi_h^*\Delta x_h\Phi\left(\overline{p}^*\right)^T\Delta\lambda_h+\chi_h^*\Delta x_h\Lambda_h^*\Delta p^{\setminus}.$$

Moreover from $(h.2)$ and $(h.3)$ in (24) we have

$$\chi_h^*\Delta x_h\Phi\left(\overline{p}^*\right)^T\Delta\lambda_h=\chi_h^*\Delta z_h\begin{bmatrix}-q^*\\ Y\end{bmatrix}^T\Delta\lambda_h=\Delta z_h\left(\gamma^{*C}(0)\Delta q-\chi_h^*\widetilde{I}_{\mathcal{B}_h}\Delta\beta_h\right).$$

and by definition of Λ_h^* and Γ^* we have

$$\chi_h^* \Delta x_h \Lambda_h^* \Delta p^{\setminus} = \Delta x_h \Gamma^* \Delta p^{\setminus}.$$

Then

$$\sum_{h=1}^{H} \chi_h^* \Delta x_h D^2 u_h(x_h^*) \Delta x_h = \left(\sum_{h=1}^{H} \Delta z_h \right) \gamma^{*C}(0) \Delta q - \sum_{h=1}^{H} \chi_h^* \Delta z_h \widetilde{I}_{\mathcal{B}_h} \Delta \beta_h + \left(\sum_{h=1}^{H} \Delta x_h \right) \Gamma^* \Delta p^{\setminus}.$$

We are going to prove (27) showing then the right hand side of the above equality is zero. Of course by $(M.z)$ in (24)

$$\left(\sum_{h=1}^{H} \Delta z_h \right) \gamma^{*C}(0) \Delta q = 0.$$

Moreover we have

$$\left(\sum_{h=1}^{H} \Delta x_h \right) \Gamma^* \Delta p^{\setminus} = \left(\sum_{h=1}^{H} \Delta x_h^{\setminus}, \sum_{h=1}^{H} \Delta x_h^{C} \right) \Gamma^* \Delta p^{\setminus} = 0$$

using $(M.x)$ in (24) and the properties of Γ^*. Finally from $(h.5)$ in (24) we have

$$\sum_{h=1}^{H} \chi_h^* \Delta z_h \widetilde{I}_{\mathcal{B}_h} \Delta \beta_h = 0$$

and the proof is completed. ∎

Lemma 14 *$H^{-1}(0)$ is compact.*

Proof. We want to show that any sequence $\left(\xi^{[k]}, \tau^{[k]}\right)_{k \in \mathbb{N}}$ included in $H^{-1}(0)$ admits a convergent subsequence inside that set.
Since $\left\{\tau^{[k]}\right\}_{k \in \mathbb{N}} \subseteq [0,1]$, up to a subsequence, we can assume $\tau^{[k]} \to \widehat{\tau} \in [0,1]$. Consequently, defined

$$\begin{aligned} e_h^{[k]} &= \left(1 - \tau^{[k]}\right) e_h + \tau^{[k]} x_h^*, & \widehat{e}_h &= (1 - \widehat{\tau}) e_h + \widehat{\tau} x_h^*, \\ e^{[k]} &= \left(1 - \tau^{[k]}\right) e + \tau^{[k]} x^*, & \widehat{e} &= (1 - \widehat{\tau}) e + \widehat{\tau} x^*, \end{aligned}$$

we have $e_h^{[k]}, \widehat{e}_h \in \mathbb{R}_{++}^{G}$, $e^{[k]}, \widehat{e} \in \mathbb{R}_{++}^{GH}$ and

$$e_h^{[k]} \to \widehat{e}_h, \quad e^{[k]} \to \widehat{e} \quad \text{as } k \to \infty.$$

Claim 1. $\left(x^{[k]}\right)_{k \in \mathbb{N}}$ admits a subsequence converging to $\widehat{x} \in \mathbb{R}_{++}^{GH}$.
It suffices to show that $\left(x^{[k]}\right)_{k \in \mathbb{N}}$ is contained in a compact subset of $\mathbb{R}^{GH}$ included in $\mathbb{R}_{++}^{GH}$. Since $\left\{x^{[k]}\right\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_{++}^{GH}$ is bounded from below from zero. Observing that Walras' law holds also for the homotopy system, we have

$$\sum_{h=1}^{H} \left(x_h^{[k]} - \left(\left(1 - \tau^{[k]}\right) e_h + \tau^{[k]} x_h^* \right) \right) = 0$$

and then for any $h' \in \mathcal{H}$

$$x_{h'}^{[k]} = \sum_{h=1}^{H} \left(\left(1 - \tau^{[k]}\right) e_h^{[k]} + \tau^{[k]} x_h^* \right) - \sum_{h \neq h'} x_h^{[k]}.$$

Therefore, since $\left(e_h^{[k]}\right)_{k \in \mathbb{N}}$ converges and $\left(x^{[k]}\right)_{k \in \mathbb{N}}$ is bounded from below, it is bounded from above as well. We are left with showing the closedness. Remember that equations $(h.1.s)$, $(h.2.0)$, $(h.2.s)$,

$(h.3.a1)$, $(h.3.a2)$, $(h.4.j)$ and $(h.5.a)$ in (19) say that for all $k \in \mathbb{N}$, $\left(x_h^{[k]}, z_h^{[k]}\right)$ solves the problem

$$\begin{aligned}
&\max_{(x_h, z_h)} \; u_h(x_h) \quad s.t. \\
&-\overline{p}^{[k]}(0)\left(x_h(0) - (1-\tau^{[k]})e_h(0) - \tau^{[k]}x_h^*(0)\right) - q^{[k]}z_h = 0, \\
&-\overline{p}^{[k]}(s)\left(x_h(s) - (1-\tau^{[k]})e_h(s) - \tau^{[k]}x_h^*(s)\right) + y(s)\,z_h = 0, \quad s = 1, ..., S \\
&r_h\left((1-\tau^{[k]})z_h + \tau^{[k]}\widetilde{z}_h(\overline{p}^{[k]}, q^{[k]}), \overline{p}^{[k]}, q^{[k]}\right) \geq 0 \\
&z_h^a = 0, \quad a \in \mathcal{B}_h
\end{aligned} \tag{28}$$

From Assumption u3 and the properties of $\widetilde{z}_h$ it is simple to prove that, for all $k \in \mathbb{N}$, we have that the vector $\left(e_h^{[k]}, 0\right)$ belongs to the constraint defined by (28), and therefore by definition of $x_h^{[k]}$, we have that

$$u_h\left(x_h^{[k]}\right) \geq u_h\left(e_h^{[k]}\right) \geq \min_{e_h \in \left\{e_h^{[k]}\right\}_{k\in\mathbb{N}} \cup \{\widehat{e}_h\}} u_h(e_h) = \underline{u}_h, \quad \forall k \in \mathbb{N}, h \in \mathcal{H},$$

where the minimum is well defined because $\{e_h^{[k]}\}_{k\in\mathbb{N}} \cup \{\widehat{e}_h\}$ is a compact set. Therefore, $\left\{x^{[k]}\right\}_{k\in\mathbb{N}} \subseteq \left\{x_h \in \mathbb{R}^G_{++} : u_h(x_h) \geq \underline{u}_h\right\}$, which is a closed set in $\mathbb{R}^G$ and contained in $\mathbb{R}^G_{++}$ from Assumption u4. Therefore, $\left\{x^{[k]}\right\}_{k\in\mathbb{N}}$ is contained in a closed set, too. We can assume then, up to a subsequence that $x^{[k]} \to \hat{x} \in \mathbb{R}^{GH}_{++}$ as $k \to \infty$.

Claim 2. $\left(\lambda^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{\lambda} \in \mathbb{R}^{(S+1)H}_{++}$.
See Claim 2 of the proof of Lemma 8.

Claim 3. $\left(p^{\backslash[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{p}^{\backslash} \in \mathbb{R}^{G-(S+1)}_{++}$.
See Claim 3 of the proof of Lemma 8.

Claim 4. $\left(z^{[k]}\right)_{k=1}^{\infty}$ *admits a subsequence converging to* $\widehat{z} \in \mathbb{R}^{AH}$.
See Claim 4 of the proof of Lemma 8.

Claim 5. $\left(q^{[k]}\right)_{k=1}^{\infty}$ *admits a subsequence converging to* $\widehat{q} \in \mathbb{R}^{A}$.
From the previous claims we know that, up to a subsequence, $\left(z^{[k]}, p^{\backslash[k]}\right)$ converges to $\left(\widehat{z}, \widehat{p}^{\backslash}\right)$. Let us fix now $a \in \mathcal{A}$ and prove $q^{a[k]} \to \widehat{q}^a$. From Assumption r5 and the properties of $\mathcal{B}$ we know there exists $h \in \mathcal{H}$ such that $a \in \mathcal{A}_h \setminus \mathcal{B}_h$ and therefore, for every k, we get

$$D_{z_h^a} r_h\left((1-\tau^{[k]})z_h^{[k]} + \tau^{[k]}\widetilde{z}_h(\overline{p}^{[k]}, q^{[k]}), \overline{p}^{[k]}, q^{[k]}\right) = 0$$

Then, $(h.3.a2)$ in (19) can be written as

$$-\lambda_h^{[k]}(0)\,q^{a[k]} + \sum_{s=1}^{S}\lambda_h^{[k]}(s)\,y^a(s) = 0$$

and then

$$q^{a[k]} = \frac{\sum_{s=1}^{S}\lambda_h^{[k]}(s)y(s)}{\lambda_h^{[k]}(0)} \to \frac{\sum_{s=1}^{S}\widehat{\lambda}_h(s)y(s)}{\widehat{\lambda}_h(0)} = \widehat{q}^a.$$

Claim 6. $\left(\mu^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{\mu} \in \mathbb{R}^{JH}$.
See Claim 6 of the proof of Lemma 8.

Claim 7. $\left(\beta^{[k]}\right)_{k\in\mathbb{N}}$ *admits a subsequence converging to* $\widehat{\beta} \in \mathbb{R}^{B}$.
It immediately follows from $(h.3.a1)$ in (19) and the already proved claims. The proof of the lemma is then completed. ■

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